The Dirichlet Problem for the Kohn Laplacian on the Heisenberg Group,

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For \((x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\), denote
\[ X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}. \]
\[ \mathcal{L} = -\frac{1}{4} \sum_{j=1}^{n} X_j^2 + Y_j^2 + ia \frac{\partial}{\partial t}. \]
When \(a = n - 2q\), \(\mathcal{L}\) represents the action of the Kohn Laplacian \(\Box_b\) on \(q\)-forms on the Heisenberg group. For \(-n < a < n\), we construct a parametrix for the Dirichlet problem in smooth domains \(D\) near non-characteristic points of \(\partial D\). A point \(w\) of \(\partial D\) is non-characteristic if one of \(X_1, \ldots, X_n, Y_1, \ldots, Y_n\) is transverse to \(\partial D\) at \(w\). This yields sharp local estimates in the Dirichlet problem in the appropriate non-isotropic Lipschitz classes. The main new tool is a "convolution calculus" of pseudo-differential operators that can be applied to the relevant layer potentials, for which the usual asymptotic composition formula is false. Characteristic points are treated in Part II.

Contents. 1. Introduction. 2. Local coordinates on a hypersurface. 3. Convolution calculus. 4. \(\Gamma_a\) spaces and restriction. 5. Single and double layer potentials. 6. An inversion on \(\mathbb{H}^{n-1} \times \mathbb{R}\). 7. Regularity in the Dirichlet problem for \(\mathcal{L}_a\). Appendix A. Appendix B. References.

1. INTRODUCTION

Background

Let \(x\) and \(y\) belong to \(\mathbb{R}^n\) and \(t\) belong to \(\mathbb{R}\). Denote
\[ X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad \text{and} \]
\[ \mathcal{L}_a = -\frac{1}{4} \sum_{j=1}^{n} X_j^2 + Y_j^2 + iaT. \]
The operator \(\mathcal{L}_a\) for \(a = n - 2q\) represents the action of the Kohn Laplacian \(\Box_b\) on \(q\)-forms on the boundary of the unbounded ball in \(\mathbb{C}^{n+1}\) (see [6]). Let

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$D$ be a smooth domain in $\mathbb{R}^{2n+1}$. Kohn's subelliptic estimate [13] says that for $-n < \alpha < n$,
\[
\text{Re}(\mathcal{L}_\alpha \phi, \phi) \geq c \|\phi\|^2_{H^s_{\alpha}} \quad \text{for} \quad \phi \in C^\infty_0(D).
\] (1.1)

Here $H_s$ denotes the Sobolev space of order $s$,
\[
\|\phi\|^2_{H^s} = \int |\hat{\phi}(\xi)|^2(1 + |\xi|^2)^s \, d\xi.
\]

It follows from the Lax–Milgram lemma (see, for instance, [14, Theorem 1]) that the Dirichlet problem
\[
\mathcal{L}_\alpha u = f \quad \text{in} \quad D,
\]
\[
u|_{\partial D} = g
\]
has a unique generalized solution for all $f \in C^\infty(\bar{D})$ and $g \in C^\infty(\partial D)$. Kohn and Nirenberg [14] showed that the solution $u$ is smooth up to the boundary near all non-characteristic points. A boundary point $w$ of $\partial D$ is characteristic if the principal symbol of $\mathcal{L}_\alpha$ annihilates the normal vector to $\partial D$ at $w$. Equivalently, the point $w \in \partial D$ is characteristic if $X_j$ and $Y_j$ are tangent to $\partial D$ at $w$ for all $j = 1, \ldots, n$. Related regularity problems at non-characteristic points are treated by Oleinik [16, 17], Baouendi [1], Derridj [12], and Hung [12].

In Part I we will examine boundary regularity near non-characteristic points of $\partial D$. We will construct a parametrix for the Dirichlet problem that will give sharp estimates on a scale of "non-isotropic" Lipschitz spaces suited to the operator $\mathcal{L}_\alpha$. The results obtained here will be used in Part II where we treat the problem of boundary regularity at characteristic points. (As we shall see, $C^\infty$ regularity may fail at those points.)

The Heisenberg group of degree $n$ is the Lie group $\mathbb{H}^n$ whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ with coordinates $(z, t) = (x + iy, t)$ and whose group law is given by $(z, t)(z', t') = (z + z', t + t' + 2 \text{Im} z \cdot z')$, $z \cdot z' = \sum_{j=1}^n z_j \bar{z}_j'$. The vector fields $X_j, Y_j, j = 1, \ldots, n$, and $T$ form a basis for the left-invariant vector fields of $\mathbb{H}^n$. The commutation relations are
\[
[Y_j, X_k] = 4 \delta_{jk} T, \quad [X_j, X_k] = [Y_j, Y_k] = [X_j, T] = [Y_j, T] = 0.
\] (1.2)

In particular, the whole Lie algebra is generated by $X_j, Y_j$ and their commutators. It follows that the set of non-characteristic points of $\partial D$ is open and dense in $\partial D$. For example, the only characteristic points on the boundary of the ball $|z|^2 + t^2 < r^2$ are the north and south poles $(0, r)$ and $(0, -r)$.

On the Heisenberg group there is a natural norm $|(z, t)| = (|z|^4 + t^2)^{1/4}$ and
a notion of dilation \( \delta(z, t) = (\delta z, \delta^2 t) \). At this point, with all due apology, we will shift our notation and represent points of \( \mathbb{H}^n \) by \( x, y, \) and \( z \). Notice that the dilation satisfies \( |\delta x| = \delta |x| \) and \( (\delta x)(\delta y) = \delta(xy) \) for \( x \) and \( y \) in \( \mathbb{H}^n \).

The operator \( \mathcal{L}_\alpha \) is not only left translation invariant with respect to the multiplication on the Heisenberg group, but homogeneous of degree \(-2\) with respect to the dilations introduced. These observations enabled Folland and Stein \([6]\) to construct a fundamental solution to \( \mathcal{L}_\alpha \) (for \( \alpha \neq \pm n, \pm(n+2), \pm(n+4), \ldots \)) and the measure the smoothing properties of this inverse in terms of the distance

\[
d(x, y) = |x^{-1}y|. \tag{1.3}
\]

The distance \( d(x, y) \) incorporates dilation invariance and left translation invariant properties:

\[
d(\delta x, \delta y) = \delta d(x, y); \quad d(zx, zy) = d(x, y)
\]

for \( \delta \in \mathbb{R} \), \( x, y, \) and \( z \) in \( \mathbb{H}^n \). Folland and Stein introduced Lipschitz (or Hölder) classes \( \Gamma_\beta \), \( 0 < \beta < \infty \), that are the analogue of the Lipschitz classes \( A_\beta \), based on ordinary Euclidean distance (see Section 4). Roughly speaking, \( \Gamma_\beta \) is the class of functions with \( \beta \) derivatives with respect to \( X_j, Y_j, j = 1, \ldots, n \) and \( \beta/2 \) derivatives with respect to \( T \).

**Plan of the Paper**

In both Parts I and II we seek estimates in the Dirichlet problem in a smooth domain \( D \) in the spaces \( \Gamma_\beta \) and their restriction to \( \partial D \), and \( \bar{D} \), denoted \( \Gamma_\beta(\partial D) \) and \( \Gamma_\beta(\bar{D}) \), respectively. Let \( \phi \) be supported in a small neighborhood of a non-characteristic point of \( \partial D \). Let \( -n < \alpha < n \). Our goal in Part I is to prove

\[
(7.1) \text{THEOREM. For } f \text{ in } \Gamma_\beta(\bar{D}) \text{ and } g \text{ in } \Gamma_{\beta+2}(\partial D) \text{ there is a unique solution to }
\]

\[
\mathcal{L}_\alpha u = f \text{ in } D \quad \text{and} \quad u|_{\partial D} = g.
\]

Moreover, \( \phi u \in \Gamma_{\beta+2}(\bar{D}) \).

In the second section we begin analysis of non-characteristic boundary points by showing that the structure of the boundary is like that of \( \mathbb{H}^{n-1} \times \mathbb{R} \). In Section 4 we deal with various characterizations of Lipschitz classes \( \Gamma_\alpha \) and their restrictions to \( \bar{D} \) and \( \partial D \). The main tool is a class of non-isotropic pseudo-differential operators introduced by Nagel and Stein \([15]\), which are also important elsewhere in the paper.

To construct a parametrix for the Dirichlet problem, we attempt to imitate the classical method of layer potentials. There are two main differences.
First, whereas the classical double layer potential (on the boundary) has the form $I + K$, where $K$ is compact, in our case the operator $K$ is a singular integral operator. We avoid this problem by trying to invert the single layer potential instead. We now arrive at a more serious difference. The operator we wish to invert is a pseudo-differential operator of class $S_{1/2,1}$, for which no symbolic calculus exists. In the case $n = 1$, we can choose special coordinates so that the operator has class $S_{1/2,0}$, recovering the symbolic calculus. For $n > 1$, we show that there is what we call a convolution calculus (Section 3) provided we view the top term of the single (or double) layer potential as a kernel on $\mathbb{H}^{n-1} \times \mathbb{R}$. Thus, there is a symbolic calculus if the principal symbol is interpreted as the group Fourier transform on $\mathbb{H}^{n-1} \times \mathbb{R}$ of a homogeneous kernel. This calculus generalizes the procedure of Folland and Stein [6] used to invert $L_a$, because it applies to the composition of two kernels as well as the composition of a kernel with a differential operator. The proof is just a tedious integration by parts, but it requires some new estimates on the way special coordinates approximate group multiplication (Propositions (2.6) and (2.10)).

In Section 5 we show that single and double layer potentials are pseudo-differential operators of Nagel–Stein type and calculate their leading terms. The inversion of the leading term is accomplished in Section 6, by an asymptotic procedure due to Geller in the case $\mathbb{H}^n$. (The main tools are the group Fourier transform and a notion called contraction; see [8–10].) In Section 7 we construct the desired parametrix. There is every reason to believe that the results and methods of Part I extend without much effort to the Dirichlet problem for $\square$ on CR manifolds equipped with a metric with respect to which the eigenvalues of the Levi form are $+1$. (See [6, Section 13].) However, the author has yet to extend results of Part II to such a context.

This paper and its sequel, Part II, are a minor revision of the author's doctoral dissertation at Princeton University. I am deeply grateful to my adviser, E. M. Stein, not only for his many valuable suggestions but for his inspiring example in research and teaching. I would also like to thank D. Geller for advice in the proof of Theorem (6.1).

2. LOCAL COORDINATES ON A HYPERSURFACE

Let $D$ be a smooth, bounded domain in $\mathbb{H}^n$ given by a $C^\infty$ defining function $r$. $D = \{ x \mid r(x) > 0 \}$. Let $\mathcal{V}$ denote the span of the first order vector fields $X_j, Y_j$, $j = 1, \ldots, n$. Let $N = \sum_{j=1}^n c_j X_j + d_j Y_j$, where $c_j = X_j r$, $d_j = Y_j r$. A point $x \in \partial D$ is characteristic if and only if

$$Nr(x) = \sum_j X_j r(x)^2 + Y_j r(x)^2 = 0.$$
Suppose that $x$ is non-characteristic. If $M \subset \partial D$ is a suitably small neighborhood of $x$, then, multiplying $r$ by a smooth function, we may as well assume that $Nr = 1$ on $M$.

Denote $V = \sum_{j=1}^{n} -d_jX_j + c_jY_j$. Notice that $Vr = 0$, so that $V$ is tangent to $M$. Denote $\mathcal{F}_0 = \mathcal{F} \cap T(M)$, the space of vector fields in $\mathcal{F}$ that are tangent to $M$.

(2.1) **Proposition.** If $W \in \mathcal{F}_0$, then $[V, W] \in \mathcal{F}_0$.

**Proof:** $W \in \mathcal{F}$ implies

$$W = \sum a_jX_j + b_jY_j, \quad a_j, b_j \in C^\infty(M).$$

Therefore, by (1.2)

$$[V, W] = \sum -d_j[X_j, W] + c_j[Y_j, W] \pmod{\mathcal{F}},$$

$$= \sum -d_jb_j[X_j, Y_j] + c_aj_j[Y_j, X_j] \pmod{\mathcal{F}},$$

$$= \left( \sum (d_jb_j + c_ja_j) \right) 4T \pmod{\mathcal{F}}.$$

Now, because $W$ is tangential,

$$\sum d_jb_j + c_ja_j = \sum a_jX_j + b_jY_j = W = 0 \quad \text{on } M.$$ 

Therefore, $[V, W] \in \mathcal{F}$. Since $V$ and $W$ belong to $T(M)$, so does $[V, W]$.

Choose $C^\infty$ functions $c^{(k)}, d^{(k)}: M \to \mathbb{R}^n$, $k = 1, \ldots, n$ so that $(c^{(k)}, d^{(k)}), (-d^{(k)}, c^{(k)})$ is an orthonormal system and $c^{(n)} = (c_1, \ldots, c_n), d^{(n)} = (d_1, \ldots, d_n)$. This is equivalent to choosing in a smooth way a unitary matrix with given unit vector $c^{(n)} + id^{(n)}$ as its last column. Denote

$$U_k = \sum_{j=1}^{n} c^k_jX_j + d^k_jY_j, \quad k = 1, \ldots, n,$$

$$W_k = \sum_{j=1}^{n} -d^k_jX_j + c^k_jY_j, \quad k = 1, \ldots, n,$$

$$B = T - (Tr)N.$$

The orthogonality relations for $c^{(k)}, d^{(k)}$ imply that $U_1, \ldots, U_{n-1}, W_1, \ldots, W_{n-1}, W_n = V$, and $B$ are tangent to $M$. Moreover, they satisfy

$$[U_i, U_j] \equiv [W_i, W_j] \equiv 0 \pmod{\mathcal{F}},$$

$$[U_i, W_j] \equiv -4\delta_{ij}T \pmod{\mathcal{F}}.$$
Since $B \equiv T \mod \mathcal{F}$, we can rewrite these relations as

\[
[U_i, U_j] \equiv [W_i, W_j] \equiv 0 \mod \mathcal{F},
\]

\[
[U_i, W_j] \equiv -4\delta_i^j B \mod \mathcal{F},
\]

\[
[U_i, V] \equiv [W_i, V] \equiv 0 \mod \mathcal{F},
\]

$i, j = 1, \ldots, n - 1$.

Let $m = n - 1$. Consider the group $\mathbb{H}^m \times \mathbb{R}$ with coordinates $u = (u^0, u^1, u^{2m+1})$, $u^0, u^{2m+1} \in \mathbb{R}$, and $u^1 \in \mathbb{C}^m$. The group law is

\[
u \cdot \nu = (u^0 + v^0 + 2\text{Im}u^c \cdot \sigma^c, u^c + v^c, u^{3m+1} + v^{3m+1}).
\]

We will also denote $u^c = \xi + i\eta$ and

\[
u^j = \xi^j, \quad u^{1+m} = \eta^j, \quad j = 1, \ldots, m.
\]

The norm for this group is

\[
|\nu| = ((u^0)^2 + |u^c|^4 + (u^{2m+1})^4)^{1/4}.
\]

In the case $n = 1, m = 0$, we use the convention $\mathbb{H}^0 = \mathbb{R}$ with coordinate $u^0$ and

\[
|\nu| = ((u^0)^2 + (\eta^0)^4)^{1/4}.
\]

The group law is just ordinary vector addition.

A basis for the left-invariant vector fields on $\mathbb{H}^m \times \mathbb{R}$ is given by

\[
S_j = \frac{\partial}{\partial \xi^j} + 2\eta^j \frac{\partial}{\partial u^0}, \quad S_{j+m} = \frac{\partial}{\partial \eta^j} - 2\xi^j \frac{\partial}{\partial u^0},
\]

\[
S_0 = \frac{\partial}{\partial u^0}, \quad S_{2m+1} = \frac{\partial}{\partial u^{2m+1}}.
\]

A smooth vector field on $\mathbb{H}^m \times \mathbb{R}$, $\sum_{j=1}^{2m+1} a_j(u) \partial/\partial u^j$, will be denoted $T^{(1)}$ if $a_0(u) = O(|u|)$. It will be denoted $T^{(0)}$ if $a_j(u) = O(|u|)$, $j = 1, \ldots, 2n - 1$, and $a_0(u) = O(|u|^2)$. The commutation relations among $U_j, W_j, V$, and $B$ given above reflect the structure of the Lie algebra of $\mathbb{H}^m \times \mathbb{R}$. Following Folland and Stein [6], we consider the exponential mapping $\phi(x, \cdot): \mathbb{H}^m \times \mathbb{R} \to M$ given by $\phi(x, u) = \exp_x(u^{2m+1}V + u^0B + \sum_{j=1}^{n} u^j U_j + u^{j+m} W_j)$. Folland and Stein have proved that one can define $\Theta: M \times M \to M \times \mathbb{R}$ by $\Theta(x, \phi(x, u)) = u$.

Moreover,

(2.2) Theorem. There exists $\Theta: M \times M \to \mathbb{H}^m \times \mathbb{R}$ satisfying

(a) $\Theta(x, y) = -\Theta(y, x) \ (\Rightarrow \Theta(x, y)^{-1})$. 


(b) For fixed \(x_0\), the mapping \(\Theta(x_0, \cdot): M \to \mathbb{H}^m \times \mathbb{R}\) is a local diffeomorphism sending \(x_0\) to the origin, and in local coordinates on \(\mathbb{H}^m \times \mathbb{R}\):

\[
U_k = S_k + T^{(0)}; \quad W_k = S_{k+m} + T^{(0)};
B = S_0 + T^{(1)}; \quad V = W_{m+1} = S_{2m+1} + T^{(0)}.
\]

(c) The pull-back under the mapping \(\Theta(x_0, \cdot)\) of the volume element \(du\) on \(\mathbb{H}^m \times \mathbb{R}\) evaluated at \(x_0\) is the volume element of \(M\) at \(x_0\).

(d) \(|\Theta(x, z)| \leq C(|\Theta(x, y)| + |\Theta(z, x)|).

We can assign coordinates in a neighborhood \(O\) of \(M\) in \(\mathbb{H}^n\) by the exponential mapping \(\phi: M \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{H}^n\) given by \(\phi: (x, \varepsilon) \to \exp_x(\varepsilon N)\). Thus \(N = \partial/\partial \varepsilon\). We can extend vector fields on \(M\) to vector fields in \(O = \phi(M \times (-\varepsilon_0, \varepsilon_0))\) merely by having them act on level sets in \(\varepsilon\) in the same way as they act on the level set \(\varepsilon = 0\). Thus the vector fields \(U_k, W_k, V, B\) are extended in such a way as to commute with \(\partial/\partial \varepsilon\). Notice that we have extended \(\mathcal{F}_0\) to ambient vector fields in such a way that \(\mathcal{F}_0\) and \(\mathcal{F}\) no longer coincide away from \(M\). However, if \(X \in \mathcal{F}\), then there are \(a, b \in C^\infty(\mathbb{H}^n)\) such that

\[
X \equiv a(x) \partial/\partial \varepsilon + b(x)B \quad \text{mod } \mathcal{F}_0 \quad (2.3)
\]

and conversely for any \(A \in \mathcal{F}_0\), there are \(c, d \in C^\infty(\mathbb{H}^n)\) such that

\[
A = c(x)X + d(x)B \quad \text{for some } X \in \mathcal{F}. \quad (2.4)
\]

We can now show that \(M\) sits in \(\mathbb{H}^n\) in essentially the same way as the group \(\mathbb{H}^m \times \mathbb{R}\) sits in \(\mathbb{H}^n\) under the inclusion \(t = u^0, (z_1, \ldots, z_{n-1}) = u^i, z_n = u^{2m+1}\).

(2.5) **Proposition.** Fix \(x_0 \in M\). Let \((\zeta^{(1)}, t^{(1)})\) and \((\zeta^{(2)}, t^{(2)})\) belong to \(\mathbb{H}^n\). Suppose that \((\zeta^{(2)}, t^{(2)}) \in M\) and \(\Theta(x_0, (\zeta^{(2)}, t^{(2)})) = u\). Also, suppose that \(\phi(x_0, \varepsilon) = (\zeta^{(1)}, t^{(1)})\). Then

\[
(a) \quad |\zeta^{(1)} - \zeta^{(2)}|^2 = \varepsilon^2 + (u^{2m+1})^2 + |u|^2 + O(|\varepsilon|^3 + |u|^3),
\]

\[
(b) \quad t^{(2)} - t^{(1)} - 2 \text{Im } \zeta^{(1)} \cdot \zeta^{(2)} = u^0 + 2u u^{2m+1} + O(|\varepsilon|^3 + |u|^3).
\]

**Proof.** We may as well assume that \(x_0 = 0 \in \mathbb{H}^n\). Note that

\[
(\zeta^{(1)}, t^{(1)}) = \exp_0(\varepsilon N),
\]

\[
(\zeta^{(2)}, t^{(2)}) = \exp_0 \left( u^0 B + \sum_{j=1}^n u^j U_j + u^{2m+1} W_{m+1} \right).
\]
Hence,

\[ \zeta^{(1)} = e(c^{(n)} + id^{(n)}) + O(e^2), \]
\[ t^{(1)} = O(|e|^3), \]
\[ \zeta^{(2)} = u^{2m+1}(-c^{(n)} + id^{(n)}) + \sum_{j=1}^{a} u^j(c^{(j)} + id^{(j)}) \]
\[ + u^{j+m}(-c^{(j)} + id^{(j)}) + O(|u|^2), \]
\[ t^{(2)} = u^0 + O(|u|^3). \]

The lemma now follows from the orthogonality relations among the \( c^{(k)}, d^{(k)} \).

In order to measure more precisely the way \( M \) inherits the group structure of \( \mathbb{H}^m \times \mathbb{R} \), we first observe that if \( u = \Theta(z, x) \) and \( v = \Theta(z, y) \), then we expect that \( \Theta(x, y) \sim u^{-1}v \). Denote

\[ \Delta = u^{-1}v\Theta(x, y)^{-1} = \Theta(x, z) \Theta(z, y) \Theta(y, x). \]

As we shall see in Section 3, \( \Delta \) measures the difference between convolution and composition of kernels. We need to show that \( \Delta \) is small in an appropriate sense. Denote \( \alpha = (\alpha_0, ..., \alpha_{2m+1}) \) and

\[ u^\alpha = \prod_{j=0}^{2m+1} (u^j)^{\alpha_j}. \]

We measure the degree of \( u^\alpha \) giving \( u^0 \) weight 2.

\[ |\alpha| = 2\alpha_0 + \alpha_1 + \cdots + \alpha_{2m+1}. \]

Fix \( x \) and consider \( \Delta \) as a function of \( v = \Theta(z, x) \) and \( u = \Theta(z, y) \). An immediate consequence of Taylor’s formula is that there are smooth functions

\[ a_{\alpha\beta} : M \to \mathbb{H}^m \times \mathbb{R}, \quad c_\gamma : \mathbb{H}^m \times (\mathbb{H}^m \times \mathbb{R})^2 \to \mathbb{H}^m \times \mathbb{R} \]

such that \( (\gamma = (\gamma_1, \gamma_2)) \)

\[ \Delta = \sum_{|\alpha + \beta| \leq N-1} a_{\alpha\beta}(x) u^\alpha v^\beta + r_N(x, u, v), \]

where

\[ r_N(x, u, v) = \sum_{|\gamma_1| + |\gamma_2| = N} c_\gamma(x, u, v) u^{\gamma_1} v^{\gamma_2}. \]
(2.6) **Proposition.** With the notations above,

(a) If $|\alpha| = 0$ or $|\beta| = 0$, then $a_{\alpha\beta}(x) = 0$.

(b) If $|\alpha| + |\beta| = 2$, then $a^{0}_{\alpha\beta}(x) = 0$ ($a^{0}_{\alpha\beta}$ denotes the $u^0$ component of $a_{\alpha\beta}$.)

(c) $|A| \leq C(|u^{-1}v| |u| |v|)^{1/2}$, assuming $|u| < 1$ and $|v| < 1$.

**Proof.** To prove (a) we need only show that $u = 0$ implies $A = 0$ and $v = 0$ implies $A = 0$, in fact, $u = \Theta(z, y) = 0$ implies $z = y$. Then Theorem (2.2a) implies $A = 0$, the proof that $v = 0$ implies $A = 0$ is similar.

For (b), recall that, by definition, $\Theta(x, y) = p(1)$, where $p(0) = v^{-1}$ and

$$p'(s) = \sum_{j=1}^{m} u^j(p) + u^{j+m}W^j(p) + u^0B(p) + u^{2m+1}V(p).$$

(Here, in an abuse of notation, the vector fields $U^j$, $W^j$, $B$ are evaluated at points of $\mathbb{H}^m \times \mathbb{R}$ under the identification $\Theta(x, \cdot): M \to \mathbb{H}^m \times \mathbb{R}$.)

Define $\bar{p}(0) = v^{-1}$, $\bar{p}'(s) = \sum_{j=0}^{m+1} u^jS_j(\bar{p})$, $0 \leq s \leq 1$. Thus $\bar{p}(1) = v^{-1}u$.

To compare $\bar{p}(1)$ with $p(1)$, note that the triangle inequality, (2.2d), implies that $|p(1)| \leq C(|u| + |v|)$. The triangle inequality on the group says [6, Lemma 8.9]

$$|u^{-1}v| \leq C(|u| + |v|). \tag{2.7}$$

Thus, $|\bar{p}(1)| \leq C(|u| + |v|)$. Next, for $j \geq 1$,

$$|p^j(s) - \bar{p}^j(s)| \leq C \max(|p|, |\bar{p}|) |u| \leq C |u| (|u| + |v|).$$

Thus,

$$|p^j(s) - \bar{p}^j(s)| \leq C(|u| + |v|)^2.$$

Consequently,

$$p^0(s) - \bar{p}^0(s) = \sum u^iO(|p|^2 + |\bar{p}|^2) + u^0O(|p| + |\bar{p}|) + \sum 2 \text{ Im } u^i(p^i(s) - \bar{p}^i(s))$$

$$= O(|u|(|u| + |v|)^2).$$

Hence,

$$|p^0(1) - \bar{p}^0(1)| = O(|u|(|u| + |v|)^3).$$
Finally,
\[ \Delta^0 = (v^{-1}u p(1)^{-1})^0 \]
\[ = \bar{p}^0(1) - p^0(1) - 2 \text{Im} \bar{p}^c(1) \cdot \bar{p}^c(1) \]
\[ = \bar{p}^0(1) - p^0(1) - 2 \text{Im} \bar{p}^c(1)(p^c(1) - \bar{p}^c(1)) \]
\[ = O((|u| + |v|)^3). \]

This proves (b).

(2.8) Remark. If \( f \in C^\infty(\mathbb{R}^k \times \mathbb{R}^{k^2}) \) and \( f(x, 0) = 0, f'(0, y) = 0 \), then
\[ f(x, y) = \sum x_j y_k h_{jk}(x, y) \text{ for some } h_{jk} \in C^\infty. \]
Indeed,
\[ f(x, y) = \sum y_k \int_0^1 \frac{\partial f}{\partial y_k} f(x, ty) \, dy \]
\[ = \sum x_j y_k \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_j \partial y_k} (sx, ty) \, ds \, dt. \]

Note that if \( u = v \), then \( x = y \), and as in part (a), \( \Delta = 0 \). Therefore, as a particular case of the remark we can write
\[ \Delta = \sum (u^l - v^l) v^k g_{jk}(x, u, v). \quad (2.9) \]
Thus, \( |\Delta^l| \leq C |u^{-1}v| |v| \). By symmetry, \( |\Delta^l| \leq C |u^{-1}v| |u| \), and (c) follows for \( |\Delta^l|, l \neq 0 \), because \( \min(|u|, |v|) \leq (|u| |v|)^{1/2} \).

We conclude with the proof of (c) for \( \Delta^0 \). From (a) and (b) we have
\[ \left| \frac{\partial}{\partial u^l} \frac{\partial}{\partial v^k} \Delta^0 \right| \leq C(|u| + |v|) \quad \text{for} \quad j \geq 1, k \geq 1. \]

Differentiating (2.9), we can write
\[ \frac{\partial}{\partial u^l} \frac{\partial}{\partial v^k} \Delta^0 = g_{jk}^0(x, u, v) + O(|u| + |v|). \]
Therefore, \( g_{jk}^0(x, u, v) = O(|u| + |v|) \). Incorporating this new estimate in (2.9),
\[ |\Delta^0| \leq C \left( \sum_j |u^j - v^j| |v^j| + |u^0 - v^0| |v^0| + (|u| + |v|) |u^j - v^j| |v^j| \right) \]
\[ \leq C(|u^{-1}v| |v|^2 + |u^{-1}v| |u| |v| + |u^{-1}v|^2 |v|). \]
By symmetry, we can obtain a similar estimate with \( u \) and \( v \) reversed, and (c) follows.

For \( u = (u^0, u^1, \ldots, u^{2m+1}) \), \( s > 0 \), we denote \( su = (su^0, su^1, \ldots, su^{2m+1}) \). Note that \(|su| = s|u|\).

(2.10) **Proposition.** For any \( \varepsilon_0 > 0 \) there exists \( r_0 > 0 \) and \( \varepsilon'_0 > 0 \) such that if \(|u| < r_0 \) and \(|u^{-1}v| > \varepsilon_0 |v|\), then \(|u^{-1}v(sA)| > \varepsilon'_0 |v|\) for all \( s \), \( 0 < s \leq 1 \).

**Proof.** The triangle inequality (2.7) implies \(|u^{-1}v(sA)| \geq C^{-1} |u^{-1}v| - |A|\). But by (2.6c),

\[
|A| \leq C(|u^{-1}v| |u| |v|)^{1/2} \leq C |u^{-1}v| \varepsilon_0 r_0^{-1} \leq \frac{1}{2} C^{-1} |u^{-1}v|
\]

for sufficiently small \( r_0 \).

3. **Convolution Calculus**

Before we introduce the sort of operator on the hypersurface \( M \) that arises in the method of layer potentials, we will examine the model case on the group \( \mathbb{H}^m \times \mathbb{R} \). Denote by \( \xi = (\xi_0, \ldots, \xi_{2m+1}) \) the dual variable to \( u = (u^0, \ldots, u^{2m+1}) \). The dual distance is \( |\xi| = (\xi_0^2 + (\xi_1^2 + \cdots + \xi_{2m+1}^2))^{1/4} \). Denote by \( S(p) \) the class of distributions \( k \) on \( \mathbb{H}^m \times \mathbb{R} \) with compact support satisfying \( |(\partial^{\alpha} / \partial x^\alpha) k(\xi)| \leq C_\alpha |\xi|^{p-|\alpha|} \). (Recall that \( |\alpha| = 2\alpha_0 + \alpha_1 + \cdots + \alpha_{2m+1} \) for \( \varphi \in C_0^\infty(\mathbb{H}^m \times \mathbb{R}) \) and \( k_1, k_2 \) distributions with compact support we define convolution \( \varphi * k \) and \( k_1 * k_2 \) by

\[
(\varphi * k)(u) = \langle \varphi_u, k \rangle, \quad \text{where} \quad \varphi_u(v) = \varphi(u^{-1}v).
\]

There is a linear map \( L_u \) such that \( L_u(u - v) = v^{-1}u \). Denote \( \bar{L}_u = L_u^{-1} \).

Then

\[
\varphi * k(u) = \int e^{-2\pi i \xi \cdot \xi} \varphi(\bar{L}_u(\xi)) \phi(v) \, dv \, d\xi
\]

Thus the mapping \( \varphi \to \varphi * k \) is represented by a pseudo-differential operator \( \sigma(u, D) \), with \( \sigma(u, \xi) = \bar{k}(\bar{L}_u(\xi)) \). Let \( \rho_0(u, \xi) = |\bar{L}_u(\xi)| \). \( \rho_0(u, \xi) \) is a distance function in the sense of Nagel and Stein. There is an associated symbol class \( S^\rho_{\sigma_0} \) introduced by Nagel and Stein. Because the definition is complicated, we
will only give properties of Nagel–Stein symbols. For the full definition, see [15, p. 56]. In the special case of $\rho_0$, $k \in S(p)$ if and only if $\tilde{k}(\tilde{L}_0(\xi)) \in S^p_{\rho_0}$.

The more general picture is as follows. Denote $L_x = -\nabla_y \Theta(x, y) \big|_{y=x}$. $L_x$ has a coordinate-free meaning as a mapping of tangent spaces. $L_x: T_x(M) \to T_0(\mathbb{H}^m \times \mathbb{R})$ is given by $L_x(X) = -X\Theta(x, y) \big|_{y=x}$, where $X$ is a vector field on $M$ acting in the $y$ variable. $\tilde{L}_x = L_x^{-1}$ acts on the cotangent space $T^*_x(M)$. Thus $\rho(x, \xi) = |\tilde{L}_x(\xi)|$ is a function on the cotangent bundle $T^*(M)$. $\rho(x, \xi)$ is a distance function in the sense of Nagel and Stein. The main example of symbols of the corresponding class $S^p_\rho$ is given by

\begin{equation}
(3.2) \text{Theorem [15]. If } k \in S(p) \text{ and }
T(f)(x) = \int \int e^{-2\pi i (T(x, y) \cdot \xi)} f(y) \, dy \, d\xi,
\end{equation}

then there exists $b(x, \xi) \in S^p_\rho$ such that

\begin{equation}
T(f)(x) = b(x, D) f(x) \equiv \int \int e^{-2\pi i (T(x, y) \cdot \xi)} b(x, \xi) f(y) \, dy \, d\xi.
\end{equation}

(Here we have omitted for simplicity the usual cut-off function in $y$. $dy$ is the volume element of $M$.) Nagel and Stein also proved a composition theorem

\begin{equation}
(3.3) \text{Theorem. If } a_j(x, \xi) \in S^p_{\rho_j}, \quad j = 1, 2, \text{ then there exists } b(x, \xi) \in S^p_{\rho_1 + \rho_2} \text{ such that }
\end{equation}

\begin{equation}
a_1(x, D) \circ a_2(x, D) = b(x, D).
\end{equation}

Our goal in this section is to describe $b(x, \xi)$ of (3.3) more accurately in the case when $a_1$ and $a_2$ are of the form given by (3.2). We will do this in terms of convolution on the group $\mathbb{H}^m \times \mathbb{R}$.

We begin with some facts about $S(p)$.

\begin{equation}
\partial^a/\partial u^n \text{ maps } S(p) \to S(p + |a|).
\end{equation}

\begin{equation}
\text{Multiplication by } u^n \text{ maps } S(p) \to S(p - |a|).
\end{equation}

\begin{equation}
\text{Convolution maps } S(p_1) \times S(p_2) \to S(p_1 + p_2).
\end{equation}

Let $\phi \in C^\infty_0(\mathbb{H}^m \times \mathbb{R})$, $\delta\phi(u) = \delta^{-a}\phi(\delta^{-1}u)$ ($a = 2m + 3$, the homogeneous dimension of $\mathbb{H}^m \times \mathbb{R}$. $\delta^{-1}u = (\delta^{-2}u^0, \delta^{-1}u^1, \ldots, \delta^{-1}u^{2m+1})$.)

If $k \in S(p)$, then $k \ast \phi_\delta \in S(p)$ with seminorm independent of $\delta$ as $\delta \to 0$. Also, $\lim_{\delta \to 0} k \ast \phi_\delta = k$ in the sense of (3.7) distributions.
Assertions (3.4) and (3.5) are easy to check. Assertion (3.6) follows from Theorem (3.3) applied to the symbol classes for the distance function \( \rho_0 \). The first part of (3.7) follows from (3.6) and the observation that \( \phi_\delta \) belongs to \( S(0) \) uniformly as \( \delta \to 0 \). The second part of (3.7) is just the dual of the approximate identity property for \( \phi_\delta \) on \( C_0^\infty \).

We add one further remark.

If \( k \in S(p) \) and \( p > -a \), then \( |k(u)| \leq C |u|^{-p-a} \), where \( C \) depends only on finitely many \( S(p) \) seminorms. \((3.8)\)

To prove (3.8), use a smooth partition of unity to partition \( \hat{k}(\xi) \) into a series of functions supported on annuli \( |\xi| \approx 2^j \) for \( j = 1, 2, 3\ldots \). Integration by parts then gives the desired estimate.

The restriction \( p > -a \) in (3.8) and property (3.6) points to the difference between the classes \( S(p) \) and homogeneous distributions. Even in Euclidean space the convolution of two homogeneous distributions need not be a homogeneous distribution. For example, the convolution of \( |x|^{-1} \phi(x) \) with itself in \( \mathbb{R}^2 \) gives a logarithmic singularity. In order to obtain property (3.6), the most convenient substitute for homogeneous distributions is to measure the size of the Fourier transform as in our definition of \( S(p) \).

One way to make sense of the formal expression for \( Tf(x) \) in (3.2) is to use (3.7). In fact,

\[
Tf(x) = \int e^{-2\pi i \Theta(x, y) \cdot \xi} \hat{k}(\xi) f(y) \, dy \, d\xi
\]

\[
= \int f(y) k(\Theta(y, x)) \, dy
\]

\[
= \lim_{\delta \to 0} \int f(y)(k * \phi_\delta)(\Theta(y, x)) \, dy.
\]

(As before, we omit the cut-off function in \( \gamma \).)

(3.9) Theorem. Let \( k_1 \in S(p_1) \) and \( k_2 \in S(p_2) \). Denote

\[
T_j f(x) = \int f(y) k_j(\Theta(y, x)) \, dy.
\]

\[
T f(x) = \int f(y)(k_1 * k_2)(\Theta(y, x)) \, dy.
\]

Then for any \( l \) one can write

\[
T_2 \circ T_1 - T = P_l(x, D) + E_l,
\]

where \( P_l(x, \xi) \in S^{p_1 + p_2 - 1}_p \) and the kernel of \( E_l \) is in \( C^{(l)}(M \times M) \).
Proof. Replacing \( k_1 \) and \( k_2 \) by \( k_1 \ast \phi \) and \( k_2 \ast \phi \), we can assume that these kernels are smooth with compact support. This smoothing gives rigorous meaning to the integrals and integrations by parts carried out below.

\[
I_2 \circ I_1 f(x) = \int K(z, x) f(z) \, dz,
\]

where

\[
K(z, x) = \int k_2(\Theta(y, x)) k_1(\Theta(z, y)) \, dy.
\]

Denote \( u = \Theta(y, x) \) and \( v = \Theta(z, x) \). Fix \( x \). Our goal is to expand \( K \) as a sum of ordinary convolution terms and errors. Recall that \( \Theta(y, x) = u^{-1}v \Delta \).

\[
K(z, x) = \int k_2(u^{-1}v \Delta) k_1(u) J(u, v) \, du,
\]

where \( J(u, v) \) is the Jacobian determinant of the change of variable \( y \mapsto u \).

By Theorem (2.2c),

\[
J(u, v) = 1 + \sum_{|\alpha| + |\beta| < N} b_{\alpha \beta} u^\alpha v^\beta + \sum_{|\alpha + \beta| = N} b_{\alpha \beta}(u, v) u^\alpha v^\beta,
\]

where \( b_{\alpha \beta} \) are smooth functions of \( x \), and also of \( u \) and \( v \) when \( |\alpha| + |\beta| = N \).

Next, apply Taylor’s theorem to \( k_2(u^{-1}v(s \Delta)) \) as a function of \( s, 0 < s \leq 1 \), to obtain

\[
k_2(u^{-1}v \Delta) = \sum_{|\alpha| < N} C_\alpha \Delta^\alpha(D^\alpha k_2)(u^{-1}v) + R_N(u^{-1}v, \Delta),
\]

where

\[
R_N(u^{-1}v, \Delta) = \int_0^1 \frac{(1 - s)^N}{N!} \sum a_{i,j} s^i(u^{-1}v)^j \Delta^j(D^i k_2)(u^{-1}v(s \Delta)) \, ds.
\]

\( D^\alpha \) is the left-invariant extension of \( \partial^\alpha/\partial u^\alpha \) at the origin. The inner sum in \( j = (j_1, j_2, j_3) \) and \( j \) satisfies \( |j_3| = |j_1| + |j_2|, 0 \leq j_1 \leq j_2 / 2, N + 1 \leq |j_2| \leq 2(N + 1), |j_1| \leq N + 1, |j_3| \leq 2(N + 1) \). Finally, write \( \Delta \) in the power series given by (2.6). It is crucial that \( \Delta^j \) vanish to second order in \( u \) and \( v \) and that \( \Delta^0 \) vanish to third order. This means that \( \Delta^{j_3} \) is written as a sum of monomials in \( u \) and \( v \) of non-isotropic degree at least \( \frac{1}{2} |j_2| \). In particular, \( |\Delta^{j_3}| \leq C(|u| + |v|)^{j_2/2} |j_1|/2 \). Consequently, \( K(z, x) \) can be written as a sum and/or integral of terms of the following forms.
(i) \[ \int k_2(u^{-1}v) k_1(u) \, du = k_1 * k_2(v). \]

(ii) \[ \text{const.} \int (D^\gamma k_2)(u^{-1}v) k_1(u) \, u^\alpha v^\beta \, du, \text{ where } |\gamma| - |\alpha_1| - |\alpha_2| \leq -1. \]

(iii) \[ \text{const.} \int (D^\gamma k_2)(u^{-1}v(s\Delta)) k_1(u) \, u^\alpha v^\beta c(u, v) \, du \text{ with } 0 \leq s \leq 1 \text{ and } c(u, v) \text{ a smooth function.} \]

If we go sufficiently far in each Taylor series and remember the gain from \( \Delta^2 \), we can suppose that in terms of type (iii) we have \( |\gamma| - |\alpha_1| - |\alpha_2| < -N \) for any large value of \( N \).

The main term (i) is the kernel of \( T \). By (3.4), (3.5), and (3.6), a kernel \( k(v) \) from a term of type (ii) belongs to \( S(p, + p_2 - 1) \). Thus, applying Theorem (3.2), the contribution from terms of type (ii) is of class \( S_{p_1 + p_2 - 1} \).

It remains to show that for any \( l \), terms of type (iii) are \( C^{(l)} \) functions of \( v \) for sufficiently large \( |\alpha_1| + |\alpha_2| - |\gamma| \).

The theorem is an easy consequence of this lemma. Derivatives in \( x \) and \( z \) of terms of type (iii) are again of type (iii) with a smaller value of \( |\alpha_1| + |\alpha_2| - |\gamma| \). So it suffices to prove that such a term is bounded. Let \( h_1(u) = u^\alpha k_1(u) \) and \( h_2(u) = D^\gamma k_2(u) \). Then \( \mu_2 = p_2 - |\gamma| \) and \( \mu_1 = p_1 + |\alpha_1| \).

Using the lemma we obtain a bound for terms of type (iii) of the form
\[
|\int h_1(u) h_2(u^{-1}v(s\Delta)) c(u, v) \, du| \leq C \max(1, |v|^{-\mu_1 - \mu_2}) |v|^{-a-2},
\]

where \( C \) depends only on the \( S(\mu_j) \) seminorms.

The proof of the lemma requires an elaborate integration by parts. The key point is the validity of Propositions (2.6) and (2.10). As preparation we prove some formulas.

For \( k_1 \) and \( k_2 \) in \( S(\mu) \) we will say that \( k_1 \approx k_2 \) if \( k_1 - k_2 \in C_0^\infty \) and the \( C_0^\infty \) seminorms of \( k_1 - k_2 \) depend only on the \( S(\mu) \) seminorms of \( k_1 \). Now let \( h_1 \in S(\mu_1) \) and \( h_2 \in S(\mu_2) \) as in Lemma 3.10. Then for any positive integers \( N_1 \) and \( N_2 \),

\[
h_1(u) \approx \sum_{\beta} \frac{\partial^\beta}{\partial u^\beta} h_1^{(\beta)}(u),
\]

where \( h_1^{(\beta)} \in S(\mu_1 - |\beta|) \), \(|\beta| = N_1 \) and \( N_1 + 1 \).

\[
h_2(\eta^{-1}v) \approx \sum v^{\gamma_1} \frac{\partial^\gamma_2}{\partial \eta^{\gamma_2}} h_2^{(\gamma_2)}(\eta^{-1}v),
\]

where \( h_2^{(\gamma_2)} \in S(\mu_2 + |\gamma_1| - |\gamma_2|) \) and \(|\gamma_2| - |\gamma_1| = N_2 \) or \( N_2 + 1 \).
Furthermore, in (3.11) the seminorms of $h^{(\beta)}_1$ in $S(\mu_1 - |\beta|)$ depend only on the seminorms of $h_1$ in $S(\mu_1)$, and similarly in (3.12).

To prove (3.11), let $S_{1}, \ldots, S_{2m+1}$ denote a basis for the left-invariant vector fields on $\mathbb{H}^{m-1} \times \mathbb{R}$ of homogeneous degree $-1$ as in Section 2. The operator $L = \sum_{j=1}^{2m+1} S_j^2$ has an inverse with homogeneous degree $-a + 2$ (see [5, 6]). If $\phi(u)$ is a smooth cut-off function that is identically 1 in a neighborhood of the origin, and $g_j = S_j(\phi L^{-1} h_1)$, then $g_j \in S(\mu_1 - 1)$ and $h_1 \simeq \sum S_j g_j$. (In the case $m = 0$, a similar equality is obtained using the inverse of the heat operator.) Iterating, we have $h_1 \simeq \sum_{|\alpha| = N_1} S^\alpha g_\alpha$, with $g_\alpha \in S(\mu_1 - N_1)$. Next, observe that $S^\alpha = \sum d_{\gamma_1 \gamma_2} (\partial^{\gamma_2}/\partial u^{\gamma_2}) \partial^{\gamma_1}$, where $|\gamma_2| - |\gamma_1| = |\alpha|$ and $|\alpha| \leq |\gamma_2| \leq 2 |\alpha|$. Thus $h_1$ can be written as a linear combination of terms

$$\frac{\partial^{\gamma_2}}{\partial u^{\gamma_2}} u^{\gamma_1} g_\alpha = \frac{\partial^{\beta}}{\partial u^{\beta}} \left( \frac{\partial^{\gamma_2 - \beta}}{\partial u^{\gamma_2 - \beta}} u^{\gamma_1} g_\alpha \right)$$

for some $\beta$ with $|\beta| = N_1$ or $N_1 + 1$. Moreover,

$$\frac{\partial^{\gamma_2 - \beta}}{\partial u^{\gamma_2 - \beta}} u^{\gamma_1} g_\alpha \in S(\mu_1 - |\beta|).$$

Equation (3.12) is a consequence of (3.11) and the observation that $((\partial^{\beta}/\partial u^{\beta})h)(\eta^{-1}v)$ is a linear combination of terms of the form

$$\nu^{\gamma_1} \frac{\partial^{\gamma_2}}{\partial \eta^{\gamma_2}} h(\eta^{-1}v), \quad \text{where} \quad |\gamma_2| - |\gamma_1| = |\beta|.$$

We now proceed with the proof of Lemma (3.10). Denote $I = \int h_1(u) h_2(u^{-1}v(sA)) c(u, v) \, du$.

**Case 1a.** If $\mu_1 > 0$ and $\mu_2 > 0$, then $|I| \leq C r^{-a - \mu_1 - \mu_2 - 1}$, where $r = |v|$.

Let $\phi \in C^\infty_0(\mathbb{H}^m \times \mathbb{R})$ be such that

$$\phi(u) = \begin{cases} 1 & |u| < \varepsilon_0 \\ 0 & |u| > 2 \varepsilon_0. \end{cases}$$

Denote

$$I_1 = \int h_1(u) c(u, v) \phi(r^{-1}(u^{-1}v)) h_2(u^{-1}v(sA)) \, du,$$

$$I_2 = \int h_1(u) c(u, v)(1 - \phi(r^{-1}(u^{-1}v))) h_2((u^{-1}v(sA)) \, du.$$
where \( h_1^{(\beta)} \in S(\mu_1 - |\beta|) \). Thus by (3.8), \( \int |h_1^{(\beta)}(u)| \, du \leq C \), with \( C \) depending only on the \( S(\mu_1) \) seminorms of \( h_1 \). Integration by parts in \( I_2 \) gives terms of the form

\[
\int h_1^{(\beta)}(u) \frac{\partial^\beta}{\partial u^\beta} (c(u, v)(1 - \phi(r^{-1}(u^{-1}v)))) h_2(u^{-1}v(sA)) \, du.
\]

We shall prove

\[
|\frac{\partial^\beta}{\partial u^\beta} (c(u, v)(1 - \phi(r^{-1}(u^{-1}v)))) h_2(u^{-1}v(sA)))| \leq Cr^{-a - N_1 - \frac{1}{2}}. \tag{3.13}
\]

Since \( h_1^{(\beta)} \) is integrable, (3.13) implies \( |I_2| < Cr^{-a - a_1 - a_2 - \frac{1}{2}} \).

On the support of \( 1 - \phi(r^{-1}(u^{-1}v)), |u^{-1}v| > \varepsilon_0 r \). Proposition 2.10 implies \( |u^{-1}v(sA)| > \varepsilon_0 r \). This is the key point of the proof. On the left hand side of 3.13, \( h_2 \) and its derivatives will be evaluated at a point whose distance from the origin is at least \( \varepsilon_0 r \). We calculate

\[
\frac{\partial}{\partial u^j} h_2(u^{-1}v(sA)) = -\frac{\partial}{\partial x^j} h_2(x) \pm 2(v(sA))^{j+m} \frac{\partial h_2}{\partial x^{(j+m)}(x)}
\]

\[
+ \sum \frac{sA^l}{x_{l+1}} \frac{\partial h_2}{\partial x^{l+1}(x)}
\]

\[
\pm 2 \sum_{l} (u^{-1}v)^{j+m} \frac{\partial h_2}{\partial x^{(j+m)}(x)} A^l_{j+m}(x). \tag{3.14}
\]

\[
\frac{\partial}{\partial u^0} h_2(u^{-1}v(sA)) = -\frac{\partial}{\partial x^0} h_2(x) + \sum \frac{sA^l}{x_{l+1}} \frac{\partial h_2}{\partial x^{l+1}(x)}
\]

\[
\pm 2 \sum_{l} (u^{-1}v)^{j+m} \frac{\partial h_2}{\partial x^{(j+m)}(x)} A^l_{j+m}(x). \tag{3.15}
\]

Proposition 2.6 gives estimates

\[
|(v(sA))^{j}| \leq Cr, \quad |A^l_{j+m}| \leq Cr, \quad j \geq 1.
\]

Hence, using (3.14) and (3.15),

\[
|\frac{\partial}{\partial u^j} h_2(u^{-1}v(sA))| \leq Cr^{-1-a - a_2}, \quad j \geq 1.
\]

\[
|\frac{\partial}{\partial u^0} h_2(u^{-1}v(sA))| \leq Cr^{-a - a_2}. \tag{3.16}
\]
Similarly, we can estimate higher derivatives

$$\left| \frac{\partial}{\partial u^i} h_2(u^{-1}v(sA)) \right| \leq Cr^{-|n| - a - \mu^2}. $$

For cases where the derivatives fall on $\phi$ we need the (easy) estimate

$$\left| \frac{\partial^\prime}{\partial u^i} \phi(r^{-1}(u^{-1}v)) \right| \leq Cr^{-|n|}.$$

Combining these last two estimates gives (3.13).

To bound $I_1$, we make a change of variable $F_v(u) = v(sA)^{-1}v^{-1}u = \eta$. Let $H_v$ be the inverse, $u = H_v(\eta)$. Let $JH_v$ denote the Jacobian of $H_v$, $JF_v$ the Jacobian of $F_v$.

$$I_1 = \int h_1(H) c(H, v) \phi(r^{-1}(H^{-1}v)) h_2(\eta^{-1}v) JH_v \, d\eta.$$

The change of variable makes sense because $F$ is invertible. More precisely, we will calculate that $JF_v = I + O(r)$.

Recall that $\Delta = \sum u_i v_j f_{ij}(u, v)$.

$$\frac{\partial}{\partial u^i} F^0_v(u) = -sA^0 + u^0 + \sum_{j \geq 1} 4(v^j (sA)^{j+m} (sA)^j) + \sum_{j \geq 1} 2((sA)^j u^{j+m} - (sA)^{j+m} u^j).$$

Hence, if $i \geq 1$,

$$\frac{\partial}{\partial u^i} F^0_v(u) = -sA^0 + 2sA^{i+m} + O(r) \Delta^{i+m} + O(|u|) \Delta^i.$$

$$= O(r), \quad \text{by (3.16)}.$$

$$\frac{\partial}{\partial u^0} F^0_v(u) = 1 + O(\Delta^0 + \Delta^i) = 1 + O(r).$$

$$\frac{\partial}{\partial u^0} F^i_v(u) = \frac{\partial}{\partial u^0} (-sA^i + u^i) = -\Delta^i = O(r).$$

Using (3.12) with $N_2 < \mu_2 < N_2 + 1$.

$$h_2(\eta^{-1}v) \approx \sum \psi_{\eta^{N_2/2}} h^\phi_2(\eta^{-1}v),$$
where \( h_2^{(9)} \in S(\mu_2 + |\gamma_1| - |\gamma_2|) \) and \(|\gamma_2| - |\gamma_1| = N_2 \) or \( N_2 + 1 \). It follows from (3.8) that \( h_2^{(9)} \) is integrable with bound depending only on the \( S(\mu_2) \) seminorms of \( h_2 \). An integration by parts allows us to write \( I_1 \) as a sum of terms of the form

\[
\int v^{*1} \frac{\partial^{\gamma_1}}{\partial \eta^{\gamma_1}} \left( c(H, v) \phi(r^{-1}(H^{-1}v)) h_1(H) JH_v \right) h_2^{(9)}(\eta^{-1}v) \, d\eta.
\]

The estimate we want is

\[
\int v^{*1} \frac{\partial^{\gamma_1}}{\partial \eta^{\gamma_1}} \left( c(H, v) \phi(r^{-1}(H^{-1}v)) h_1(H) JH_v \right) \leq C r^{-a-\mu_1-|\gamma_1|}.
\] (3.17)

As in the case of \( I_2 \), an immediate consequence of (3.17) is that \(|I_1| \leq C r^{-a-\mu_1-\mu_2-1} \).

On the support of \( \phi(r^{-1}(u^{-1}v)) \), \(|u^{-1}v| < 2\epsilon_0 r \). If \( \epsilon_0 \) is sufficiently small, the triangle inequality implies that \(|u| > \frac{1}{2} r \). Also, \(|\Delta| \leq Cr \) implies \(|\eta^{-1}v| = |u^{-1}v(sA)| \leq Cr \). Finally, note that the inverse matrix of \( JF \), namely, \( JH_v \), has the form \( JH_v = I + O(r) \) because \( JF \) does. Therefore, by the chain rule, \(|(\partial^\beta / \partial \eta^\alpha) h_1(H)| \leq C r^{-a-\mu_1-|\beta|} \) for \(|H| > \frac{1}{2} r \), and finally

\[
\left| \frac{\partial^\beta}{\partial \eta^\alpha} \phi(r^{-1}(H^{-1}v)) \right| \leq C r^{-|\beta|}.
\]

These estimates combine to prove (3.17) and Case 1a is complete.

\textit{Case 1b.} Denote \( \mu^+ = \max(\mu, 0) \). Then for any values of \( \mu_1 \) and \( \mu_2 \),

\(|I| \leq C r^{-a-\mu_1-\mu_2-1} \).

This is an immediate consequence of (1a), since \( S(\mu_1) \subset S(\mu_1^+) \).

\textit{Case 2.} If \( \mu_1 < 0 \) and \( \mu_2 \geq 0 \). Then

\(|I| \leq C r^{-(\mu_1-\mu_2) + 1} r^{-a-2} \).

Choose \( N_2 \) as before, \( \mu_2 < N_2 \leq \mu_2 + 1 \) and integrate by parts as above for \( I_1 \), but without introducing the cut-off function \( \phi \). This yields terms of the form

\[
\int v^{*1} \left[ \frac{\partial^{\gamma_1}}{\partial \eta^{\gamma_1}} \left( c(H, v) h_1(H) \right) \right] h_2^{(9)}(\eta^{-1}v) \, d\eta.
\]

Recall that the derivative of \( H \) with respect to \( \eta, JH_v = I + O(r) \). This can be expressed using Taylor's formula as \( JH_v = I + \sum c_j v^j \), where \( c_j \) is a smooth
function of $H$ and $v$. Using the chain rule we can then rewrite the integral above as a sum of terms
\[ \int v^\gamma \left( \frac{\partial^2}{\partial u^2} h_1 \right) (H) \ c(H, v) \ h_2^{\beta}(\eta^{-1} v) \, d\eta, \]
where $c \in C^\infty$, $h_2^{\beta} \in S(\mu_2 - N_2) |\gamma_4| - |\gamma_3| = N_2$ or $N_2 + 1$. Since $(\partial^2/\partial u^2) h_1 \in S(\mu_1 + |\gamma_4|)$, we can change variables back to $du$ so that the integral we get is of the type arising in Case 1b. We obtain the estimate
\[ C r^{\beta} \left| r - a - (\mu_1 + |\gamma_4|) \right| - (\mu_2 - N_2)^{+} - 1 \leq C r^{\beta} \left| r - a - (\mu_1 + |\gamma_4|) \right| - 1 \]
\[ \leq C(r^{-\mu_1 - \mu_2 + 1}) r^{-a - 2}. \]

**Case 3.** If $\mu_1 \geq 0$ and $\mu_2 < 0$, then $|I| \leq C(r^{-\mu_1 - \mu_2 + 1}) r^{-a - 2}$.

The argument here is similar, but simpler than that in Case 2. Integrate by parts as we did in Case 1 for $I_2$, but without introducing the cut-off function $1 - \phi$. This reduces matters to terms of the form
\[ \int h_1^{(b)}(u) \, \frac{\partial}{\partial u^\beta} \left( c(u, v) \ h_2(u^{-1}v(sA)) \right) \, dn, \]
with $h_1^{(b)} \in S(\mu_1 - |\beta|)$, $|\beta| = N_1$ or $N_1 + 1$, $\mu_1 < N_1 \leq \mu_1 + 1$. In light of (3.14) and (3.15), $(\partial^\beta/\partial u^\beta) h_2(u^{-1}v(sA))$ can be written as a sum of terms
\[ v^\gamma k^{(b)}(u^{-1}v(sA)) \, c(u, v), \]
where $k^{(b)} \in S(\mu_2 + |\gamma| + |\beta|)$. Applying 1b, we have the estimate
\[ C \sum_{\gamma} r^{\gamma} r^{-a - 1 - (\mu_2 + |\gamma| + |\beta|)^+} \leq C r^{-a - 1} r^{-(\mu_2 + |\beta|)^+} \]
\[ \leq C(r^{-\mu_1 - \mu_2 + 1}) r^{-a - 2}. \]

**Case 4.** If $\mu_1 < 0$ and $\mu_2 < 0$, then
\[ |I| \leq C(r^{-\mu_1 - \mu_2 + 1}) r^{-a - 2}. \]
This follows immediately from Case (1b).

Q.E.D.

4. $l_\beta$ SPACES AND RESTRICTION

Recall that the distance function on $H^n$ such that for any $x \in H^n$, there exists a polynomial in $y$, $P_{(x,y)}(y)$ such that $|f(y) - P_{(x,y)}(y)| \leq C \delta^3$ for $d(x, y) < \delta$. (See [15, Sect. 4.1].) Let $X_j$ denote a monomial of length $|\gamma|$ in $X_j$.
and $Y_j$, $j = 1, \ldots, n$, a basis for the left-invariant vector fields on $\mathbb{H}^n$ of homogeneous degree $-1$.

\begin{equation}
(4.1) \text{THEOREM \ [15].} \quad \Gamma_\beta = \{ f \mid f = \sum_j f_j, \quad \text{where } f_j \in C^\infty \text{ and } |Xf_j| \leq C 2^{-\beta - |\gamma|} \text{ for } |\gamma| \leq \beta + 2 \}.
\end{equation}

Remarks. As a consequence of Theorem 4.1, $\Gamma_\beta$ is a real interpolation scale and $\Gamma_{\beta+1} = \{ f \mid X_j f, Y_j f \in \Gamma_\beta \}$. This definition is not the same for integer values of $\beta$ as that given in Folland and Stein [6], but they are equivalent because both families of spaces are real interpolation scales (see [6, 10.5]).

There is a more general approach to the spaces $\Gamma_\beta$ also introduced in [15] that allows us to define $\Gamma_\beta(M)$. Corresponding to the distance function $\rho(x, \xi)$ on $T^*(M)$ defined in Section 3, there is a dual distance on $M$ itself (see [15, p. 361]). In fact, it is easy to calculate that the dual distance is just the restriction of $d(x, y)$ to $M$. We can now define $\Gamma_\beta(M)$ as the class of function $f$ on $M$ such that for all $x \in M$ and all $\delta > 0$, there exists a polynomial $Q_{\delta}^{(\xi)}(\gamma)(y)$ in local coordinates on $M$ such that $|f(y) - Q_{\delta}^{(\xi)}(\gamma)(y)| < C\delta^\beta$ whenever $d(x, y) < \delta$.

\begin{equation}
(4.2) \text{THEOREM \ [15].} \quad \text{If } a(x, \xi) \in S^\rho_\beta, \text{ then } a(x, D) \text{ is a bounded operator from } \Gamma_\beta(M) \text{ to } \Gamma_{\beta - \rho}(M), \text{ provided } \beta - \rho > 0.
\end{equation}

In this section we will sketch the

\begin{equation}
(4.3) \text{THEOREM.} \quad \text{The restriction of } \Gamma_\beta \text{ to } M \text{ equals } \Gamma_\beta(M).
\end{equation}

(We are only concerned with local properties here, so any problem at the edges of $M$ will be ignored.) The proofs of the main lemmas will be given in Appendix A.

The reader can check that

$U_j, W_j$, and $V$ have symbols of class $S^1_\rho$.

$B$ has a symbol of class $S^2_\rho$.

\begin{equation}
(4.4)
\end{equation}

Denote by $A^\gamma$ a monomial in $U_1, \ldots, U_m, W_1, \ldots, W_m, V$ of length $|\gamma|$. An operator $p_\epsilon(x, D)$ on $M$ is called a Poisson-type operator of order $k$ if for $\delta > 0$,

\begin{equation}
e^\delta \frac{\partial^j}{\partial \xi^j} A^\gamma p_\epsilon(x, \xi)(y) \in S^{k - \beta + |\gamma| + j}_\rho
\end{equation}

uniformly in $\epsilon$. Consider

\begin{equation}
P_\epsilon f(x) = \mathcal{I} \int e^{-2\pi i \Theta(x, y)} \cdot \epsilon^{-\frac{1}{2}} f(y) dy d\xi.
\end{equation}
If \( k(\xi) = e^{\delta (\partial^j / \partial \xi^j)} e^{-\epsilon |\xi|} \), then \( k \in S(j - \delta) \) uniformly in \( \epsilon \). Therefore, as a result of (4.4) and Theorems (3.2) and (3.3), \( P_\epsilon f(x) = p_\epsilon(x, D)f \), where \( p_\epsilon(x, D) \) is a Poisson-type operator of order 0.

The action of \( X' \) in coordinates \( (\epsilon, x), x \in \mathcal{M} \) is given by (2.3). In light of (2.3), (2.4), (4.4), and Theorem (3.3), the definition of a Poisson-type operator is equivalent to

\[
e^{-\delta |X'|} p_\epsilon(x, \xi) \in S^{k-\delta}_{p-1/\gamma} \]

uniformly in \( \epsilon \).

The fact that \( \Gamma_\beta(\mathcal{M}) \subset \text{Restr}. \Gamma_\beta |\mathcal{M} \) now follows from two lemmas. (The inclusion for integer values of \( \beta \) follows from real interpolation.)

(4.5) **Lemma.** If \( f \in \Gamma_\beta(\mathcal{M}) \) and \( p_\epsilon(x, D) \) is a Poisson-type operator of order 0 and \( g(w) = p_\epsilon(x, D)f \), where \( w \) has coordinates \( (\epsilon, x) \), then \( g \) is smooth for \( \epsilon > 0 \) and \( |X'g(w)| \leq C \max(1, d(w, \mathcal{M}))^{p-1/\gamma} \) for \( \beta \neq |\gamma| \).

Let \( D \) be a smooth domain in \( \mathbb{H}^n \). Denote by \( \Gamma_\beta(\overline{D}) \) the restriction of \( \Gamma_\beta \) to \( \overline{D} \).

(4.6) **Lemma.** Suppose that \( k < \beta < k + 1, g \in C^\infty(D) \) and

\[
|X'g(w)| \leq C \max(1, d(w, D))^{p-1/\gamma} \quad \text{for} \quad |\gamma| \leq 3(k + 1).
\]

Then \( g \) belongs to \( \Gamma_\beta(\overline{D}) \).

These lemmas, as well as the opposite inclusion \( \text{Restr}. \Gamma_\beta |\mathcal{M} \subset \Gamma_\beta(\mathcal{M}) \) will be proved in Appendix A.

We would like to mention the analogous symbol classes on the ambient space \( \mathbb{H}^n \) to the classes \( S^p_\alpha \) on \( \mathbb{H}^m \times \mathbb{R} \). We will denote by \( p(x, \xi) \) the function on the cotangent bundle of \( \mathbb{H}^n \) defined in the same way as \( p_\alpha \) on \( \mathbb{H}^m \times \mathbb{R} \). Theorems (3.2), (3.3), and (4.2) are valid for \( S^p_\beta \), too. \( \tilde{X}_j, Y_j \) have class \( S^1_\beta \) and \( T = \partial / \partial t \) has class \( S^2_\beta \). Thus \( \mathcal{L}_\alpha \) has class \( S^2_\beta \). The fact that these symbol classes and the spaces \( \Gamma_\beta \) are well suited to the study of \( \mathcal{L}_\alpha \) is expressed by the fact that if \( -n < \alpha < n \), then the homogeneous fundamental solution to \( \mathcal{L}_\alpha \) is (locally) an operator of class \( S^{-2}_\beta \).

5. SINGLE AND DOUBLE LAYER POTENTIALS

Recall that \( u = (u^0, u^1, ..., u^{2m+1}) \) and \( a = 2m + 3 \). Denote \( u' = (u^1, ..., u^{2m+1}) \). Choose \( \alpha \) so that \( -n < \alpha < n \) \( (n = m + 1) \).

(5.1) **Proposition.** Denote

\[
g_\epsilon(u) = \phi(u) u^\beta(|u'|^2 + \epsilon^2 + u^0)^{-(n+\alpha)/2} j\epsilon(|u'|^2 + \epsilon^2 + u^0)^{-(n-\alpha)/2} j\epsilon.
\]
where \( \phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}) \). Let \( \lambda = |\beta| - 2j_1 - 2j_2 + 1 \). If \( \lambda + \delta \geq 0 \), then 
\( \varepsilon^\delta g_\varepsilon(u) \in S(-\delta - \lambda) \) uniformly as \( \varepsilon \to 0 \).

**Proof.** We can assume that \( \lambda < a \). Otherwise, we can reduce to that case by integration by parts. Denote

\[
f_\varepsilon(u) = u^0(|u|^2 + \varepsilon + iu^0)^{-\frac{n+a}{2} - 1/2}(|u|^2 + \varepsilon - iu^0)^{-\frac{n-a}{2} - 1/2}.
\]

If \( 0 < \lambda < a \), then \( |\hat{f}_1(\xi)| \leq C \min(|\xi|^{-\lambda}, |\xi|^{-N}) \)

If \( \lambda < 0 \), then \( |\hat{f}_1(\xi)| \leq C \min(1, |\xi|^{-N}) \).

The only non-trivial part of (5.2) is the estimate as \( |\xi| \to 0 \) when \( 0 \leq \lambda < a \). If \( \phi \) is a smooth cut-off function that is 1 near the origin we can write \( f_1 \) as

\[
f_1 = \phi f_0 + (1 - \phi)(f_1 - f_0) + f_0.
\]

The main point is that \( f_0 \) is a homogeneous distribution of degree \(-a + \lambda\). This is true even when \( \lambda = 0 \), because in that case \( |\beta| \) is odd so that \( f_0(u) \) has mean value zero. Therefore, \( |\hat{f}_0(\xi)| \leq C |\xi|^{-\lambda} \). Furthermore, we can write \( (1 - \phi)(f_1 - f_0) = (1 - \phi) \int f_\varepsilon \eta \; \phi' \; d\eta \). The integrand \( (\partial/\partial \eta)f_\varepsilon \) decreases like \( |u|^{-a+\lambda-2} \) as \( |u| \to \infty \). Hence an easy integration by parts shows

\[
|((1 - \phi)(f_1 - f_0))\xi(\xi)| \leq C |\xi|^{-\lambda} \quad \text{as} \quad |\xi| \to 0.
\]

Finally, it is easy to see that \( (\phi f_1)\xi(\xi) \) and \( (\phi f_0)\xi(\xi) \) are bounded when \( \lambda \geq 0 \). (When \( \lambda = 0 \), we need again the fact that \( f_0(u) \) has mean value zero.)

Now to prove (5.1), notice that \( g_\varepsilon(u) = \varepsilon^{\lambda-a} f_\varepsilon u \phi(u) \). Thus

\[
\hat{g}_\varepsilon(\xi) = \varepsilon^\lambda \int \hat{f}_1(\varepsilon(\xi - \eta)) \hat{\phi}(\eta) \; d\eta.
\]

(Recall that \( \varepsilon^\xi = (\varepsilon^{\xi_0}, \varepsilon^{\xi_1}, ..., \varepsilon^{\xi_{2m+1}}) \).)

**Part (a) \( \delta = 0 \)**

Split the integration in \( \eta \) into two regions

\[
(I) \quad |\xi - \eta| > \frac{1}{10} |\xi|,
\]

\[
(II) \quad |\xi - \eta| \approx 2^{-j} |\xi|, \quad j = 3, 4, ....
\]

\[
\varepsilon^\lambda \int_{(II)} |\hat{f}_1(\varepsilon(\xi - \eta))| |\hat{\phi}(\eta)| \; d\eta \leq C \varepsilon^{-\lambda} \int_{(II)} (\varepsilon |\xi|)^{-\lambda} |\hat{\phi}(\eta)| \; d\eta \leq C |\xi|^{-\lambda}, \quad \text{for} \quad \lambda \geq 0.
\]
\( \varepsilon^{\lambda} \int_{|x|} |f_1(\varepsilon(\xi - \eta))| |\hat{\phi}(\eta)| \, d\eta \)

\( \leq C_N \sum_j \varepsilon^{\lambda} \int_{|x|-\eta| < 2^{-j}|x|} (2^{-j} |\xi| |\varepsilon|)^{-\lambda} |\xi|^{-N} \, d\eta, \quad \text{for} \quad \lambda \geq 0. \)

This is because \( |\xi - \eta| \approx 2^{-j} |\xi| \) implies \( |\eta| \approx |\xi| \), and \( |\hat{\phi}(\eta)| \leq C_N |\eta|^{-N} \). The sum is dominated by

\[
C \sum_j \varepsilon^{\lambda} |\xi|^{-N} (2^{-j} |\xi| |\varepsilon|)^{-\lambda} \int_{|x|-\eta| < 2^{-j}|x|} \, d\eta \leq C |\xi|^{-\lambda-N} \sum_{j=3}^{\infty} 2^j (2^{-j} |\xi|)^{\lambda} \leq C |\xi|^{-\lambda-N+\delta},
\]

because \( \lambda < \alpha \). If we choose \( N = \alpha \), we obtain \( |\hat{g}_\varepsilon(\xi)| \leq C |\xi|^{-\lambda} \), for \( \lambda \leq 0 \). Next, \( |\hat{g}_\varepsilon(\xi)| \leq \int |g_\varepsilon(u)| \, du \leq C \) uniformly in \( \varepsilon \) provided \( \lambda > 0 \). This completes the estimate on \( \hat{g}_\varepsilon(\xi) \) when \( \delta = 0 \). The estimate on \( (\partial/\partial \xi^n) \hat{g}_\varepsilon(\xi) \) is actually contained in the previous one because \( (\partial/\partial \xi^n) \hat{g}_\varepsilon(\xi) = \) const. \( u^n g_\varepsilon \hat{g}_\varepsilon(\xi) \). Thus \( g_\varepsilon \in \mathcal{S}(-\lambda) \).

\textbf{Part (b) \( \delta > 0 \)}

\textbf{Case (i).} \( \varepsilon |\xi| < 1 \). Then \( \varepsilon < |\xi|^{-1} \). If \( \lambda < 0 \), then

\[
e^{\delta} |\hat{g}_\varepsilon(\xi)| \leq e^{\delta} \int |g_\varepsilon(u)| \, du \leq Ce^{\delta+\lambda} \leq C \min(1, |\xi|^{-\delta-\lambda}).
\]

since \( \delta + \lambda \geq 0 \).

If \( \lambda \geq 0 \), then from part (a)

\[
e^{\delta} |\hat{g}_\varepsilon(\xi)| \leq Ce^{\delta} \min(1, |\xi|^{-\lambda}) \leq C \min(1, |\xi|^{-\lambda-\delta})
\]

since \( \delta \leq 0 \).

\textbf{Case (ii).} \( \varepsilon |\xi| > 1 \). Use the formula

\[
e^{\delta} \hat{g}_\varepsilon(\xi) = e^{\delta} \int f_1(\varepsilon(\xi - \eta)) \hat{\phi}(\eta) \, d\eta.
\]

Partition the integration in three regions

\begin{align*}
(A) \quad |\xi - \eta| &> \frac{1}{10} |\xi|, \\
(B) \quad |\xi - \eta| &> 2^{-j} |\xi|, \quad j = 1, 2, 3, \ldots, 2' < \varepsilon |\xi|, \\
(C) \quad |\xi - \eta| &> 2^{-j} |\xi|, \quad 2' > \varepsilon |\xi|.
\end{align*}

\( (A) \quad e^{\delta+\lambda} \int_A (\varepsilon |\xi|)^{-\lambda-\delta} |\hat{\phi}(\eta)| \, d\eta \leq C |\xi|^{-\lambda-\delta}. \)
provided $N_1 > a$ and $N_2$ is large.

(C) When $\lambda < 0$, the third part is dominated by

$$
\varepsilon^{\delta + \lambda} \sum_{2^j > \varepsilon |\xi|} \int_{|\xi - \eta| \geq 2^{-j} |\xi|} (\varepsilon 2^{-j} |\xi|)^{-N_1} |\xi|^{-N_2} d\eta
$$

$$
\leq C \varepsilon^{\delta + \lambda - N_1} |\xi|^{-N_1 + a} \sum_{2^j > \varepsilon |\xi|} 2^{-j(a - \lambda)}
$$

$$
\leq C \varepsilon^{\delta + \lambda - a} |\xi|^{-N_1} \leq C |\xi|^{-N}
$$

for large $N_1$.

When $\lambda \geq 0$, the third part is dominated by

$$
\varepsilon^{\delta + \lambda} \sum_{2^j > \varepsilon |\xi|} \int_{|\xi - \eta| \geq 2^{-j} |\xi|} (\varepsilon 2^{-j} |\xi|)^{-\lambda} |\xi|^{-N_1} d\eta
$$

$$
\leq C \varepsilon^{\delta} |\xi|^{-\lambda - N_1 + a} \sum_{2^j > \varepsilon |\xi|} 2^{-j(a - \lambda)}
$$

$$
\leq C \varepsilon^{-\delta + \lambda - a} |\xi|^{-N_1} \leq C |\xi|^{-N}
$$

since $\lambda < a$, $N_1$ large.

Let $M$ be a non-characteristic piece of $\partial D$ as in Section 2.

(5.3) Theorem. Let $w \in O \cap D$ have coordinates $(\theta, x)$, $x \in M$. Let $y \in M$. Let $K(y, w)$ be the (homogeneous) fundamental solution of $L_\alpha$, $-n < \alpha < n$. Then for every $l$, there exist $k_\varepsilon$, $h_\varepsilon$, $r_0$, $r_1$, such that

$$
K(y, w) = k_\varepsilon(\Theta(y, x)) + r_0(y, w).
$$

$$
\frac{\partial}{\partial \varepsilon} K(y, w) = h_\varepsilon(\Theta(y, x)) + r_1(y, w).
$$

where $r_0$ and $r_1$ are $C^1(M \times O)$. For $\delta \geq 0$

$$
\varepsilon^\delta k_\varepsilon \in S(-1 - \delta); \quad \varepsilon^\delta h_\varepsilon \in S(-\delta)
$$

uniformly in $\varepsilon$. 


Let
\[ \tilde{k}_e(u) = C_{n, \alpha} |u'|^2 \times e^{2 - i(u^0 + 2\varepsilon u^{2m+1})^{-(n+\alpha)/2}}. \]
\[ \times \left( |u'|^2 + e^{2 + i(u^0 + 2\varepsilon u^{2m+1})^{-(n+\alpha)/2}} \right). \]

(5.4) **Corollary.** \( K(y, (\varepsilon, x)) \) is a kernel representing an operator of Poisson type of order \(-1\). Moreover \( K(y, (\varepsilon, x)) - \tilde{k}_e(\Theta(y, x)) \) represents an operator of Poisson type of order \(-2\).

Theorem (5.3) is a consequence of Propositions (2.5) and (5.1). Using (2.5), \( K(y, x) \) can be expanded in a Taylor series about \((0, 0)\) of (truncated) homogeneous distributions in \( \varepsilon \) and \( u = \Theta(y, x) \). The top terms is \( \tilde{k}_e(u) \). (The formula for \( K \) is const. \( \Phi_\varepsilon(y^{-1}x) \), where \( x, y \in \| \varepsilon \| \) and \( \Phi_\varepsilon \) is given in Appendix B. See also [6].)

In the notation of (5.1),
\[ \tilde{k}_e(u) = \text{const. } g_e(u^0 + 2\varepsilon u^{2m+1}, u'). \]

Hence,
\[ \tilde{k}_e(\xi) = \text{const. } \tilde{g}_e(\xi_0, ..., \xi_{2m}, \xi_{2m+1} - 2\varepsilon \xi_0). \]

It is easy to check that \( \tilde{k}_e(\xi) \) therefore satisfies the same estimates as \( \tilde{g}_e(\xi) \) as in (5.1). For example,
\[
\left| \frac{\partial}{\partial \xi_0} \tilde{k}_e(\xi) \right| = \left| \left[ \frac{\partial}{\partial \xi_0} \tilde{g}_e - 2\varepsilon \frac{\partial}{\partial \xi_{2m+1}} \tilde{g}_e \right] (\xi_0, ..., \xi_{2m}, \xi_{2m+1} - 2\varepsilon \xi_0) \right| 
\leq C |\xi|^{-3}, \quad \text{uniformly in } \varepsilon.
\]

The gain of \( |\xi|^{-2} \) over \( \tilde{k}_e(\xi) \) is made possible by the factor \( \varepsilon \) in the second term.

Lower terms in the Taylor expansion for \( K \) are also covered by (5.1), and they have better decrease. The remainder in Taylor's formula is easily seen to be of class \( C^{(l)}(M \times M) \), where \( l \) is as large as we like depending on the length of the Taylor formula.

The procedure for \( \partial/\partial \varepsilon \) \( K(y, x) \) is similar. We need only examine the top term:
\[ \frac{\partial}{\partial \varepsilon} \tilde{k}_e(u) = h_e^{(1)}(u) + h_e^{(2)}(u), \]
where
\[ h_e^{(1)}(u) = 4(\varepsilon^2 + 2\varepsilon u^{2m+1})^2 + 2\varepsilon |u'|^2 \]
\[ \times (a - ib)^{-(n+\alpha)/2 - 1}(a + ib)^{-(n-\alpha)/2 - 1}, \]
\[ h_e^{(2)}(u) = 4u^{2m+1}u^0(a - ib)^{-(n+\alpha)/2 - 1}(a + ib)^{-(n-\alpha)/2 - 1}, \]
with
\[ a = |u'|^2 + \varepsilon^2, \quad b = u^0 + 2\varepsilon u^{2m+1}. \]

Both of these terms are of the kind studied in (5.1). \( h^{(1)}_\varepsilon(u) \) is a multiple of an approximate identity. (This is not surprising for a double layer potential.) The second term tends to a singular kernel as \( \varepsilon \to 0 \), unlike the integrable kernel obtained in the classical case.

Theorems (5.3) and (3.2) imply that \( \varepsilon^\delta K(y, (\varepsilon, x)) \) represents an operator of class \( S^{-1-\delta} \) uniformly in \( \varepsilon \), and \( \varepsilon^\delta (\partial/\partial \varepsilon) K(y, (\varepsilon, x)) \) represents an operator of class \( S^{-\delta} \) uniformly in \( \varepsilon \). Also, \( \varepsilon^\delta (K(y, (\varepsilon, x)) - \text{const.} \, k_\varepsilon(\Theta(y, x))) \) represents an operator of class \( S^{-2-\delta} \) uniformly in \( \varepsilon \).

The proof of the corollary requires us to understand higher derivatives in \( \varepsilon \). Recall from (4.4) that if \( A \in \mathcal{G}_0 \), then \( A \) has class \( S^1_\rho \). The bad direction \( B \) is only of class \( S^2_\rho \). Using (2.3) and the fact that \( \mathcal{L}_\alpha \) is non-characteristic on \( M \),

\[
\mathcal{L}_\alpha = c(w) \frac{\partial^2}{\partial \varepsilon^2} + \sum_i c_i(w) A_i A_j + \varepsilon \sum_i c_i(w) B A_i
\]

with \( A_i \in \mathcal{F}_0 \), and \( c(w) \neq 0 \) in a neighborhood of \( M \). We now use \( \mathcal{L}_\alpha K(y, w) = 0 \) for \( w \neq y \). It follows that for \( \varepsilon > 0 \),

\[
\mathcal{L}_\alpha K(y, w) = -c(w)^{-1} \left( \sum_i c_i(w) A_i A_j + \varepsilon \sum_i c_i(w) B A_i + \varepsilon^2 d(w) B^2 + e(w) B \right) K(y, w).
\]

Note that \( \varepsilon \) commutes with \( A_j, B \). Thus \( (\partial^2/\partial \varepsilon^2)K = L_1 K + L_2 \varepsilon K + L_3 \varepsilon^2 K \), where \( L_1, L_2, \) and \( L_3 \) are operators of class \( S^2_\rho, S^3_\rho, \) and \( S^4_\rho \), respectively. Hence by Theorem (3.3), \( (\partial^2/\partial \varepsilon^2) K(w, y) \) corresponds to an operator of class \( S^4_\rho \) uniformly in \( \varepsilon \). In general, using this formula for \( (\partial^2/\partial \varepsilon^2)K \) repeatedly, any expression

\[
\varepsilon^\delta A^j \frac{\partial^j}{\partial \varepsilon^j} K(y, (\varepsilon, x))
\]

can be written as a sum of terms of the same kind with different \( A^j \), for which \( j = 0, 1 \) only. (Recall that \( A^j \) and \( \partial/\partial \varepsilon \) commute.) The corollary then follows from Theorem (3.3) and the properties of \( K \) and \( (\partial/\partial \varepsilon)K \) above.
6. AN INVERSION ON $\mathbb{H}^{n-1} \times \mathbb{R}$

We will now invert the main term of the single layer potential restricted to the boundary.

(6.1) **Theorem.** Suppose that $-n < \alpha < n,$

$$f_\alpha(u) = \left(|u'|^2 - iu^0\right)^{(n+\alpha)/2}\left(|u'|^2 + iu^0\right)^{-(n-\alpha)/2}.$$  

There exists a homogeneous distribution $g_\alpha(u)$ that is smooth away from the origin such that $f_\alpha \ast g_\alpha = g_\alpha \ast f_\alpha = \delta.$ (Convolution on $\mathbb{H}^m \times \mathbb{R}, m = n - 1$.)

**Proof.** We will use notations in this section that conform to the usual ones for the Heisenberg group $H^m$. $z = (u^1, ..., u^m) + i(u^{m+1}, ..., u^{2m})$. $t = u^0$, $x = u^{2m+1}$. The variables $(\zeta, \tau, \xi)$ with $\zeta \in \mathbb{R}^{2m}$, $\tau \in \mathbb{R}$, $\xi \in \mathbb{R}$ will be dual to $(u^1, ..., u^{2m})$, $t$, and $x$, respectively. Denote $z_j = u^j + iu^{j+m}$ and $r = (r_1, ..., r_m) = (|z_1|, ..., |z_m|)$. Suppose that $f$ depends on $r$, $t$, and $x$ only, then the group Fourier transform $\mathcal{F}(f)$ is diagonal and its entries can be represented by the formula [8, 2.10; 9]

$$\mathcal{F}(f) = F(\beta, \tau, \xi), \quad \beta = (\beta_1, ..., \beta_m),$$

$$F(\beta, \tau, \xi) = c_m a^{-1}_{\beta l} \int_0^\infty \int_{-\infty}^{\infty} e^{i\xi \cdot \eta} e^{i\tau \cdot t} f(r, t, x) \times e^{-|\tau||r|^2} L_{m-1}^{m-1}(2|\tau| |r|^2) |r|^{2m-1} d|r|,$$

where $L_{m-1}^{m-1}$ is the Laguerre polynomial

$$L_{m-1}^{m-1}(s) = \sum_{k=0}^{|\beta|} \frac{(|\beta| + m - 1)}{(|\beta| - (n - 1))} \frac{(-s)^k}{k!}, \quad |\beta| = \beta_1 + \cdots + \beta_m.$$  

$$a_k = \#\{\beta \mid k = \beta_1 + \cdots + \beta_m\} = \binom{m+k-1}{k}.$$  

Here and elsewhere $c_m$ will denote a constant depending only on dimension that may vary from formula to formula.

The normalized Laguerre function is $l_k(s) = e^{-s/2}L_k^0(s)$. We will use

$$\left| \frac{d^j}{ds^j} l_k(s) \right| \leq C_j k^j, \quad \text{for all } k.$$  

(6.3)

This follows from [3, Vol. 2, p. 189 (15), p. 207 (14)].

The inversion formula [8, 2.2] is
\[ f(r, \tau, x) = c_m \sum_{\beta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\tau t} e^{-ix \cdot \xi} F(\beta, \tau, \xi) l_\beta(\tau, r) \times |\tau|^m \, d\tau \, d\xi, \]  

(6.4)

where \( l_\beta(\tau, r) = (l_\beta(2 |\tau| r_1^2) l_\beta(2 |\tau| r_2^2) \cdots l_\beta_m(2 |\tau| r_m^2)). \)

(6.5) **Lemma.** Suppose that \( k_1 \) and \( k_2 \) are homogeneous distributions, \( C^\infty \) away from 0 on \( \mathbb{H}^m \times \mathbb{R} \), of degree \(-a + \lambda_1, -a + \lambda_2\), respectively. If \( \lambda_1 + \lambda_2 < a \), then \( k_1 * k_2 \) is a homogeneous distribution \( C^\infty \) away from 0 of degree \(-a + \lambda_1 + \lambda_2\), and \( \mathcal{F}(k_1 * k_2) = \mathcal{F}(k_1) \cdot \mathcal{F}(k_2) \). (See [10, Theorem 4.4] for a more general result.)

In Appendix B we calculate that \( \mathcal{F}(f_\alpha) = F_\alpha(\beta, \tau, \xi) \), where

\[
F_\alpha(\beta, \tau, \xi) = c_{m, \alpha} |\tau|^{-1/2} \int_0^1 (1 - s)^{(n - a) \text{sgn } \tau/2 + |\beta| - 1} \\
\times (1 + s)^{(n + a) \text{sgn } \tau/2 - n - |\beta|} e^{-\xi^2/4|\tau|s - 1/2} \, ds,
\]

and \( c_{m, \alpha} \neq 0 \). Note that \( F_\alpha \neq 0 \), because the integrand is positive. Hence, \( F_\alpha^{-1} \) exists. The obvious choice for \( g_\alpha \) is \( \mathcal{F}^{-1}(F_\alpha^{-1}) \). It is a homogeneous distribution of the right order, provided it is well-defined. By Lemma (6.5), \( g_\alpha * k_\alpha = k_\alpha * g_\alpha = \delta \). The main point is to prove that \( \mathcal{F}^{-1}(F_\alpha^{-1}) \) is \( C^\infty \) away from 0.

We will now replace \( F_\alpha \) by an appropriate constant multiple and drop the subscript \( \alpha \). Let

\[
F = \pi^{-1/2} |\tau|^{-1/2} \int_0^1 (1 - s)^{(n - a) \text{sgn } \tau/2 + |\beta| - 1} \\
\times (1 + s)^{(n + a) \text{sgn } \tau/2 - n - |\beta|} e^{-\xi^2/4|\tau|s - 1/2} \, ds.
\]

Denote \( \rho = |\tau| (2 |\beta| + m) + \xi^2/4; E = 1 - \rho^{1/2} F. \)

(6.7) **Lemma.**

(a) \( |F| \leq C |\tau| \rho^{-1}. \)

(b) \[ \left| \frac{\partial^j}{\partial \xi^j} \frac{\partial^k}{\partial \tau^k} E \right| \leq C \rho^{-j/2} |\tau|^{-k}. \]

(c) \[ \left| \frac{\partial^j}{\partial \xi^j} \frac{\partial^k}{\partial \tau^k} F \right| \leq C \rho^{-j/2 - 1/2} |\tau|^{-k}. \]

(d) \( 0 < c_1 < |\rho^{1/2} F| < c_2. \)
Proof. Let \( h = \zeta^2/4 \mid \tau \mid, b = \mid \beta \mid \). Big \( O \) below is independent of \( h \) and \( b \).

\[
\pi^{1/2} \rho^{1/2} F = \int_0^1 (1 - s)^{(n + \alpha)/2 + b - 1} (1 + s)^{(n - \alpha)/2 - n - b} \times e^{-hs}(2b + m + h)^{1/2} s^{1/2} \frac{ds}{s} \\
= \int_0^{1/2} \exp \left\{ (\log(1 - s)) \left( \frac{n + \alpha}{2} + b - 1 \right) \\
+ (\log(1 + s)) \left( \frac{n - \alpha}{2} - n - b \right) - hs \right\} \\
\times ((2b + m + h)s)^{1/2} \frac{ds}{s} + O((2b + m + h)^{-10})
\]

\[
= \int_0^{1/2} \exp\{- (2b + m + h)s + O(s) + O(s^3b)\} \\
\times ((2b + m + h)s)^{1/2} \frac{ds}{s} + O((2b + m + h)^{-10})
\]

\[
= \int_0^{1/2} \exp\{- (2b + m + h)s\} (1 + O(s) + O(s^3b)) \\
\times ((2b + m + h)s)^{1/2} \frac{ds}{s}
\]

\[
= \int_0^{(1/2)(2b + m + h)} e^{-u} \left( 1 + O \left( \frac{u}{(2b + m + h)} \right) \\
+ O \left( \frac{u^3}{2b + m + h} \right) \right) u^{1/2} \frac{du}{u}
\]

\[
= \int_0^{\infty} e^{-u} u^{1/2} \frac{du}{u} + O \left( \frac{1}{2b + m + h} \right)
\]

\[
= \pi^{1/2} + O(\mid \tau \mid \rho^{-1}).
\]

This proves (a). To prove (c), note that

\[
\left| \frac{\partial^j}{\partial y^j} \frac{\partial^k}{\partial \tau^k} \left( \mid \tau \mid^{-1/2} e^{-t^{2/4} \mid \tau \mid} \right) \right| \\
\leq C \left( \left| \frac{\xi_s}{\tau} \right| \mid \tau \mid^{-k - 1/2} + \left| \frac{\xi_s}{\tau} \right| \left( \frac{\xi_s^2}{\tau^2} \right)^k \mid \tau \mid^{-1/2} \right).
\]
Thus,
\[
\left| \frac{\partial^j}{\partial s^j} \frac{\partial^k}{\partial \tau^k} F \right| \leq C \int_0^\infty \exp\left(-2b+m+h\right)s^{j+1/2} \frac{ds}{s} \\
+ \int_0^\infty \exp\left(-2b+m+h\right)s^{j+1/2} \frac{ds}{s} \right| \left| \frac{\partial^j}{\partial \tau^j} \left( \frac{\partial^k}{\partial s^k} \right) \right| \left| \tau \right|^{-k-1/2} \left(2b+m+h\right)^{j+1/2} \frac{ds}{s}
\]
\[
\leq C \int_0^\infty \exp\left(-2b+m+h\right)s^{j+1/2} \left(2b+m+h\right)^{j+1/2} \frac{ds}{s} \\
\times \left| \tau \right|^{-k} \rho^{-j/2-1/2}
\]
\[
+ C \int_0^\infty \exp\left(-2b+m+h\right)s^{j+1/2} \left(2b+m+h\right)^{j+1/2} \frac{ds}{s} \\
\times s^{k+j+1/2} \frac{ds}{s} \left| \tau \right|^{-k-1/2} \rho^{-j/2-1/2}
\]
\[
\leq C \int_0^\infty \exp\left(-2b+m+h\right)s^{j+1/2} \frac{ds}{s} \left( \frac{du}{u} \right) \left| \tau \right|^{-k} \rho^{-j/2-1/2}
\]
\[
\leq C \rho^{-j/2-1/2} \left| \tau \right|^{-k}.
\]

A similar proof gives (b).

If \( \left| \tau \right| \rho^{-1} \) is less than a small constant, then (a) implies (d). If not, then \( 2b+m+h<C \). Because \(-n<\alpha<n, \) the integral for \( F \) is absolutely convergent and positive. The formula for \( \rho^{1/2}F \) on the first line of the proof shows that \( \rho^{1/2}F \) is a continuous positive function of \( h \) and \( b \) on a compact set: \( h \geq 0, b \geq 0; 2b+m+h<C \). Therefore, (d) holds.

Let \( \phi_0 \in C_0^\infty(\mathbb{R}) \) with
\[
\phi_0(s) = \begin{cases} 
1 & |s| < \varepsilon_0 \\
0 & |s| > 2\varepsilon_0.
\end{cases}
\]

Define \( \phi(\beta, \tau, \xi) = \phi_0(\tau \rho^{-1}) \).

(6.8) **Lemma.** \( \mathcal{F}^{-1}(1-\Phi) \) and \( \mathcal{F}^{-1}((1-\Phi)F^{-1}) \) are \( C^\alpha \) homogeneous distributions away from 0.

**Proof:** Let \( \psi(\beta, \tau, \xi) = \phi_0(\rho) \). The support of \( 1-\Phi \) is the set where \( \left| \tau \right| \geq \varepsilon_0 \rho. \) Using (6.4), \( \mathcal{F}^{-1}(1-\Phi) = A_1 + A_2 \), where
\[
A_1 = \sum_\beta \int \left( 1 - \Phi \right) \psi e^{-i\pi \tau} e^{-i\pi \cdot \xi \cdot \beta(\tau, r)} \left| \tau \right|^m \, dt \, d\xi,
\]
\[
A_2 = \sum_\beta \int \left( 1 - \Phi \right)(1-\psi) e^{i\pi \tau} e^{i\pi \cdot \xi \cdot \beta(\tau, r)} \left| \tau \right|^m \, dt \, d\xi.
\]
Any derivative of $A_i$ in $t, x, z$ is dominated by (see (6.3))

$$
\sum_\beta \iint (1 - \phi)\psi |\tau|^j |\xi|^k |\beta|^\gamma \, dt \, d\xi
$$

which is finite because the support of $(1 - \phi)\psi$ is compact.

For $A_2$ we must introduce the operator $A$ with the property

$$
A \mathcal{F}(g) = \mathcal{F}((|z|^2 + it + x^2) \, g(t, \xi, x)).
$$

If we can show that for any $k$ there exists $N$ such that

$$
\mathcal{F}^{-1}(A^N(1 - \phi)(1 - \psi))
$$

has $k$ derivatives away from 0 then $\mathcal{F}^{-1}((1 - \phi)(1 - \psi))$ is $C^\infty$ away from 0.

Let $e_j$ denote the multi-index with entry 1 in the $j$th place and zero elsewhere. Denote

$$
D_{j,S}(\beta, t, \xi) = S(\beta + e_j, t, \xi) + S(\beta - e_j, t, \xi) - 2S(\beta, t, \xi).
$$

Let $DS(\beta, t, \xi) = \sup_j |D_j S(\beta, t, \xi)|$. Geller [8] has calculated that

$$
AS = \frac{\partial}{\partial t} S + \frac{1}{4} \tau^{-2} \sum_{j=1}^{m} 2 |\tau| (\beta_j + 1) D_j S - \frac{\partial^2}{\partial \xi^2} S.
$$

Thus

$$
|AS| \leq \left| \frac{\partial}{\partial t} S \right| + \rho |\tau|^{-2} DS + \left| \frac{\partial^2}{\partial \xi^2} S \right|.
$$

Note that

$$
\left| \frac{\partial}{\partial t} \rho \right| \leq C \rho |\tau|^{-1}, \quad \left| \frac{\partial^2}{\partial \xi^2} \rho \right| \leq C.
$$

Hence,

$$
|A^M(1 - \psi)(1 - \phi)| \leq C(\rho^N |\tau|^{-N-M} + \rho^{-M}).
$$

On the support of $(1 - \psi)(1 - \phi)$, $|\tau| > \epsilon_0^1 \rho$ and $\rho > \epsilon_0$. Therefore, $|\beta| < \epsilon_0^{-1}$.

We thus obtain

$$
\left| \frac{\partial^j}{\partial x^j} \frac{\partial^\nu}{\partial z^\nu} \frac{\partial^k}{\partial t^k} (|z|^2 + x^2 + it)^M A_2 \right| \leq C \sum_{|\beta| < \epsilon_0^{-1}} \int \int |\beta| |\tau|^{k+m} |\xi|^j (\rho^N |\tau|^{-N-M} + \rho^{-M}) \, dt \, d\xi
$$

$$
\leq \sum_{|\beta| < \epsilon_0^{-1}} \int \int \rho^{-M} \, dt \, d\xi < \infty, \quad \text{since } |\tau|^{-1} < \epsilon_0^1 \rho^{-1}.
$$
This concludes the proof that $\mathcal{F}^{-1}(1 - \phi)$ is $C^\infty$ away from the origin. The same proof using estimates from Lemma 6.7(c) and (d) shows $\mathcal{F}^{-1}((1 - \phi) F^{-1})$ is $C^\infty$ away from zero.

(6.9) Lemma. $\mathcal{F}^{-1}(\rho^{1/2})$ is a homogeneous distribution $C^\nu$ away from 0 (of degree $-a - 1$).

This lemma is a consequence of a theorem of Folland [5]. Denote by $\mathcal{L}_0$, the sub-Laplacian on $\mathbb{R}^m$ ($\mathcal{L}_0 = -\frac{1}{4} \sum_{j=1}^m X_j^2 + Y_j^2$). The symbol of $\mathcal{L}_0$ is $(2|\beta| + m)|\tau|$ (see [9]). Thus the symbol of

$$\mathcal{L}_0 - \frac{1}{4} \frac{\partial^2}{\partial x^2}$$

is $\rho$. Folland constructed a homogeneous kernel $R_1$, $C^\nu$ away from 0 such that

$$\left(\mathcal{L}_0 - \frac{1}{4} \frac{\partial^2}{\partial x^2}\right)^{1/2} * R_1 = \delta.$$

Thus $\mathcal{F}(R_1) = \rho^{-1/2}$.

The kernel we want is

$$\mathcal{F}^{-1}(\rho^{1/2}) = \left(\mathcal{L}_0 - \frac{1}{4} \frac{\partial^2}{\partial x^2}\right) R_1.$$

It is $C^\infty$ away from zero because it is obtained from $R_1$ by differentiation.

(6.10) Lemma. $\mathcal{F}^{-1}(E^{N+1}F^{-1})$ is a homogeneous distribution that is $C^\nu$ away from 0 for sufficientlarge $N$.

Proof. As in Lemma (6.8), we split the integral into two parts.

$$A_1 = \sum_{\beta} \left\{ \int \psi \phi E^{N+1}F^{-1}e^{-it\tau}e^{-ix\cdot l_\beta(\tau, r)}|\tau|^m \, dt \, d\xi \right\}.$$

$A_2$ is the same as $A_1$ with $\psi$ replaced by $1 - \psi$. Using (6.3)

$$\left| \frac{\partial^j}{\partial t^j} \frac{\partial^k}{\partial t^k} \frac{\partial^\gamma}{\partial x^\gamma} A_1 \right| \leq C \int \int \psi \phi |E^{N+1}|F^{-1}|\beta|^N(1 + |\tau|)^N(1 + |\xi|)^N \, dt \, d\xi,$$

where $N_1$ depends only on $j$, $k$, $\gamma$. From Lemma (6.7a), $|E^{N+1}| \leq \ldots$
\[|\tau|^{N+1}\rho^{-(N+1)} \text{ and Lemma (6.7d), } |F^{-1}| \leq \rho^{1/2}. \] Notice also that \[|\tau| \rho^{-1} \leq (2|\beta| + m)^{-1}. \] Therefore, the sum above is dominated by

\[C \sum_{\beta} (2|\beta| + m)^{-N_2} \int \int \psi \phi(1 + \rho)^{2N_1 + 1/2} \, d\tau \, d\xi < \infty\]

because \(\rho < \varepsilon_0\) on the support of \(\psi\).

For \(A_2\), we need to act by \(A^{N_2}\). By reasoning similar to that in Lemma (6.8), this gives rise to a bound of the form

\[\sum_{\beta} \int \int (1 - \psi) \phi(\tau p^{-1})^{N+1} |\tau|^{-N_2} |\beta|^{1/2} |\tau|^{1/2} |\xi|^{k} \, d\tau \, d\xi,\]

where we can assume \(N_2 \gg 2|\gamma| + k\) and \(N \gg N_2\). Using \(\tau p^{-1} \leq (2|\beta| + m)^{-1}\), this reduces to

\[\sum_{\beta} (2|\beta| + m)^{-M_1} \int \int \rho^{-M_2} \, d\tau \, d\xi < \infty,\]

because \(\rho > \varepsilon_0\) on the support of \(1 - \psi\). This concludes the proof of the lemma.

Let us now prove Theorem (6.1). \(F^{-1} = B_1 + B_2 + B_3\), where \(B_1 = \phi \rho^{1/2}(1 + E + \cdots + E^N)\), \(B_2 = \phi E^{N+1}F^{-1}\), and \(B_3 = (1 - \phi)F^{-1}\). By Lemmas (6.8), (6.9), and (6.5), \(F^{-1}(\phi)\) (and hence \(F^{-1}(\phi))\), \(F^{-1}(\rho^{1/2})\), \(F^{-1}(E)\) are homogeneous distributions that are \(C^\infty\) away from zero. By Lemma (6.5), \(B_1\) is \(C^\infty\) away from zero. The same is true for \(B_3\) (Lemma (6.8)). Finally, Lemma (6.10) says that \(B_2\) is \(C^k\) away from zero. Since this is true for any \(k\), \(F^{-1}(\rho)\) is \(C^\infty\) away from zero. Q.E.D.

Remark. This inversion procedure enables one to write down a formula for the symbol of the Poisson kernel for \(\mathcal{L}_a\) in the upper half space \(\{(z, t) | \text{Im } z_n > 0\}\) by taking the composition of \(g_a\) with the single layer potential. In \(H^1\) this formula is completely explicit and very easy to obtain, since the boundary group is just Euclidean space \(R^2\) and the symbol for the single layer potential was already calculated by Greiner and Steiner [11]. In all cases it is possible to deduce sharp estimates from the formula. Another formula for the case of this upper half plane in \(H^1\) and for the operator \(\mathcal{L}_a\) was given by Gaveau [7]. There appears to be an error in his formula—it is not invariant under left translation.

7. Regularity in the Dirichlet Problem for \(\mathcal{L}_a\)

Let \(D \subset \mathbb{H}^n\) be a smooth, bounded domain. Let \(\phi \in C_0^\infty(\mathbb{H}^n)\) be supported in a small neighborhood of a non-characteristic point of \(\partial D\). Let \(-n < a < n\).
(7.1) Theorem. For \( f \in I_0(\overline{D}) \) and \( g \in I_{\beta+2}(\partial D) \), there is a unique solution to
\[
L_\alpha u = f \text{ in } D; \quad u \mid_{\partial D} = g.
\]
Moreover, \( \phi u \in I_{\beta+2}(\overline{D}) \).

Proof. Note that \( f - L_\alpha g \in I_\beta \subset L^2(D) \). From Kohn's subcoercive estimate (1.1), the problem
\[
L_\alpha w = f - L_\alpha g \text{ in } D \quad \text{and} \quad w \mid_{\partial D} = 0
\]
has a unique solution \( w \in L^2(D) \). Let \( u = w + g \). Then \( u \) is the unique solution to (*) in \( L^2(D) \). Let \( \psi \) be a smooth function with compact support that is 1 on the support of \( \phi \). The theorem of Kohn and Nirenberg [14, Theorem 4] shows that if \( \psi f \) and \( \psi g \) are \( C^\infty \), then \( \phi u \in C^\infty(\overline{D}) \). What we need is an estimate a priori that if \( \psi f \) and \( \psi g \) are \( C^\infty \), then
\[
\| \phi u \|_{H^1(D)} \leq C(\| \psi g \|_{H^1(D)} + \| \psi f \|_{H^1(D)} + \| \psi u \|_{L^2(D)}).
\]
(7.2)

The fact that \( \phi u \in I_{\beta+2}(\overline{D}) \) when \( \psi f \) and \( \psi g \) are not \( C^\infty \) follows from a routine limiting argument based on (7.2).

To prove (7.2) we will construct a parametrix for the Dirichlet problem. The method is analogous to that given for the ordinary Dirichlet problem by Greiner and Stein [11]. Let \( V \) be a small ball about a non-characteristic point; \( M = V \cap \partial D \). As in Theorem (5.3) let \( K \) denote the homogeneous fundamental solution to \( L_\alpha \).

(7.3) Lemma. There is an operator \( P: C^\infty(M) \to C^\infty(V \cap \overline{D}) \) such that for any \( g \in C^\infty(M) \),
\[
\begin{align*}
(a) & \quad L_\alpha P(g) = 0 \text{ in } V \cap D, \\
(b) & \quad \lim_{\epsilon \to 0} P(g) = g \text{ on } M. \quad (P(g) \text{ is a function on } V \cap D \text{ with coordinates } (\epsilon, x), x \in M.) \\
(c) & \quad \text{For any } N \text{ there is a Poisson-type operator of order zero, } \tilde{P}, \text{ such that } P - \tilde{P} \text{ has a kernel in } C^N(V \cap \overline{D} \times M).
\end{align*}
\]
(For a similar theorem, see 7.57 of [11]).

Proof. According to Corollary (5.4), the main term of the single layer potential \( K((\epsilon, x), y) \) is \( \overline{K}(\Theta(y, x)) \). In Theorem (6.1) we found \( g_\alpha \) so that \( \overline{K} \ast g_\alpha = \delta \) on \( \mathbb{R}^{n-1} \times \mathbb{R} \). The kernel \( \tilde{P}(x, y) \) given by composition of operators with kernel \( K((\epsilon, x), y) \) and \( g_\alpha(\Theta(y, x)) \) is a Poisson-type operator of order zero. Moreover, by Theorem (3.9), \( \lim_{\epsilon \to 0} P_\epsilon(x, y) = \delta(y) + E(x, y) \), where \( E \) is the sum of a kernel of class \( S_{-1} \) and a kernel in \( C^N \) for arbitrarily large \( N \).

\( g \) is identified with an extension belonging to \( I_{\beta+2}(\overline{D}) \).
Let $\tilde{P}_\epsilon$, $E$ denote operators with kernels $\tilde{P}_\epsilon(x, y)$ and $E(x, y)$. By a standard Neumann series argument (possibly shrinking $M$) we can replace $\tilde{P}_\epsilon$ by $P_\epsilon(I - E + E^2 + \cdots + E^j)$ and $E$ by the composition of $E$ with itself $j + 1$ times and assume that $E$ has a kernel in $C^N$ for any large $N$. For a sufficiently small neighborhood $M$, the kernel of $(I + E)^{-1} - I$ belongs to $C^N$. Thus $P_\epsilon = \tilde{P}_\epsilon(I + E)^{-1}$ satisfies Lemma (7.3).

For $f \in C_0^\infty(V)$, denote

$$G(f) = K(f) - P((Kf) \mid_M)$$

($K$ is the fundamental solution to $\mathcal{L}_\alpha$ as above.) From (7.3) we deduce

(7.4) **Theorem.** If $f \in C_0^\infty(V)$, $g \in C_0^\infty(M)$, and $u = G(f) + P(g)$, then $\mathcal{L}_\alpha u = f$ in $V \cap D$ and $u \mid_M = g$ (see [11, 7.62]).

(7.5) **Theorem.** If $u \in C_0^\infty(V)$, then $u = G_1(\mathcal{L}_\alpha u) + P_1(u \mid_M)$ in $V \cap \overline{D}$, where $G_1$ and $P_1$ differ from $G$ and $P$ by integral operators with $C^\infty$ kernels.

Theorem (7.5) follows from Theorem (7.4) as in 7.67 of [11]. The only change is that Green’s formula is replaced by its analogue (see [7])

$$\int_U v^\alpha w - (\mathcal{L}_\alpha v) \bar{w} = \int_{\partial U} (v(\bar{D}_r \cdot \bar{D}w) - \bar{w}(\bar{D}_r \cdot \bar{D}v)) \frac{d\sigma}{\|\nabla r\|} + i\alpha \int_{\partial U} v \bar{w} \text{Tr} \frac{d\sigma}{\|\nabla r\|},$$

where $r$ is a defining function for $U$,

$$\bar{D}_r = (X_1 r, \ldots, X_n r, Y_1 r, \ldots, Y_n r);$$

$\|\nabla r\|$ is the ordinary length of the ordinary gradient, and $d\sigma$ is surface measure on $\partial U$.

Let $u \in C^\infty(V)$ and $f = \mathcal{L}_\alpha u$. For $\phi \in C_0^\infty(V)$ we have $\mathcal{L}_\alpha(\phi u) = \phi f + D_1(u \nabla \phi)$, where $D_1$ is an operator of class $S_0^1$ (see Sect. 4). Hence, by Theorem (7.5),

$$\phi u = G_1(\phi f + D_1(u \nabla \phi)) + P_1(\phi u \mid_M).$$

We can rewrite this as

$$\phi u = G_1(\psi f) + P_1((\psi g)) + R(\psi u),$$

where $g = u \mid_M$ and $\psi \in C_0^\infty$, $\psi = 1$ on the support of $\phi$ and

$$R(\psi u) = K_1(\psi u) + P((K_2(\psi u)) \mid_M),$$

(7.6)
where $K_1$ and $K_2$ are of class $S^0_{-1}$. For sufficiently large $s$, $H_s \subset L^\infty_{\text{loc}}$. The operators $K_1$ and $K_2$ map $H_s$ to $H_{s+1/2}$ [15, Theorem 11] and it is easy to show that $P$ maps $H_s(M)$ to $H_s(V \cap D)$. It follows from an iteration of (7.6) with nested cut-off functions that

$$\|\phi_1 u\|_{L^\infty} \leq C(\|\psi g\|_{H^{s+1/2}(M)} + \|\psi f\|_{H^s(M)} + \|\psi u\|_{L^2(D)}),$$

where $\phi_1 = 1$ on the support of $\phi$ and $\psi = 1$ on the support of $\phi_1$. Next, $K_1$ and $K_2$ are bounded from $\Gamma_\beta$ to $\Gamma_{\beta+1}$ (Theorem (4.2)) and from $L^\infty$ to $\Gamma_1$ (see [15, Chap. III]). Furthermore, $P$ maps $\Gamma_\beta(M)$ to $\Gamma_\beta(V \cap D)$. An iteration with nested cut off functions shows that

$$\|\phi u\|_{\Gamma_{\beta+1}(\overline{D})} \leq C(\|\psi g\|_{\Gamma_{\beta+1}(M)} + \|\psi f\|_{\Gamma_\beta(\overline{D})} + \|\phi_1 u\|_{\Gamma_1(D)}).$$

This, combined with the estimate above on $\|\phi_1 u\|_{L^\infty}$ gives (7.2).

APPENDIX A

In this appendix we will prove the theorems and lemmas stated in Section 4. For notation, see Section 4.

(4.5) **Lemma.** If $f \in \tilde{F}_\beta(M)$, and $p_\varepsilon(x, D)$ is a Poisson-type operator of order 0 and $g(w) = p_\varepsilon(x, D)f$, where $w$ has coordinates $(\varepsilon, x)$, then $g$ is smooth for $\varepsilon > 0$ and

$$|X^\gamma g(w)| \leq C \max(1, d(w, M)^{\beta-|\gamma|}).$$

for $\beta \neq |\gamma|$.

**Proof:** First of all, for $|\gamma| < \beta$, the estimate follows immediately from Theorem (4.2). The dyadic decomposition of a symbol $a(x, \xi) \in S^k_\rho$ is as follows [15, p. 49]. Choose $\eta \in C^\infty(\mathbb{R})$ so that $\eta \geq 0$ and

$$\eta(t) = \begin{cases} 1 & t \leq 1 \\ 0 & t \geq 2. \end{cases}$$

Denote $a_0(x, \xi) = \psi(\rho(x, \xi)) a(x, \xi)$.

$$a_j(x, \xi) = (\psi(2^{-j} \rho(x, \xi)) - \psi(2^{-j+1} \rho(x, \xi))) a(x, \xi),$$

$$a(x, \xi) = \sum_{j \geq 0} a_j(x, \xi).$$
The kernel $k_j(x, x - y) = \int a_j(x, \xi) e^{-2\pi i L \cdot (x - y)} \, d\xi$ is well-defined because $a_j$ has compact support in $\xi$. The kernels $k_j(x, x - y)$ satisfy the estimate [13]

$$|k_j(x, x - y)| \leq C_{N_1} 2^{(k + a)(1 + 2^l d(x, y))^{-N_1}},$$

where $a = 2n + 1$ ($a$ is the homogeneous dimension of $\mathbb{H}^m \times \mathbb{R}$). Moreover, for $j \geq 1$, $\int k_j(x, x - y) Q(y) \, dy = 0$ for any polynomial $Q(y)$ in local coordinates on $M$. This is because $a_j(x, \xi)$ vanishes in a neighborhood of $\xi = 0$.

In our case, we can write

$$X^\sigma p_\epsilon(x, D) f = \sum_j \int k^j_\epsilon(x, x - y) f(y) \, dy.$$

$k^j_\epsilon$ depends on $y$, of course. Since

$$e^\delta \frac{\partial^N_2}{\partial e^N_2} X^\sigma p_\epsilon(x, \xi) S^{\delta + N_2 + |y|},$$

uniformly in $\epsilon$, we have the estimates

$$|e^\delta \frac{\partial^N_2}{\partial e^N_2} k^j_\epsilon(x, x - y)| \leq C 2^{j(-\delta + N_2 + |y| + a)} (1 + d(x, y) 2^j)^{-N_1},$$

where $C$ depends on $N_2$, $\gamma$, and $\delta$.

We will now estimate

$$X^\sigma g(\nu) = \sum g^j_\epsilon(x),$$

where

$$g^j_\epsilon(x) = \int k^j_\epsilon(x, x - y) f(y) \, dy.$$

For $j = 0$, the term $g^0_\epsilon(x)$ is evidently bounded independent of $\epsilon$. For $j \geq 1$, by the orthogonality,

$$e^\delta \frac{\partial^N_2}{\partial e^N_2} g^j_\epsilon(x) = \int e^\delta \frac{\partial^N_2}{\partial e^N_2} k^j_\epsilon(x, x - y)(f(y) - Q_{(x, \nu)}(y)) \, dy$$

for $j \geq 1$. There exists a dimensional constant $A$ such that (see [13]),

$$|f(y) - Q_{(x, \nu)}(y)| \leq C 2^{4l \epsilon 2^l},$$

where $d(x, y) \simeq 2^l \epsilon$, $l = 1, 2, \ldots$. Hence

$$|e^\delta \frac{\partial^N_2}{\partial e^N_2} g^j_\epsilon(x)| \leq C \sum_{l \geq 0} 2^{l(-\delta + N_2 + |y| + a)} (1 + 2^{l/2^l} \epsilon) - N_1 2^{l/2^l} \epsilon \int_{d(x, y) \simeq 2^l \epsilon} \, dy$$

$$< C \sum_{l \geq 0} 2^{l(-\delta + N_2 + |y| + a)} (1 + 2^{l/2^l} \epsilon) - N_1 2^{l/2^l} \epsilon^{\delta(2^l \epsilon)^a}.$$
Case 1. $2^j \geq \varepsilon^{-1}$. Let $N_2 = 0, N_1 > A + a$. Then

$$|g^j(x)| \leq C e^{-\delta + \beta + a} 2^{k(-\delta + \beta + \gamma + a)} \sum_{l \geq 0} (1 + (2^j \varepsilon)^{2l})^{-N_1} 2^{l(A + a)}$$

$$\leq C e^{-\delta + \beta + a} 2^{k(-\delta + \beta + \gamma + a)}.$$

Now choose $\delta > |\gamma| + a$.

$$\sum_{2^j \geq \varepsilon^{-1}} |g^j(x)| \leq C e^{-\delta + \beta + a} \sum_{2^j \geq \varepsilon^{-1}} 2^{j(-\delta + |\gamma| + a)} \leq C e^{-|\gamma|}.$$

Case 2. $2^j < \varepsilon^{-1}$. This time, let $\delta = 0, N_1 > A + a$.

$$\left| \frac{\partial^{N_2}}{\partial \varepsilon^{N_2}} g^j(x) \right| \leq C e^{\beta + a} 2^j(N_2 + |\gamma| + a) \sum_{l \geq 0} (1 + (2^j \varepsilon)^{2l})^{-N_1} 2^{l(A + a)}$$

$$\leq C e^{\beta + a} 2^j(N_2 + |\gamma| + a) \sum_{l \geq 0} 2^{-lN_1} e^{-N_1} 2^{l(A + a) - l - |\gamma|}$$

$$\leq C e^{\beta + a - N_1} 2^j(N_2 + |\gamma| + a - N_1).$$

By Taylor's formula,

$$|g^j(x)| = \left| \sum_{i=0}^{N_2-1} \frac{(\varepsilon - 1)^i}{i!} \frac{\partial^i}{\partial \varepsilon^i} g^j(x) \right|$$

$$+ \frac{1}{(N_2 - 1)!} \int_{\varepsilon}^{1} (\varepsilon - s)^{N_2 - 1} \frac{\partial^{N_2}}{\partial s^{N_2}} g^j(x) ds \right|$$

$$\leq \sum_{i \leq N_2 - 1} \left| \frac{\partial^i}{\partial \varepsilon^i} g^j(x) \right| + \int_{\varepsilon}^{1} (\varepsilon - s)^{N_2 - 1} \left| \frac{\partial^{N_2}}{\partial s^{N_2}} g^j(x) \right| ds.$$

The first term is estimated trivially by $2^{-j}$ as in Case 1. For the second term we have the bound

$$\left| \int_{\varepsilon}^{1} (\varepsilon - s)^{N_2 - 1} s^{\beta + a - N_1} ds \right|$$

$$\leq \int_{\varepsilon}^{1} s^{N_2 + \beta + a - N_1 - 1} ds \leq C e^{N_2 + \beta + a - N_1 + |\gamma| - a - N_1},$$

provided $N_2 < N_1 - (\beta + a)$. We will also choose $N_2 > N_1 - (|\gamma| + a)$. (Remember, $|\gamma| > \beta$.) Finally,
\[ \sum_{2^j \geq \varepsilon^{-1}} |g^j(x)| \leq C \sum_{2^j \geq \varepsilon^{-1}} 2^{-j} + C \sum_{2^j \geq \varepsilon^{-1}} e^{N_2 + \alpha - N_1} 2^{j(N_2 + |\gamma|) + \alpha - N_1} \leq C(1 + e^{\beta - \varepsilon^2}). \]

Let \( D \) be a smooth domain in \( \mathbb{R}^n \).

(4.6) \textbf{Lemma.} \textit{Suppose that} \( k < \beta < k + 1 \), \( g \in C^\infty(D) \) and \( |X^\gamma g(x)| \leq C \max(1, d(x, \partial D)^{\beta - |\gamma|}) \) for \( |\gamma| \leq 3(k + 1) \); \textit{then} \( u \in \Gamma_\beta(D) \).

\textbf{Proof.} \ Let \( D^\gamma \) denote the left-invariant operator that agrees with \( \partial^\gamma / \partial x^\gamma \) at 0. For \( w \in D \), denote \( d = d(w, \partial D) \). If \( d(x, w) < \frac{1}{4}d \), then Taylor’s formula can be written \( g(x) = P(x, w) + R(x, w) \), where

\[
P(x, w) = \sum_{|\gamma| \leq k} \frac{1}{\gamma!} D^\gamma g(w)(w^{-1}x)^\gamma,
\]

\[
R(x, w) = \int_0^1 (1 - s)^k \sum_{k + 1 \leq |\gamma| \leq 2(k + 1)} c_{n, j} s^j (w^{-1}x)^\gamma \times D^\gamma g(w(s(w^{-1}x))) ds.
\]

The hypothesis of the lemma implies

\[ |R(x, w)| \leq C |w^{-1}x|^{k + 1} d^{\beta - (k + 1)} \leq C \min(d(w, x), d)^{\beta} \]

if \( d(x, w) < \frac{1}{4}d \). More generally, if \( |\gamma| \leq k \), then

\[ |D^\gamma R(x, w)| \leq C d^{\beta - 1|\gamma|}. \]

Choose \( b \in D \) such that \( |w^{-1}b| < \frac{3}{4}d \).

\[
P(x, b) = P(x, w) = \sum_{|\gamma| \leq k} \frac{1}{\gamma!} (D^\gamma b)(b)(x^{-1}b)^\gamma.
\] (2)

This identity holds because each side is a polynomial of (non-isotropic) degree \( k \) whose Taylor series at \( b \) coincide:

\[ D^\gamma(P(x, b) - P(x, w)) |_{x = b} = D^\gamma(g(x) - P(x, w)) |_{x = b} = D^\gamma R(x, w) |_{x = b}. \]

Formula (2) implies

\[ |D^\gamma_x(P(x, b) - P(x, w))| \leq C d^{\beta - 1|\gamma|} \] (3)
provided \(|b^{-1}w| < \frac{3}{2}d\) and \(|x^{-1}w| < 100d\). (Note that we have extended the range of \(x\) far beyond \(\partial D\).)

Because \(D\) is a domain with bounded second derivatives, one can construct a chain of balls \(B_j = \{z: d(z, x_j) < d_j\}D\) such that \(x_1 = w, x_n = x\); \(d(B_j, \partial D) > cd_j\), and \(d_j = (1 + r)^{-j}d\), whenever \(x \in D, \; d(x, w) < 100d\). Here \(r > 0\) depends only on \(D\). Hence using (3) for \(\gamma = 0\),

\[
|g(x) - P(x, w)| \leq \sum_{j=1}^{N-1} |P(x, x_j) - P(x, x_{j+1})| \leq C \sum_{j=1}^{N} ((1 + r)^{-j}d)^\beta \leq Cd^\beta.
\]

for all \(x \in D, \; d(x, w) < 100d\).

This procedure shows that \(g(x)\) is continuous in \(\bar{D}\). Moreover, for any point \(x \in \bar{D}\), distance \(\delta > 0\) and any point \(w\) such that \(d(w, a) < \delta\) and \(d(w, \partial D) > c\delta\), we can show that

\[
|g(x) - P(x, w)| \leq C\delta^\beta \quad \text{for} \quad d(x, w) < 100\delta.
\]  (4)

Take a Whitney decomposition of \(\partial D\) relative to the distance function \(d\). (This is possible because \(d\) satisfies the approximate triangle inequality.) Let \(\phi_j\) be a partition of unity subordinate to the "cubes" \(Q_j\). We can demand that

\[
|X\phi_j| \leq C\text{diam}(Q_j)^{-1/\gamma}.
\]

Let \(p_j\) be a point of \(D\) such that \(d(p_j, \partial D) > c\text{diam} Q_j\), and \(d(p_j, Q_j) < C\text{diam} Q_j\).

Define

\[
Eg(x) = g(x) \quad x \in \bar{D}
\]

\[
= \sum_j P(x, p_j) \phi_j(x) \quad x \in \partial D.
\]

Estimate (3) and the estimates on \(X\phi_j\) above imply that

\[
|D^\gamma Eg(x)| \leq C \max(1, d(x, \partial D))^{\beta - 1/\gamma}
\]

for \(x \in \partial D\). By the same argument as for \(g\), \(Eg\) satisfies (4) in all \(\mathbb{H}^n\) and hence belongs to \(\Gamma_\beta\).

**Proposition.** For \(k < \beta < k + 1, f \in \Gamma_\beta\) if and only if for every \(x\) there exists \(P(y, x)\), a polynomial in \(y\) of homogeneous degree \(k\) such that

\[
|f(y) - P(y, x)| \leq C |y^{-1}x|^\beta.
\]

Furthermore, \(|D^\gamma (f(y) - P(y, x))| \leq C |y^{-1}x|^\beta - 1/\gamma\), for all \(|y| \leq k\).
Proof: As in Lemma (4.6), denote the Taylor polynomial of homogeneous degree $k$.

$$P(y, x) = \sum_{|\gamma| \leq k} \frac{1}{\gamma!} D^\gamma f(x)(y^{-1}x)^\gamma.$$ 

We will show that $|f(y) - P(y, x)| \leq C |y^{-1}x|^{\beta}$. The estimate for $D^\gamma (f(y) - P(y, x))$ is the same because $D^\gamma f \in \Gamma_{\beta+1}$. Theorem (4.1) implies $F = \sum f_j$, where $f_j \in C^\infty$ and $|D^\gamma f_j| \leq C 2^{j(-\beta+1)}$. Let $f_\delta = \sum_{j > \delta} f_j$. The sum is finite, so $f_\delta \in C^\infty$. Denote the Taylor polynomial of $f_\delta$ of degree $k$ by $P_{(x, \delta)}(y)$. Nagel and Stein have shown

$$|f(y) - P_{(x, \delta)}(y)| \leq C \delta^\beta \quad \text{for} \quad |x^{-1}y| < \delta.$$ 

Hence it suffices to prove

$$|P(y, x) - P_{(x, \delta)}(y)| \leq C \delta^\beta \quad \text{for} \quad |x^{-1}y| < \delta.$$ 

This reduces to a comparison of the coefficients of the two polynomials. We must check that for $|\gamma| \leq k$,

$$|D^\gamma f(x) - D^\gamma f_\delta(x)| \leq C \delta^{\beta - |\gamma|}.$$ 

In fact,

$$|D^\gamma f(x) - D^\gamma f_\delta(x)| = C \sum_{2^j < \delta} |D^\gamma f_j(x)|$$

$$\leq \sum_{2^j < \delta} 2^{j(-\beta+1)} \leq C \delta^{\beta - |\gamma|},$$

since $\beta > k$. This proves the proposition.

We will now prove Theorem (4.3) that restr. $\Gamma_\beta |_M = \Gamma_\beta (M)$. We need only prove the theorem for non-integer values of $\beta$, because both collections of spaces are real interpolation scales. Choose $k$ so that $k < \beta < k + 1$. Begin with $f \in \Gamma_\beta |_M$. By definition, $f$ is the restriction of a function of $\Gamma_\beta$, which we will also denote by $f$.

Let $x \in M$. Let $P(y, x)$ be the Taylor polynomial of $f$ at $x$ in coordinates on $\|\cdot\|_n$:

$$P(y, x) = \sum_{|\gamma| < k} D^\gamma f(x)(y^{-1}x)^\gamma.$$ 

As a function of $y$, $P(y, x)$ has bounded $C^N$ norm for any $N$ (say, $N = 2k$) in a neighborhood of $x$ of size 1. (Indeed, the coefficients $D^\gamma f(x)$ are bounded.) Therefore, if $Q_\delta(y)$ is the Taylor polynomial of $P(y, x)$ in local coordinates
on $M$ we can assume that $|P(y, x) - Q_x(y)| < C\delta^N$ for $d(x, y) < \delta$. Now by the proposition, for $d(x, y) < \delta$,

$$|f(y) - Q_x(y)| \leq |P(y, x) - Q_x(y)| + |f(y) - P(y, x)|$$

$$\leq C\delta^N + C\delta^N \leq C\delta^N.$$

Thus restr. $\Gamma_\beta |_M \subset \Gamma_\beta(M)$. The opposite inclusion is an easy consequence of Lemmas (4.5) and (4.6).

**APPENDIX B**

Our goal is to calculate the group Fourier transform $F_\alpha(\beta, \tau, \zeta) = \mathcal{F}(f_\alpha)^{1}$ on $H^{m} \times \mathbb{R}$, for

$$f_\alpha(z, t, x) = (|z|^2 + x^2 - it)^{-(n+a)/2}(|z|^2 + x^2 + it)^{-(n-a)/2}$$

($m = n - 1$). Recall that $(z, t) \in C^m \times \mathbb{R}, \zeta \in \mathbb{R}^m, \tau \in \mathbb{R}, \xi \in \mathbb{R}$ are dual to $z, t, x$. We allow the possibility $m = 0$, in which case the group is $H^0 \times \mathbb{R} = \mathbb{R} \times \mathbb{R}$ with homogeneity 2 in $t$ and one in $x$. We can view $H^m \times \mathbb{R}$ as a hyperplane in $H^n$ with $z_\alpha = (x + iy)$. The dual variable $y$ is denoted $\eta$.

We begin with a formula of Greiner and Stein for the ordinary Fourier transform of the fundamental solution of $\mathcal{L}_\alpha$ on $H^n$.

Denote

$$f_\alpha(z, t, x) = (|z|^2 + |z_n|^2 - it)^{-(n+a)/2}(|z|^2 + |z_n|^2 + it)^{-(n-a)/2}.$$

Then ([11, p. 34]):

$$\hat{\Phi}_\alpha(\zeta, \tau, \xi, \eta) = C_{m,a} |\tau|^{-(n-a)sgn(\alpha)/2 - 1} \int C_{m,a} |\tau|^{-(n-a)sgn(\alpha)/2 - 1} \times (1 + s)^{-(n+\alpha)sgn(\alpha)/2 - 1} e^{-t(|\xi|^2 + |\xi + \eta|^2)/4} \tau_1 ds.$$

Also, $C_{m,a} \neq 0$ for $-n < \alpha < n$.

Notice that $f_\alpha$ is the restriction of $\Phi_\alpha$ to $H^m \times \mathbb{R}$.

$$f_\alpha(z, t, x) = \Phi_\alpha(z, x, t).$$

Thus,

$$\hat{f}_\alpha(\zeta, \tau, \xi) = \int_{-\infty}^{\infty} \hat{\Phi}_\alpha(\zeta, \tau, \xi, \eta) d\eta$$

$$= C_{m,a} \int_0^1 h(s) e^{-t|\xi|^2/4|\tau|} ds,$$

1 See Section 6, formulas (6.6) and (6.2).
where
\[ h(s) = |\tau|^{-1/2}(1 - s)^{(n - \alpha \text{ sgn } \tau)/2 - 1}(1 + s)^{(n + \alpha \text{ sgn } \tau)/2 - 1} \times e^{-\nu |\tau|^{4/\nu} s^{-1/2}}. \]

When \( m = 0 \), our calculation is complete. We have a formula for \( \hat{f}_\alpha(\tau, \xi) \) on \( \mathbb{R}^2 \). (Just omit the factor \( e^{-|\xi|^{2/4}|\tau|^{1/4}} \).)

For \( m \geq 1 \), formula (6.6) follows from

**Proposition.**

\[ F_{\alpha}(\beta, \tau, \xi) = c_{m, \alpha} \int_0^1 (1 - s)^{|\beta|}(1 + s)^{-m - |\beta|} h(s) \, ds. \]

**Proof.** We are using the convention

\[ \hat{g}(\xi) = \int_{\mathbb{R}^2m} e^{iz \cdot \xi} g(z) \, dz = F.T. g. \]

Recall that [18, p. 155, 4, Vol. 2, p. 42 (3); 4, Vol. 1, p. 174 (28)] if \( g(z) = g_0(|z|), \ z \in \mathbb{R}^2m, \) then

\[ \hat{g}(\xi) = c_m |\xi|^{-(m-1)} \int_0^{\infty} g_0(u) J_{m-1}(u |\xi|) u^m \, du \quad (1) \]

\[ \int_0^{\infty} x^m e^{-(1/2)x^2} L^m_{|\beta|} I_{m-1}(x^2) J_{m}(xy) \, dy = (-1)^{|\beta|} e^{-(1/2)y^2} y^m L^{m-1}_{|\beta|}(y^2) \quad (2) \]

\[ \int_0^{\infty} u^m L^m_{|\beta|} (x^2) e^{-|x| u} \, du = \frac{\Gamma(m + |\beta|)}{|\beta|!} \frac{(p - 1)^{|\beta|}}{p^{m + |\beta|}} \quad (3) \]

provided \( \text{Re } m > 0, \text{ Re } p > 0. \)

Applying (1), a change of variable, and (2),

\[ \text{F.T. } L^m_{|\beta|} (2 |\tau| |x|^2) e^{-|x| |\tau|^2} = C_m |\xi|^{-(m-1)} \int_0^{\infty} L^m_{|\beta|} (2 |\tau| u^2) e^{-|x| u} J_{m-1}(u |\xi|) u^m \, du \]

\[ = C_m |\xi|^{-(m-1)} \int_0^{\infty} L^m_{|\beta|} (x^2) e^{-(1/2)x^2} \]

\[ \times J_{m-1} \left( \frac{|\xi|}{\sqrt{2 |\tau|}} x \right) x^m \, dx \left| \frac{x}{|\tau|} \right|^{-(m+1)/2} \]

\[ = C_m (-1)^{|\beta|} \frac{e^{-|\xi|^{2/4}|\tau|^{1/4}}}{|\tau|^{-m} L^m_{|\beta|} (|\xi|^2/2 |\tau|)}. \]
Denote
\[ f^{(1)}_a(z, \xi, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ix_1 t} e^{it \tau} f(z, x, t) \, dx \, dt. \]

From (6.2),
\[ F_a(\beta, \xi, \tau) = C_{m, \alpha} a_{|\beta|}^{-1} \int_{0}^{\infty} f^{(1)}(z, \xi, \tau) L_{|\beta|}^{m-1}(2 |\tau| |r|^2) \]
\[ \times e^{-|\tau|^2 |r|^2} |r|^{2m-1} \, d|r|, \]
where \( r = (|z_1|, \ldots, |z_m|) \). This integral is the inner product on \( R^{2m} \) of radial functions \( f^{(1)}_a(z, \xi, \tau) \) and
\[ L_{|\beta|}^{m-1}(2 |\tau| |z|^2) e^{-|\tau|^2 |z|^2}. \]

By Plancherel's theorem with \( \omega = |\xi| \),
\[ F_a(\beta, \xi, \tau) = C_{m, \alpha} a_{|\beta|}^{-1} \int_{0}^{\infty} f_a(\xi, \tau, \xi) (-1)^{|\beta|} e^{-\omega^2/4 |\tau|} |\tau|^{-m} \]
\[ \times L_{|\beta|}^{m-1}(\omega^2/2 |\tau|) \omega^{2m-1} \, d\omega \]
\[ = C_{m, \alpha} a_{|\beta|}^{-1} \int_{0}^{\infty} \int_{0}^{\infty} (-1)^{|\beta|} e^{-\omega^2 + 1} \omega^{2m-1} \omega^{2m-1} \, d\omega \, h(s) \, ds \]
\[ = C_{m, \alpha} a_{|\beta|}^{-1} \int_{0}^{\infty} (-1)^{|\beta|} e^{-\omega^2} L_{|\beta|}^{m-1}(u) u^{m-1} \, du \, h(s) \, ds \]
\[ = C_{m, \alpha} a_{|\beta|}^{-1} \frac{(m + |\beta|)}{|\beta|!} (-1)^{|\beta|} \int_{0}^{1} (s - 1)^{\frac{|\beta|}{m + |\beta|}} h(s) \, ds \]
\[ = C_{m, \alpha} \int_{0}^{1} (1 - s)^{|\beta|}(1 + s)^{-m - |\beta|} h(s) \, ds, \]
because
\[ a_{|\beta|}^{-1} \frac{\Gamma(m + |\beta|)}{|\beta|!} = (m - 1)! \]
and (3).

The constant \( C_{m, \alpha} \) may change from line to line, but it depends only on \( m \) and \( \alpha \) and does not vanish for \(-n < \alpha < n\).
REFERENCES


