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Convergence and Gibbs' phenomenon in cubic spline interpolation of discontinuous functions

Zhimin Zhang*,[1], Clyde F. Martin[2]

Department of Mathematics, Texas Tech University, Lubbock, Texas 79409-1042, United States

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Abstract

Convergence of cubic spline interpolation for discontinuous functions are investigated. It is shown that the complete cubic spline interpolation of the Heaviside step function converges in the L^p-norm at rate $O(h^{1/p})$ for quasi-uniform meshes when $1 \leqslant p < \infty$, and diverges in the L^∞-norm when the uniform meshes are used. No matter how small the uniform mesh size is, the complete cubic spline interpolation always oscillates near the discontinuity. Although this oscillation decays exponentially away from the discontinuous point, the maximum overshoot is not decreasing. Especially, we obtain the asymptotic maximum overshoot when the uniform mesh size goes to zero. The knowledge on the Heaviside function is utilized to discuss convergence properties of cubic spline interpolation for functions with isolated discontinuous points.

Keywords: Cubic spline; Gibbs' phenomenon; Interpolation; Convergence; p-Norms

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1. Introduction

Convergence of spline interpolation for smooth functions has been investigated intensively in the literature, see, e.g., [2, 4, 6], and references therein. However, we know very little about approximation properties of spline interpolation for functions with discontinuity. It has been observed from numerical computation that the complete cubic spline interpolation oscillates near a discontinuous point and has an overshoot when uniform meshes are used [7, p. 122]. Since this behavior is similar to Gibbs' phenomenon in Fourier's series (see, e.g., [3]), it is called Gibbs' phenomenon of splines.

* Corresponding author.

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The periodic spline with equal knots spacing that best approximates the square wave function in the norm of $L^2[-1,1]$ was investigated in [5]. The overshoot at the discontinuity was found for splines of degree $k \leqslant 8$ when the knots number goes to infinity.

Aforementioned periodic spline does not really "interpolate" the function at "knots", rather, it approximates the function in a "Fourier" sense. A more practically interesting problem would be a spline that interpolates the function at knots, the complete spline interpolation. In the current work, we shall analyze convergence of the complete cubic spline interpolation in the L^p-norm $(1 \leqslant p < \infty)$ for the Heaviside step function. An optimal convergence rate of $O(h^{1/p})$ is established under quasi-uniform meshes. Moreover, we provide an explanation for the Gibbs' phenomenon under uniform meshes. The asymptotic overshoot when the mesh size goes to zero equals $(\sqrt{6}+\sqrt{3})/4 - 5(\sqrt{2}+1)/12 \approx 4\%$. It differs from that of the periodic cubic spline obtained in [5] which is about 10%. The results obtained for Heaviside function is used to study the behavior of cubic spline interpolation for functions with isolated discontinuous points. Furthermore, we indicate that there is no oscillation and overshoot in the B-spline interpolation.

2. The complete cubic spline interpolation

The construction of the complete cubic spline interpolation can be found in many standard numerical analysis textbooks (cf., e.g., [7]). For the convenience of our analysis, we outline an approach based on the Hermite interpolation.

Given interpolating points $a = t_0 < t_1 < \cdots < t_n = b$, and data $f_i = f(t_i)$, $\beta_0 = f'(a)$, $\beta_n = f'(b)$, we want to construct a cubic spline $s_n \in C^2[a,b]$ such that on each subinterval $[t_{i-1}, t_i]$,

$$s_n(t) = f_{i-1}p_i(t) + f_i q_i(t) + \beta_{i-1}u_i(t) + \beta_i v_i(t), \tag{2.1}$$

with

$$p_i(t) = \frac{(t-t_i)^2}{h_i^3}[h_i + 2(t-t_{i-1})], \qquad q_i(t) = \frac{(t-t_{i-1})^2}{h_i^3}[h_i - 2(t-t_i)],$$
$$u_i(t) = \frac{(t-t_i)^2(t-t_{i-1})}{h_i^2}, \qquad v_i(t) = \frac{(t-t_{i-1})^2(t-t_i)}{h_i^2},$$

where $h_i = t_i - t_{i-1}$, and β_i, $i = 1,\ldots,n-1$ are parameters to be decided. It is easy to verify that

$$s_n(t_j) = f_j, \quad s_n'(t_j) = \beta_j, \quad j = 0,1,\ldots,n,$$

and hence $s_n \in C^1[a,b]$. In order that $s_n \in C^2[a,b]$, we enforce the continuity condition $s_n''(t_i - 0) = s_n''(t_i + 0)$. As a consequence, we have the following system of linear equations for β_i, $i = 1,\ldots,n-1$:

$$\frac{1}{h_i}\beta_{i-1} + 2\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)\beta_i + \frac{1}{h_{i+1}}\beta_{i+1} = 3f_i\left(\frac{1}{h_i^2} - \frac{1}{h_{i+1}^2}\right) + 3\left(\frac{f_{i+1}}{h_{i+1}^2} - \frac{f_{i-1}}{h_i^2}\right), \tag{2.2}$$

or $A\boldsymbol{\beta} = \boldsymbol{f}$ with $A = D + B$, where $D = 2\,\mathrm{diag}(1/h_i + 1/h_{i+1})_{i=1}^{n-1}$, and B is a symmetric tridiagonal matrix with $b_{ii} = 0$, $b_{i+1,i} = 1/h_{i+1}$. Obviously, A is diagonally dominant. Hence, $\boldsymbol{\beta}$ can be uniquely solved. Once $\boldsymbol{\beta}$ is decided, the cubic spline can be constructed from (2.1). We have the following theorem regarding the norm of A^{-1} which will be used later in the convergence analysis.

Theorem 2.1. *For* $1 \leqslant p < \infty$,

$$\|A^{-1}\|_p \leqslant \max_{1 \leqslant i \leqslant n-1} \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right)^{-1}.$$

Proof. It is easy to verify that all rows of $D^{-1}B$ add up to $\frac{1}{2}$ except the first and the last (which are less than $\frac{1}{2}$); and all columns of BD^{-1} add up to $\frac{1}{2}$ except the first and the last (which are less than $\frac{1}{2}$). Therefore, $\|D^{-1}B\|_\infty = \frac{1}{2}$ and $\|BD^{-1}\|_1 = \frac{1}{2}$. Now $A = D + B = (I + BD^{-1})D$, $A^{-1} = D^{-1}(I + BD^{-1})^{-1}$, and hence

$$\|A^{-1}\|_1 \leqslant \|D^{-1}\|_1 \|(I + BD^{-1})^{-1}\|_1 \leqslant \frac{1}{2} \max_{1 \leqslant i \leqslant n-1} \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right)^{-1} \frac{1}{1 - \|BD^{-1}\|_1}$$

$$= \max_{1 \leqslant i \leqslant n-1} \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right)^{-1}.$$

Similarly, we can verify that

$$\|A^{-1}\|_\infty \leqslant \max_{1 \leqslant i \leqslant n-1} \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right)^{-1}.$$

Therefore, by the Riesz–Thorin interpolation theorem [1, p. 2], for $0 < \theta < 1$,

$$\|A^{-1}\|_p \leqslant \|A^{-1}\|_1^{1-\theta} \|A^{-1}\|_\infty^{\theta} \leqslant \max_{1 \leqslant i \leqslant n-1} \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right)^{-1},$$

where $1/p = 1 - \theta$, or $p = 1/(1 - \theta)$. This complete the proof. □

3. L^p convergence

We interpolate the step function

$$f(t) = \begin{cases} 0, & -1 \leqslant t < 0, \\ \frac{1}{2}, & t = 0, \\ 1, & 0 < t \leqslant 1, \end{cases}$$

by the complete cubic spline and discuss convergence in the L^p-norm ($1 \leqslant p < \infty$). For simplicity, we use $t_m = 0$ as an interpolating point and assume that the derivative is interpolated exactly at the boundary, i.e., $\beta_0 = 0$, $\beta_n = 0$.

Since s_n interpolates f at all nodal points, we have on $(t_0, t_m) = (-1, 0)$:

$$s_n(t) - f(t) = \beta_{k-1} u_k(t) + \beta_k v_k(t), \quad t_{k-1} \leqslant t \leqslant t_k, \quad k = 1, \ldots, m-1,$$

$$s_n(t) - f(t) = \tfrac{1}{2} q_m(t) + \beta_{m-1} u_m(t) + \beta_m v_m(t), \quad t_{m-1} \leqslant t < t_m.$$

Therefore,

$$\int_{-1}^{0} |s_n(t) - f(t)|^p \,\mathrm{d}t = \sum_{k=1}^{m-1} \int_{t_{k-1}}^{t_k} |\beta_{k-1}u_k(t) + \beta_k v_k(t)|^p \,\mathrm{d}s + \int_{t_{m-1}}^{t_m} |\tfrac{1}{2}q_m(t) + \beta_{m-1}u_m(t) + \beta_m v_m(t)|^p \,\mathrm{d}t. \tag{3.1}$$

Introduce a change of variable $t - t_{k-1} = h_k s$, $k = 1, \ldots, m$, and we have

$$u_k(t) = h_k u(s), \quad v_k(t) = h_k v(s), \quad q_m(t) = q(s),$$

where u, v, q satisfies

$$u(0) = u(1) = v(0) = v(1) = 0, \quad u'(1) = v'(0) = 0, \quad u'(0) = 1, \quad v'(1) = 1,$$

$$q(0) = 0, \quad q(1) = 1, \quad q'(0) = q'(1) = 0.$$

It is easy to verify that

$$\|u\|_{p,I} = \|v\|_{p,I}, \quad I = (0,1).$$

Now, we have

$$\sum_{k=1}^{m-1} \int_{t_{k-1}}^{t_k} |\beta_{k-1}u_k(t) + \beta_k v_k(t)|^p \,\mathrm{d}t = \sum_{k=1}^{m-1} h_k^{1+p} \int_0^1 |\beta_{k-1}u(s) + \beta_k v(s)|^p \,\mathrm{d}s \leqslant 2^{p-1}\|u\|_{p,I}^p \sum_{k=1}^{m-1} h_k^{1+p}(|\beta_{k-1}|^p + |\beta_k|^p). \tag{3.2}$$

Here, we have applied an inequality

$$\|u + v\|_p^p \leqslant 2^{p-1}(\|u\|_p^p + \|v\|_p^p), \quad 1 \leqslant p < \infty.$$

We have also,

$$\int_{t_{m-1}}^{t_m} |\tfrac{1}{2}q_m(t) + \beta_{m-1}u_m(t) + \beta_m v_m(t)|^p \,\mathrm{d}t = h_m \int_0^1 |\tfrac{1}{2}q(s) + \beta_{m-1}h_m u(s) + \beta_m h_m v(s)|^p \,\mathrm{d}s \leqslant h_m 3^{p-1}[2^{-p}\|q\|_{p,I}^p + h_m^p(|\beta_{m-1}|^p + |\beta_m|^p)\|u\|_{p,I}^p]. \tag{3.3}$$

In the last step, the inequality

$$\|u + v + w\|_p^p \leqslant 3^{p-1}(\|u\|_p^p + \|v\|_p^p + \|w\|_p^p), \quad 1 \leqslant p < \infty,$$

is used. Combining (3.1)–(3.3), we obtain

$$\int_{-1}^{0} |s_n(t) - f(t)|^p \,\mathrm{d}t \leqslant 3^{p-1}\|u\|_{p,I}^p \sum_{k=1}^{m} h_k^{1+p}(|\beta_{k-1}|^p + |\beta_k|^p) + \frac{h_m^{1+p}}{3}\left(\frac{3}{2}\right)^p \|q\|_{p,I}^p.$$

Similarly, we can estimate the error in $(0,1)$. Denote $h=\max_{1\leqslant i\leqslant n} h_i$, and we have established:

Theorem 3.1. *$\|s_n - f\|_p \to 0$ with rate $h^{1/p}$ if $(\sum_{k=1}^{n} |h_k\beta_{k-1}|^p + |h_k\beta_k|^p)^{1/p}$ is bounded uniformly with respect to n, $1\leqslant p<\infty$.*

A sequence of meshes are called quasi-uniform if there exists $\sigma>0$ independent of n, such that

$$\frac{\max_i h_i}{\min_j h_j} \leqslant \sigma. \tag{3.4}$$

Theorem 3.2. *Assume that the mesh is quasi-uniform. Then $\|s_n - f\|_p \to 0$ with rate $h^{1/p}$ for $1\leqslant p<\infty$.*

Proof. A sufficient condition for Theorem 3.1 to hold is $h\|\boldsymbol{\beta}\|_p \leqslant C$, where C is a constant independent of h. We need to estimate

$$\|\boldsymbol{\beta}\|_p \leqslant \|A^{-1}\|_p \|\boldsymbol{f}\|_p.$$

We have an upper bound for $\|A^{-1}\|_p$ by Theorem 2.1. From (2.2), it is easy to verify that

$$\boldsymbol{f} = \left(0,\ldots,0,\frac{3}{2h_m^2},\frac{3}{2}\left(\frac{1}{h_m^2}+\frac{1}{h_{m+1}^2}\right),\frac{3}{2h_{m+1}^2},0,\ldots,0\right)^{\mathrm{T}},$$

$$\|\boldsymbol{f}\|_p = \frac{3}{2}\left[\frac{1}{h_m^{2p}} + \left(\frac{1}{h_m^2}+\frac{1}{h_{m+1}^2}\right)^p + \frac{1}{h_{m+1}^{2p}}\right]^{1/p}$$

$$\leqslant \frac{3}{2}\left[2\left(\frac{1}{h_m^2}+\frac{1}{h_{m+1}^2}\right)^p\right]^{1/p} = \frac{3}{2}2^{1/p}\left(\frac{1}{h_m^2}+\frac{1}{h_{m+1}^2}\right).$$

Therefore,

$$\|\boldsymbol{\beta}\|_p \leqslant \frac{3}{2}2^{1/p}\max_i\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)^{-1}\left(\frac{1}{h_m^2}+\frac{1}{h_{m+1}^2}\right).$$

Applying the quasi-uniform condition (3.4), we then have

$$h\|\boldsymbol{\beta}\|_p \leqslant \tfrac{3}{2}2^{1/p}\sigma^2.$$

The assertion is proved by applying Theorem 3.1. □

Remark 3.1. Theorem 3.1 does not include the case $p=\infty$. In fact, the complete cubic spline interpolation does not converge to f in the L^∞-norm under the uniform mesh. See details in the next section.

Remark 3.2. The convergence rate $\mathrm{O}(h^{1/p})$ is optimal. Indeed, in the case of uniform mesh, we will establish a lower bound for the approximation error in the next section.

4. Gibbs' phenomenon

In this section, we shall study the Gibbs' phenomenon of the complete cubic spline interpolation for the step function f when the equidistance nodes are used.

Set $n=2m$ in (2.2), then $h=1/m$, $t_m=0$. Note that β_k values are symmetric with respect to β_m, i.e., $\beta_{m+k}=\beta_{m-k}$, $k=1,\dots,m$. Therefore, we only need to consider half of them by examining the following system of equations:

$$4\beta_1+\beta_2=0, \tag{4.1}$$

$$\beta_{k-1}+4\beta_k+\beta_{k+1}=0, \quad k=2,\dots,m-2, \tag{4.2}$$

$$\beta_{m-2}+4\beta_{m-1}+\beta_m=\frac{3}{2h}, \tag{4.3}$$

$$\beta_{m-1}+4\beta_m+\beta_{m-1}=\frac{3}{h} \quad (\beta_{m+1}=\beta_{m-1}). \tag{4.4}$$

Theorem 4.1. *β_k's have the following properties*:

(a) *Alternating sign*:

$$\beta_{m-k}(-1)^k<0, \quad k=1,\dots,m-1. \tag{4.5}$$

(b) *Exponential decay*:

$$\tfrac{1}{4}|\beta_{k+1}|<|\beta_k|<(2-\sqrt{3})|\beta_{k+1}|, \quad k=2,\dots,m-2. \tag{4.6}$$

(c) *Upper and Lower bounds*:

$$\frac{3-\sqrt{3}}{2}<\beta_m h<\frac{33}{52}, \tag{4.7}$$

$$\frac{3}{13}<\beta_{m-1}h<\sqrt{3}-\frac{3}{2}. \tag{4.8}$$

(d) *Asymptotical behavior*:

$$\lim_{m\to\infty}\beta_m h=\frac{3-\sqrt{3}}{2}, \quad \lim_{m\to\infty}\beta_{m-1}h=\sqrt{3}-\frac{3}{2}, \quad \lim_{m\to\infty}\frac{\beta_{m-2}}{\beta_{m-1}}=\sqrt{3}-2. \tag{4.9}$$

Proof. We first show that $\beta_1\neq 0$. Suppose $\beta_1=0$, then $\beta_2=0$ from (4.1), and consequently $\beta_3=0,\dots,\beta_{m-1}=0$ from (4.2). Therefore, $\beta_m=3/2h$ by (4.3), and $\beta_m=3/4h$ by (4.4). This is a contradiction.

It is easy to see that $\beta_1\neq 0$ yields $\beta_1\beta_2<0$ and $|\beta_2|=4|\beta_1|$ from (4.1). We then deduce from $\beta_1+4\beta_2+\beta_3=0$ that $\beta_2\beta_3<0$, and

$$|4\beta_2|=|-\beta_3-\beta_1|=|\beta_3|+|\beta_1|=|\beta_3|+\tfrac{1}{4}|\beta_2|\begin{cases}<|\beta_3|+(2-\sqrt{3})|\beta_2|,\\ >|\beta_3|,\end{cases}$$

which yields

$$\frac{1}{4}|\beta_3| < |\beta_2| < \frac{1}{2+\sqrt{3}}|\beta_3| = (2-\sqrt{3})|\beta_3|.$$

By mathematical induction, we can deduce for $k = 1, \ldots, m-2$ that $\beta_k\beta_{k+1} < 0$, and

$$|4\beta_k| = |\beta_{k+1}| + |\beta_{k-1}| \begin{cases} < |\beta_{k+1}| + (2-\sqrt{3})|\beta_k|, \\ > |\beta_{k+1}|, \end{cases}$$

which yields (4.6).

Now we estimate β_{m-1} and β_m. Solving (4.3) and (4.4) yields

$$\beta_{m-1}h = \tfrac{3}{14} - \tfrac{2}{7}\beta_{m-2}h, \tag{4.10}$$

$$\beta_m h = \tfrac{9}{14} + \tfrac{1}{7}\beta_{m-2}h. \tag{4.11}$$

Recall $\beta_{m-2}\beta_{m-1} < 0$ and $|\beta_{m-1}|/4 < |\beta_{m-2}| < (2-\sqrt{3})|\beta_{m-1}|$, we see from (4.10) that $\beta_{m-1} > 0$, $\beta_{m-2} < 0$, and furthermore,

$$\frac{3}{14h} = \beta_{m-1} + \frac{2}{7}\beta_{m-2} = \beta_{m-1} - \frac{2}{7}|\beta_{m-2}| \begin{cases} > \beta_{m-1} - \frac{2}{7}(2-\sqrt{3})\beta_{m-1} = \frac{3+2\sqrt{3}}{7}\beta_{m-1}, \\ < \beta_{m-1} - \frac{2}{7}\frac{1}{4}\beta_{m-1} = \frac{13}{14}\beta_{m-1}, \end{cases} \tag{4.12}$$

which yields (4.8). Using (4.4) and (4.8), we have

$$\beta_m h = \frac{3}{4} - \frac{\beta_{m-1}h}{2} \begin{cases} > \frac{3}{4} - \frac{\sqrt{3}}{2} + \frac{3}{4} = \frac{3-\sqrt{3}}{2}, \\ < \frac{3}{4} - \frac{3}{26} = \frac{33}{52}. \end{cases}$$

This proves (4.7). (4.5) is a direct consequence of $\beta_m > 0$, $\beta_{m-1} > 0$ and $\beta_{k-1}\beta_k < 0$, $k = 2, \ldots, m-1$.

For the asymptotic limits (4.9c), we observe that $\sqrt{3} - 2$ is the root with modular less than 1 of the difference equation

$$\beta_{k-1} + 4\beta_k + \beta_{k+1} = 0.$$

Further, we can show that $\beta_m h$ is a monotonically decreasing sequence with a lower bound, and $\beta_{m-1}h$ is a monotonically increasing sequence with an upper bound. Hence, they both have limits, and so does $\beta_{m-2}h$. We denote these limits as β_i^*, $i = m-2,\ m-1,\ m$. Taking limit in (4.10) and (4.11), using the relation

$$\frac{\beta_{m-2}^*}{\beta_{m-1}^*} = \sqrt{3} - 2,$$

we then have

$$\beta_{m-1}^* = \tfrac{3}{14} - \tfrac{2}{7}(\sqrt{3}-2)\beta_{m-1}^*, \tag{4.13}$$

$$\beta_m^* = \tfrac{9}{14} + \tfrac{1}{7}(\sqrt{3}-2)\beta_{m-1}^*. \tag{4.14}$$

Solving (4.13) and (4.14) yields (4.9a) and (4.9b). □

For the complete cubic spline interpolation with uniform mesh

$$u(s)=s(1-s)^2, \quad v(s)=-s^2(1-s), \quad q(s)=s^2(3-2s).$$

Since signs of β_k's alternate, we have for each $k=1,2,\ldots,m-1$,

$$\max_{t_{k-1}\leqslant t\leqslant t_k}|s_n(t)-f(t)|=\max_{t_{k-1}\leqslant t\leqslant t_k}|\beta_{k-1}u_k(t)+\beta_k v_k(t)|=|\beta_k|h\max_{0\leqslant s\leqslant 1}\left(-\frac{\beta_{k-1}}{\beta_k}u(s)-v(s)\right). \tag{4.15}$$

From (4.6), we see that

$$4^{-m+1+k}\beta_{m-1}<|\beta_k|<(2-\sqrt{3})^{m-1-k}\beta_{m-1}, \quad \frac{1}{4}\leqslant-\frac{\beta_{k-1}}{\beta_k}<2-\sqrt{3},$$

for $k=1,\ldots,m-1$. Let

$$g(s)=\tfrac{1}{4}s(1-s)^2+s^2(1-s), \qquad G(s)=(2-\sqrt{3})s(1-s)^2+s^2(1-s),$$

then a simple calculation shows that

$$\max_{0\leqslant s\leqslant 1} g(s)=g\left(\frac{\sqrt{13}+2}{9}\right)>g\left(\frac{1}{2}\right)=\frac{5}{32},$$

$$\max_{0\leqslant s\leqslant 1} G(s)=G\left(\frac{3-\sqrt{3}+\sqrt{6}}{6}\right)=\frac{\sqrt{2}+1}{6}-\frac{\sqrt{6}+\sqrt{3}}{18}.$$

Hence, for $k=1,\ldots,m-1$,

$$\frac{5}{32}<-\frac{\beta_{k-1}}{\beta_k}u(s)-v(s)<\frac{\sqrt{2}+1}{6}-\frac{\sqrt{6}+\sqrt{3}}{18}. \tag{4.16}$$

The estimate of $|\beta_k h|$ comes from (4.6) and (4.8):

$$4^{-m+1+k}\tfrac{3}{13}<|\beta_k h|<(2-\sqrt{3})^{m-1-k}(\sqrt{3}-\tfrac{3}{2}). \tag{4.17}$$

Substituting (4.16) and (4.17) into (4.15), we have for $k=1,\ldots,m-1$,

$$\begin{aligned}\max_{t_{k-1}\leqslant t\leqslant t_k}|s_n(t)-f(t)| &< (2-\sqrt{3})^{m-1-k}\left(\sqrt{3}-\frac{3}{2}\right)\left(\frac{\sqrt{2}+1}{6}-\frac{\sqrt{6}+\sqrt{3}}{18}\right)\\ &=(2-\sqrt{3})^{m-1-k}\left(\frac{\sqrt{6}+\sqrt{3}}{4}-\frac{5(\sqrt{2}+1)}{12}\right)\\ &<(2-\sqrt{3})^{m-1-k}0.0395,\end{aligned} \tag{4.18}$$

$$\max_{t_{k-1}\leqslant t\leqslant t_k}|s_n(t)-f(t)| > 4^{-m+1+k}\tfrac{3}{13}\tfrac{5}{32}>4^{-m+1+k}0.036. \tag{4.19}$$

Setting $k=m-1$ in (4.18) and (4.19), we have

$$0.036<\max_{t_{m-2}\leqslant t\leqslant t_{m-1}}|s_n(t)-f(t)|<0.0395. \tag{4.20}$$

Furthermore, we know asymptotically,

$$\lim_{m\to\infty}\frac{\beta_{m-2}}{\beta_{m-1}}=\sqrt{3}-2,\quad \lim_{m\to\infty}\beta_{m-1}h=\sqrt{3}-\frac{3}{2}. \tag{4.21}$$

Therefore,

$$\begin{aligned}\lim_{m\to\infty}\max_{-1\leqslant t\leqslant t_{m-1}}|s_n(t)-f(t)| &= \lim_{m\to\infty}\max_{t_{m-2}\leqslant t\leqslant t_{m-1}}|s_n(t)-f(t)|\\ &=\lim_{m\to\infty}\beta_{m-1}h\max_{0\leqslant s\leqslant 1}\left(-\frac{\beta_{m-2}}{\beta_{m-1}}u(s)-v(s)\right)\\ &=\frac{\sqrt{6}+\sqrt{3}}{4}-\frac{5(\sqrt{2}+1)}{12}\approx 0.0394628199\ldots .\end{aligned} \tag{4.22}$$

We see that asymptotically, the maximum overshoot is about 4%.

In case of uniform meshes, we have a more accurate error estimate in the L^p-norm. First, it is easy to verify that

$$\begin{aligned}\tfrac{1}{2}q(s)+\beta_{m-1}hu(s)+\beta_m hv(s) &= (\tfrac{3}{2}-s)s^2+\beta_{m-1}hs(1-s^2)-\beta_m hs^2(1-s)\\ &= s^2[\tfrac{1}{2}+(1-s)(1-\beta_m h)]+\beta_{m-1}hs(1-s)^2\begin{cases}\leqslant 1/2,\\ \geqslant s^2/2.\end{cases}\end{aligned}$$

Hence,

$$\int_{t_{m-1}}^{t_m}|s_n(t)-f(t)|^p\,\mathrm{d}t = h\int_0^1|\tfrac{1}{2}q(s)+\beta_{m-1}hu(s)+\beta_m hv(s)|^p\,\mathrm{d}s\begin{cases}\leqslant h2^{-p},\\ \geqslant h2^{-p}(2p+1)^{-1}.\end{cases} \tag{4.23}$$

On the other hand, from (4.18), we have

$$\sum_{k=1}^{m-1}\int_{t_{k-1}}^{t_k}|s_n(t)-f(t)|^p\,\mathrm{d}t<h\sum_{k=1}^{m-1}(2-\sqrt{3})^{(m-1-k)p}0.0395^p<h2^{-p}. \tag{4.24}$$

Adding up (4.23) and (4.24) yields,

$$\frac{h}{2^p(2p+1)}<\int_{-1}^{0}|s_n(t)-f(t)|^p\,\mathrm{d}t<\frac{2h}{2^p},$$

or

$$\frac{h^{1/p}}{2(2p+1)^{1/p}}<\|s_n-f\|_p<\frac{h^{1/p}}{2^{1-1/p}}. \tag{4.25}$$

Setting $p\to\infty$ in (4.25), we will have

$$\|s_n-f\|_\infty=\tfrac{1}{2},$$

which is precisely the error in the maximum norm (it appears in the subinterval (t_{m-1},t_m)).

Summing up, we have proved the following theorem.

Theorem 4.2. *When uniform meshes are used, the complete cubic spline interpolation converges to the step function f in the L^p-norm* ($1 \leqslant p < \infty$) *with an optimal rate* $O(h^{1/p})$; *it diverges in the L^∞-norm and oscillates near the discontinuous point with a maximum overshoot estimated by* (4.20). *In the limit $h \to 0$, this overshoot is given by* (4.22). *Moreover, the oscillation decays exponentially away from the discontinuity in a pattern estimated by* (4.18) *and* (4.19).

Remark 4.1. The discussion for the cubic spline gives us some insights for other polynomial splines. Indeed, the numerical tests indicates that the complete quintic spline interpolation for the step function f behaves very much like the cubic spline.

Remark 4.2. The Gibbs' phenomenon occurs for many other complete spline interpolation such as classical exponential splines. But every different spline may have a different overshoot value.

We plot the complete cubic spline interpolation for the step function f with uniform meshes when $n = 10, 20, 40, 80$ in Fig. 1. It clearly indicates a 4% overshoot.

5. Convergence for functions with isolated discontinuities

In this section, we discuss spline interpolation for functions with isolated discontinuous points. Let F be such a function, then it can be expressed as

$$F(t) = g(t) + \sum_i c_i f(t - \bar{t}_i), \tag{5.1}$$

where $g \in C[a,b]$ and f is the step function defined in Section 3. Clearly, c_i is the jump at the discontinuous point $\bar{t}_i$. Here, we take the liberty to define the function value at the discontinuity as the average of the limits from two sides. For simplicity, we consider only one discontinuous point which is located at the center of the interval: $F(t) = g(t) + cf(t)$. Again, the interpolating interval is assumed to be $[-1, 1]$, since an arbitrary interval $[a, b]$ can be transfered to it by a linear mapping.

Recall the construction of the complete cubic spline interpolation, parameters β_i, $1 \leqslant i \leqslant n-1$, can be solved uniquely from β_0, β_n, and f_j, $0 \leqslant i \leqslant n$. Therefore, the spline interpolation (2.1) can be written symbolically as

$$s_n(t) = \beta_0 \psi_0(t) + \beta_n \psi_n(t) + \sum_{i=0}^{n} f_i \phi_i(t),$$

where ψ_0, ψ_n, and ϕ_i are some piecewise cubic polynomials. Denote by s_f, the cubic spline interpolation of f, we then have $s_F = s_g + cs_f$ where

$$s_g(t) = \beta_0 \psi_0(t) + \beta_n \psi_n(t) + \sum_{i=0}^{n} g_i \phi_i(t), \qquad s_f(t) = \sum_{i=0}^{n} f_i \phi_i(t).$$

Recall the interpolation property of the complete cubic spline (cf. [2, p. 61, Problem 7(d)]):

$$\|g - s_g\|_\infty \leqslant \tfrac{5}{2} h \|g'\|_\infty,$$

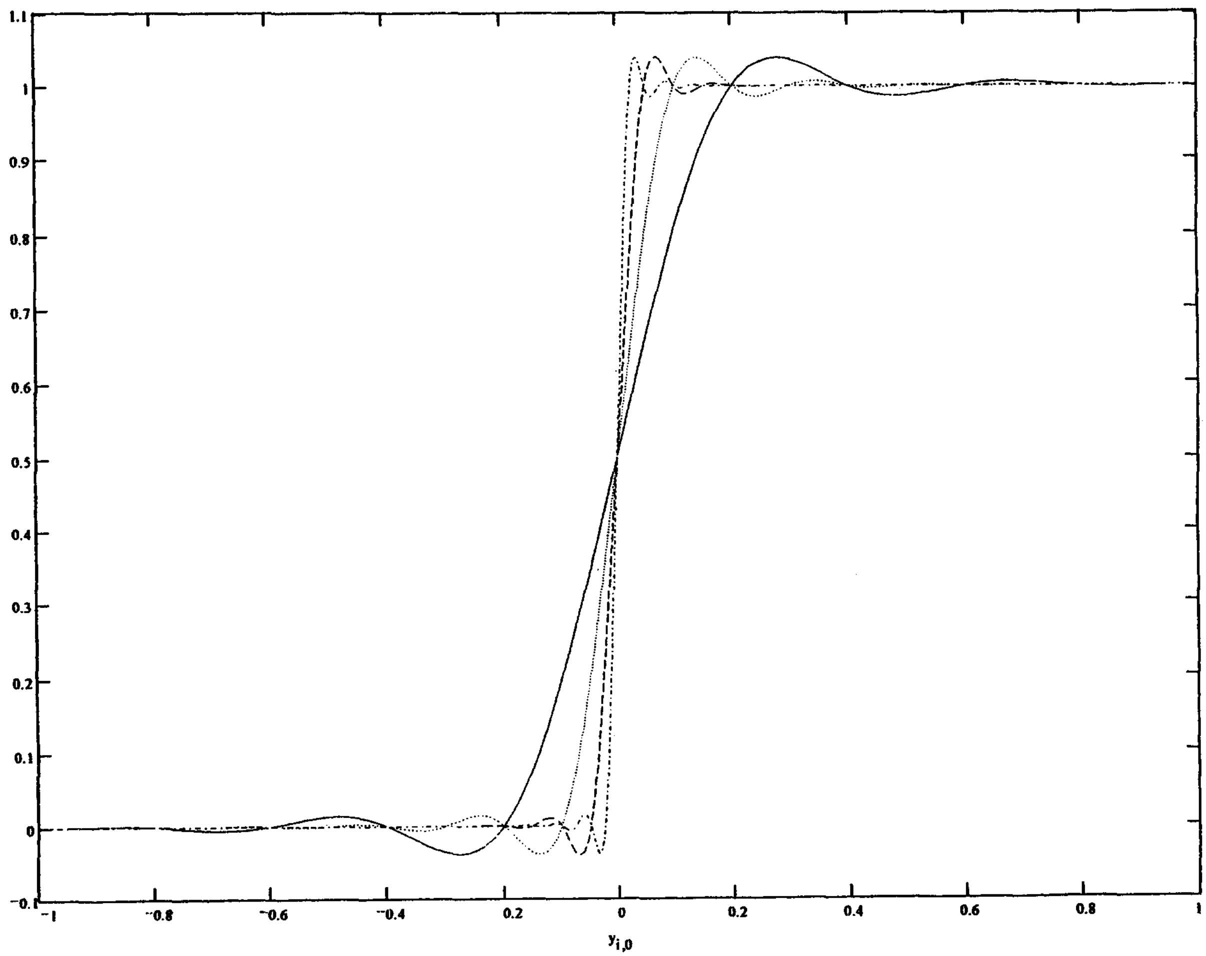

Fig. 1.

for $g \in C^1[-1,1]$. We then have

$$\|F - s_F\|_\infty \leqslant \tfrac{5}{2}h\|g'\|_\infty + c\|f - s_f\|_\infty.$$

The analysis of the last term on the right-hand side was discussed in previous sections. Hence, we conclude: *For a function with isolated discontinuity, its complete cubic spline interpolation oscillates near the discontinuous points with a maximum overshoot about* 4% *in the limit* $h \to 0$; *in the region away from the discontinuity, the oscillation decays exponentially and the standard interpolation error estimate applies.*

It is interesting to know that the B-spline interpolation does not oscillate when the function "jumps". We provide a brief explanation in the following. For simplicity, we again consider F has one "jump" only. We make the discontinuous point as a nodal point t_k and assume that the nodes are equally spaced. We denote by N_j, the normalized B-spline that centered at the node t_j, and by

B_F, the B-spline interpolation of F. Notice that $\sum_j N_j = 1$, then

$$F(t) - B_F(t) = \sum_j [g(t) - g(t_j)]N_j(t) + c\sum_j [f(t-t_k) - f(t_j - t_k)]N_j(t).$$

By the standard theory (cf., [4, p. 159]),

$$|g(t) - B_g(t)| = \left|\sum_j [g(t) - g(t_j)]N_j(t)\right| \leqslant C\omega(g;h),$$

where $\omega(g;h)$ is the modulus of continuity of g, and C is a constant independent of g and h. We need to examine the B-spline interpolation for the step function. To fix the idea, we use the cubic B-spline as a model in which case $N_i(t)$ has a support (t_{i-2}, t_{i+2}). Note that f has only three different values, therefore,

$$\begin{aligned} f(t) - B_f(t) &= \sum_j [f(t-t_k) - f(t_j - t_k)]N_j(t) \\ &= [f(t-t_k) - f(t_{k-1} - t_k)]N_{k-1}(t) + [f(t-t_k) - f(t_k - t_k)]N_k(t) \\ &\quad +[f(t-t_k) - f(t_{k+1} - t_k)]N_{k+1}(t) \\ &= f(t-t_k)N_{k-1}(t) + [f(t-t_k) - \tfrac{1}{2}]N_k(t) + [f(t-t_k) - 1]N_{k+1}(t) \\ &= \begin{cases} -N_k(t)/2 - N_{k+1}(t), & t_{k-2} < t < t_k, \\ N_{k-1}(t) + N_k(t)/2, & t_k < t < t_{k+2}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Note that

$$N_j(t_j) = \tfrac{2}{3}, \quad N_j(t_{j-1}) = N_j(t_{j+1}) = \tfrac{1}{6}, \quad N_j(t_i) = 0 \quad \text{if } |i-j| > 1,$$

we then have

$$\lim_{t \to t_k + 0} [f(t) - B_f(t)] = N_{k-1}(t_k) + N_k(t_k)/2 = \tfrac{1}{2},$$

$$\lim_{t \to t_k - 0} [f(t) - B_f(t)] = -N_k(t_k) - N_{k+1}(t_k)/2 = -\tfrac{1}{2},$$

$$f(t_{k+1}) - B_f(t_{k+1}) = N_{k-1}(t_{k+1}) + N_k(t_{k+1})/2 = \tfrac{1}{12},$$

$$f(t_{k-1}) - B_f(t_{k-1}) = -N_k(t_{k-1})/2 - N_{k+1}(t_{k-1}) = -\tfrac{1}{12}.$$

We see that there is no oscillation and overshoot. $B_f(t)$ equals $f(t)$ on most part of the domain except on a small subinterval of length $4h$ that centered at the discontinuous point t_k. In this small subinterval, $B_f(t)$ approximates $f(t)$ smoothly, its value increases monotonically from 0 to 1, and the graph passes through $(t_{k-1}, \frac{1}{12})$, $(t_k, \frac{1}{2})$, and $(t_{k+1}, 1 - \frac{1}{12})$.

B-splines of order other than three can be analyzed similarly.

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