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Law of large numbers for a general system of stochastic differential equations with global interaction

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A model for the activities of N agents in an economy is presented as the solution to a system of stochastic differential equations with stochastic coefficients, driven by general semimartingales and displaying weak global interaction. We demonstrate a law of large numbers for the empirical measures belonging to the systems of processes as the number of agents goes to infinity under a weak convergence hypothesis on the triangular array of starting values, coefficients and driving semimartingales which induces the systems of equations. Further it is shown that the limit can be uniquely characterized as the weak solution to a further (nonlinear) stochastic differential equation.

interacting stochastic processes * empirical distributions * law of large numbers * propagation of chaos

Introduction

The model

In this paper we will be investigating laws of large numbers for systems of stochastic processes with weak global interaction. The type of law of large numbers to be considered is formalized in a concept we refer to as point convergence. Let $(\Omega, \mathscr{A}, \boldsymbol{P})$ be a probability space and E a topological space. For every $N \in \mathbb{N}$, let $(\xi_1^N, \ldots, \xi_N^N): \Omega \to E^N$ be a vector of random elements in E. The sequence of vectors $\Delta = (\xi_i^N)_{i=1,\ldots,N}^{N \in \mathbb{N}}$ can be seen as a 'triangular array' of random elements in E. Such arrays arise in a natural fashion in those areas of probability theory that deal with asymptotic results, the classical example being of a sequence of sample vectors of size N, $N \in \mathbb{N}$. Let $(\eta_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables and $f: \mathbb{R} \to \mathbb{R}$ a bounded continuous function. Defining for every $i = 1, \ldots, N$, $N \in \mathbb{N}$, $\xi_i^N = \eta_i$, it follows from the classical weak law of large numbers that the normed sum $(1/N) \sum_{i=1}^N f(\xi_i^N)$ converges in probability to the constant $E(f(\eta_1))$. For $x \in \mathbb{R}$ denote with $\mathscr{E}_x$ the Dirac measure on x. Then, for every $N \in \mathbb{N}$, the 'empirical measure' belonging to the vector $(\xi_1^N, \ldots, \xi_N^N)$ is given by $\varphi_\Delta^N = (1/N) \sum_{i=1}^N \mathscr{E}_{\xi_i^N}$.

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One can interpret $(\varphi_\Delta^N)_{N\in\mathbb{N}}$ as a sequence of random elements in the space of Borel measures on $\mathbb{R}$ (equipped with the weak topology). The normed sum given above is then the integral of f with respect to φ_Δ^N for every $N\in\mathbb{N}$. Since f was arbitrarily chosen, the convergence in probability noted above has as consequence, that the sequence $(\varphi_\Delta^N)_{N\in\mathbb{N}}$ converges weakly to the point $\mathscr{L}\{\eta_1\}$ (i.e. denoting with $\xrightarrow{w}$ the weak convergence of measures, $\mathscr{L}\{\varphi_\Delta^N\}\xrightarrow{w}\mathscr{E}_{\mathscr{L}\{\eta_1\}}$ for $N\to\infty$, where $\mathscr{L}\{\xi\}$ denotes the law of a random element ξ (see Section 1)).

Let $\Delta=(\xi_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ be a triangular array of random elements in some topological space E and μ a Borel probability on E. We say that Δ is *point convergent with limit* μ if the sequence of empirical measures induced by the array Δ converges weakly to the point μ. This type of convergence is referred to in the theory of large deviations as 'level II convergence' (see Ellis, 1985), and with respect to arrays of interacting stochastic processes as 'propagation of chaos' (see Sznitman, 1984a, or Dawson, 1983). We refrain from using these titles since they sometimes appear rather unmotivated in our context.

Let d, $m\in\mathbb{N}$. The model we will be considering describes the activity of N agents in an economy, as a vector $(X_1^N,\dots,X_N^N)$ of stochastic processes. These processes are derived as the solution to the following stochastic differential equation:

$$X_i^N(t)=K_i^N+\int_0^t g_i^N(s,X_i^N,\varphi^N)\,\mathrm{d}Z_i^N(s),\quad i=1,\dots,N, \tag{N}$$

where $\varphi_\Delta^N=(i/N)\sum_{i=1}^N \mathscr{E}_{X_i^N}$ is the empirical measure belonging to the vector $(X_i^N)_{i=1,\dots,N}$. Here, for every $N\in\mathbb{N}$, $i=1,\dots,N$, K_i^N is a square integrable $\mathbb{R}^d$ valued (starting) variable and g_i^N is an $\mathbb{R}^{d\times m}$ valued process which depends on the paths of the process X_i^N and (weakly) on the paths of the other processes X_j^N, $j=1,\dots,N$, $j\neq i$ through the empirical measure φ_Δ^N. Finally, Z_i^N is an $\mathbb{R}^m$ valued semimartingale (driving process).

The starting values represent the initial value of some observable microeconomic data of the respective agents (for example age, wealth, consumption of some good up to time 0, etc.). The driving processes describe the basic behavioral patterns of the respective agents which can't be explained through the systematic influence of observable economic data. These systematic influences are captured for a given agent-i by the coefficient process g_i^N, which records the dependency of the agent's behavior on his own past behavior and microeconomic starting values (given by X_i^N and $K_i^N=X_i^N(0)$), and the influences of macroeconomic variables (such as prices, rate of inflation etc.) given implicitly through the empirical measure φ_Δ^N.

Assume now that a triangular array $\Delta=((K_i^N,g_i^N,Z_i^N))_{i=1,\dots,N}^{N\in\mathbb{N}}$ of starting values, coefficients and driving processes as described above is given, representing such values for the agents $i=1,\dots,N$ in an economy of N agents for ever larger economies (i.e. $N\to\infty$). Our goal is to determine whether, under weak assumptions on the array Δ, there exists for every $N\in\mathbb{N}$, a unique solution $(X_1^N,\dots,X_N^N)$ to (N) so that the array formed by these solutions $(X_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ is point convergent with some limit μ. Moreover, in what way can this limit μ be characterized?

The economic interpretation of these questions is as follows: It is assumed that an economy of N agents can be described at the microeconomic level by the solution $(X_1^N, \dots, X_N^N)$ to an equation (N) for every $N \in \mathbb{N}$. Further, the same economy can be described at the macroeconomic level by the empirical measure $(1/N)\sum_{i=1}^N \mathscr{E}_{X_i^N}$. Then, one wants to know whether, for N 'large', (i.e. for $N \to \infty$), there exists a macroeconomic model for this economy that is essentially deterministic (this description being given by the limit measure to the point convergent array $(X_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$). There has been considerable debate in recent years as to whether the dynamics of macroeconomic variables are deterministic or stochastic (see Grandmont and Malgrange, 1986, and related articles appearing in the same volume). Our results provide some insight into the conditions under which the deterministic hypothesis can be given a rigorous microeconomic foundation.

Considering the case without interaction, a fairly minimal condition for the point convergence of an array of solutions $(X_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ is that the array Δ be point convergent. Assume that the limit of Δ is given by the law of a triple (K, g, Z), which is defined in an analog fashion to the elements (K_i^N, g_i^N, Z_i^N) of the array Δ. The sort of limit that one can expect for the array of solutions can be derived with the following heuristic considerations. First, assume that the elements of the array Δ are i.i.d. with law $\mathscr{L}\{(K, g, Z)\}$. Then Δ is point convergent with limit $\mathscr{L}\{(K, g, Z)\}$ (see Section 1). Now, assume for a moment that for every $i = 1, \dots, N$, $N \in \mathbb{N}$, the coefficient g_i^N only depends on the path of the process X_i^N but not on the empirical measure φ^N. Then, since there is no interaction, the elements in the array of solutions are i.i.d. as well, with law $\mathscr{L}\{X\}$, where X is solution to the following equation:

$$X(t) = K + \int_0^t g(s, X)\, \mathrm{d}Z(s). \tag{$\tilde{\infty}$}$$

If one reintroduces the empirical measure in the coefficient g_i^N (i.e. $g_i^N = g_i^N(\cdot, \cdot, \varphi_\Delta^N)$), then one would expect, given the preceding remarks, that the limit of the array of solutions could be characterized as the law of a process X, where X is solution to the following equation

$$X(t) = K + \int_0^t g(s, X, \mathscr{L}\{X\})\, \mathrm{d}Z(s). \tag{∞}$$

Therefore, our task will be as follows: Given a point convergent array Δ of starting values, coefficients and driving processes with limit $\mathscr{L}\{(K, g, Z)\}$, find conditions under which a unique solution $(X_1^N, \dots, X_N^N)$ (resp. X) exists to (N) for $N \in \mathbb{N}$ (resp. to (∞)) so that the array of solutions is point convergent with limit $\mathscr{L}\{X\}$.

Comparison of methods and hypotheses

In the last decade there has been a great deal of work done concerning the asymptotic properties of stochastic systems with weak global interaction of the type described above. In statistical physics interaction through the empirical measure is referred to as 'mean field interaction'. The limiting procedure given above is then called a 'Vlasov' or 'McKean-Vlasov' limit (see Brown and Hepp, 1977; Dobrushin, 1979; Dawson, 1983; Spohn, 1980).

The systems that have been most thoroughly investigated (and are furthermore a special case of the systems we consider) are multidimensional diffusion processes. The first results concerning the point convergence of such systems were derived under very strong hypotheses by McKean (1967). Later, results of this type have been shown under weaker assumptions and in different situations (see Meleard and Roelly-Coppoletta, 1988; Dawson, 1983; Leonard, 1986; Nagasawa and Tanaka 1987a,b,c; Oelschläger, 1982, 1984; Sznitman 1984a,b; Graham, 1988; Graham and Metivier, 1988; Gärtner, 1988). For related work, including a variety of applications in physics, chemistry and population dynamics, consult Bose (1986), Borde-Boussion (1990), Dawson and Gärtner (1987), Finnoff (1989, 1990), Grunbaum (1971), Maruyama (1977), Murata (1977), Nappo and Orlandi (1988), Nagasawa (1980), Scheutzow (1986), Shiga (1980), Sznitman (1984a, 1986), Tanaka (1984) and Wang (1975).

All of these authors working with stochastic differential equations in recent years follow a similar program to derive their results. We have followed a somewhat different program. The primary difference lies in the fact that we make a detour from the systems of equations (N), $N \in \mathbb{N}$ by going over first to discrete time approximations, which we feel are interesting in their own right. Our results go in many respects beyond those found in the literature to take into account the economic background. We recall the interpretation of the equation (N), $N \in \mathbb{N}$, as description of the microeconomic activity of N agents in an economy. Due to the different backgrounds (age, education, etc.) of the agents, one cannot presume the vectors of the array Δ to be identically (much less symmetrically) distributed. Neither can one make any of the usual independence or decay of correlation assumptions, since one must assume a high degree of (unobservable) interaction between individual agents, which must be collected in the stochastic elements of the array Δ. (For an economic justification of the point convergence condition consult Finnoff, 1989).

Further, for $N \in \mathbb{N}$ and $t \in [0, \infty)$ the coefficients $(g_1^N, \ldots, g_N^N)$ at time t depend on the entire past $(X_1^N(h), \ldots, X_N^N(h))_{h \le t}$ of the solution to (N) (since people, in contrast to particles, do have a memory), and are themselves stochastic (reflecting for example stochastic lags, individual optimization procedures or utility functions, etc., see McFadden, 1981). Finally, the Lipschitz condition that we require of our coefficients refers only to discrete measures, which makes verification much simpler for concrete examples. A precise description of the hypotheses that we make is given at the beginning of Section 3 (see (CM), (CL) and (CP)).

The things that we didn't consider were first, coefficients with singularities or growth that is nonlinear (as in Nagasawa and Tanaka, 1987a,b; Gärtner, 1988; Leonard, 1986), and second, systems with boundary conditions (as in Graham and Metivier, 1988; Graham, 1988; Sznitman, 1984b).

Our method for demonstrating point convergence goes as follows: First, show for every $N \in \mathbb{N}$, that there exists a unique solution $(X_1^N, \dots, X_N^N)$ to (N) and define a sequence of 'approximative solutions' $(({}^nX_1^N, \dots, {}^nX_N^N))_{n\in\mathbb{N}}$ to (N) by time discretization. Second, show that for every $n \in \mathbb{N}$, the array $\Delta^n = ({}^nX_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ of approximative solutions is point convergent with limit $\mu^n = \mathscr{L}\{X^n\}$. Third, demonstrate that the arrays of approximative solutions converge 'uniformly in N' to the array of genuine solutions. Finally, use this uniform convergence to show that the array of solutions $(X_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ is point convergent with limit $\mathscr{L}\{X\}$. Here X is identified as the unique limit point of the sequence $(\mu^n)_{n\in\mathbb{N}}$, and as such, solution to the equation (∞) using a stability theorem due to J. Jacod and J. Memin.

What one gains by using our methods is, first of all, the very general result for discrete time processes contained in Theorem 3.3, which requires no martingale theoretic, boundedness or Lipschitz hypotheses. This result, together with the approximation Theorem 2.5 may be of independent interest to those working with discrete time models, or performing Monte Carlo simulations and 'approximation through simulation' using Monte Carlo methods for certain types of nonlinear partial differential equations, (see Babovski, 1989; Lecót, 1989; Griffiths and Mitchell, 1988; Engquist and Hou, 1989; Seidman, 1988). Second, by the application of our methods, one gains insight into the close relationship between point convergence and the classical stability theory for stochastic differential equations.

Formal organization of the paper

In the first paragraph definitions are given, a number of topological results are collected and several necessary and sufficient criteria for the point convergence of an array are derived. Further, we show that one can transform a point convergent array in a number of different ways without losing this property and we prove an approximation theorem for point convergent arrays. In the second paragraph we present the martingale theoretic concepts and results we require in the sequel. The final paragraph is devoted to proving the point convergence of arrays of solutions to stochastic different equations.

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Notation

Here we list the notation and conventions we will be using in the sequel. In the following E will always denote a topological space and $(\Omega, \mathscr{A}, \boldsymbol{F}, \boldsymbol{P})$ a fixed filtered probability space on which all random elements that appear are defined. Here, the filtration $\boldsymbol{F} = (\mathscr{F}_t)_{t\in[0,\infty)}$, $\mathscr{F}_t \subset \mathscr{A}$ for every $t \in [0, \infty)$ will be assumed to be right continuous (i.e. $F_s = \bigcap_{t>s} F_t$ for every $s \in [0, \infty)$) and complete w.r.t. $\boldsymbol{P}$.

General

(1) $\mathbb{N}$ (resp. $\mathbb{R}$) will denote the natural (resp. real) numbers. $\mathbb{N}^\infty$ (resp. $\mathbb{Z}$, resp. $\mathbb{Z}_+$) will denote $\mathbb{N} \cup \{\infty\}$ (resp. the integers, resp. the positive integers).

Let $t\in[0,\infty)$, I some finite set and $\{x_i: i\in I\}\subset E$.

(2) Then we will denote by $[t]=\max\{n\in\mathbb{Z}_+: n\leq t\}$ the integer part of t, by $|I|$ the cardinality of the set I and by x_I the vector $(x_i)_{i\in I}$.

Probability spaces

(3) (i) Let $\xi:\Omega\to E$ be a Borel measurable mapping (random element) in E. The Borel probability induced by such ξ is denoted by $\mathcal{L}\{\xi\}$.

(ii) For $p\in[1,\infty)$, $\|\cdot\|_p$ will denote the $L_p(P)$ norm. We denote with $\|\cdot\|_0$ the L_0 pseudonorm defined for any random variable ξ by setting $\|\xi\|_0=E(|\xi|\wedge 1)$.

Let I be a countable index set. For every $i\in I$ let X_i be a set, $\mathcal{A}_i$ a sigma algebra on X_i and μ_i a probability measure on $\mathcal{A}_i$. Finally, let i_1, $i_2\in I$ and let $f: X_{i_1}\to X_{i_2}$ be a $\mathcal{A}_{i_1}$, $\mathcal{A}_{i_2}$ measurable function.

(4) (i) We denote with $\bigotimes_{i\in I}\mathcal{A}_i$ (resp. $\bigotimes_{i\in I}\mu_i$) the product sigma algebra on $\Pi_{i\in I}X_i$ (resp. product measure on $(\Pi_{i\in I}X_i, \bigotimes_{i\in I}\mathcal{A}_i)$).

(ii) $\tilde{f}\mu_{i_1}$ denotes the image measure of μ_{i_1} under the mapping f.

Topological Spaces

Let A, $B\subset E$, $x\in E$ and I some index set. Then, we denote with

(5) (i) $\mathcal{G}(E)$ (resp. $\mathcal{K}(E)$, resp. $\mathcal{F}(E)$, resp. $\mathcal{B}(E)$) the family of open (resp. compact, resp. closed, resp. Borel measurable) sets in E,

(ii) E^I the product space equipped with the product topology,

(iii) $M(E)$ (resp. $M_1(E)$) the set of Borel measures (resp. Borel probabilities) on E equipped with the weak topology,

(iv) A^a (resp. A^o) the closure (resp. open kernel) of A in E,

(v) $\mathcal{E}_x$ the Dirac measure on the point x,

(vi) $\bar{A}=E\backslash A=\{x\in E: x\notin A\}$ the complement of A in E.

Let μ be a Borel measure on E. Then:

(6) (i) $\mathcal{M}(E)$ (resp. $\mathcal{M}_b(E)$) denotes the set of Borel measurable functions $f: E\to\mathbb{R}$ (resp. $\{f\in\mathcal{M}(E): f \text{ bounded}\}$).

(ii) Let $f\in\mathcal{M}(E)$ be such that the integral $\int_E f\,d\mu$ exists. We write $\langle f,\mu\rangle=\int_E f\,d\mu$.

(iii) A function $f\in\mathcal{M}(E)$ with $\{x\in E: f \text{ is continuous in } x\}\in\mathcal{B}(E)$ is called μ a.s. continuous iff $\mu(x\in E: f \text{ is continuous in } x)=\mu(E)$. Then $\mathcal{C}(E)$ (resp. $\mathcal{C}_b(E)$, resp. $\mathcal{C}_b^{\mu}(E)$) denotes the family of continuous (resp. bounded continuous, resp. bounded μ a.s. continuous) functions $f: E\to\mathbb{R}$.

Let E be metrizable with metric m.

(7) $L_1^m(E)$ denotes the family of $f\in\mathcal{C}_b(E)$ so that for every $x, y\in E$: $|f(x)-f(y)|\leq m(x,y)$ and $|f|\leq 1$. $\hat{m}$ denotes the induced metric on $M(E)$. This is defined for λ, $\mu\in M(E)$ by setting $\hat{m}(\lambda,\mu)=\sup_{f\in L_1^m(E)}|\int f\,d\mu-\int f\,d\lambda|$.

(8) Let X and Y be topological spaces. Then $C(X, Y)$ denotes the space of continuous functions $f: X\to Y$ equipped with the compact open topology.

The spaces C^d and D^d
Let $d \in \mathbb{N}$.

(9) (i) $C^d = C^d[0,\infty)$ denotes the space of continuous functions $x:[0,\infty) \to \mathbb{R}^d$ with the compact open topology.

(ii) $D^d = D^d[0,\infty)$ denotes the d-dimensional Skorokhod space of functions $x:[0,\infty) \to \mathbb{R}^d$ so that x is right continuous and has left hand limits. D^d is equipped with the Skorokhod topology.

Let X be either D^d or C^d and $\mathscr{F} \subset \mathscr{M}(X)$.

(10) We define $\mathscr{F}^{\mathrm{L}} = \{g \in \mathscr{F}$: There exists a $t \in [0,\infty)$ so that for every $x, y \in X$, $x|_{[0,t]} = y|_{[0,t]}$, then $g(x) = g(y)\}$.

Let $x \in D^d$ and $t \in [0,\infty)$.

(11) Denote with $x(t-) = \lim_{s \uparrow t} x(s)$ and with $\mathscr{J}x(t) = x(t) - x(t-)$.

Let Z be a random element in D^d (stochastic process).

(12) Define $S(Z) = \{t \in (0,\infty)$: $\mathscr{L}\{Z\}(x(t) \neq x(t-)) = 0\}$.

(13) (i) Define on D^d the canonical σ-algebra $\tilde{\mathscr{A}}^d = \mathscr{B}(D^d)$ and filtration $\tilde{F}^d = (\tilde{F}_s)_{s>0}$, where $\tilde{F}_s = \bigcap_{t>s} \sigma\{X(h): h \leq t\}$ for $s \in (0,\infty)$.

(ii) The product space $\bar{\Omega}^d = \Omega \times D^d$ equipped with the σ-algebra $\bar{\mathscr{A}}^d = \mathscr{A} \otimes \tilde{\mathscr{F}}^d$ and filtration $\bar{F}^d = (\bar{F}_s)_{s>0}$ (where $\bar{F}_s = \bigcap_{t>s} F_t \otimes \tilde{F}_t$) is referred to as the (d-dimensional) *canonical extension* of $(\Omega, \mathscr{A}, F)$.

1. Topological preliminaries

In this paragraph we introduce the concepts we will be dealing with and collect a number of general results from the field of topological measure theory. Further, we will derive the results we need to demonstrate the point convergence of arrays of approximative solutions. We will frequently be concerned with questions as to the continuity of mappings defined on a space of measures. Let I be a finite index set, X a topological space and $f: E \to X$ a continuous mapping. We recall then that the following mappings are continuous (see Topsøe 1970, p. 68, p. 48; resp. Schief, 1986, p. 5):

$$M(E) \to M(X), \quad \mu \mapsto \tilde{f}\mu \quad \text{(see (6)(ii))},$$

$$E^I \to M_1(E)^I, \quad x_I \mapsto (\mathscr{E}_{x_i})_{i \in I},$$

$$M(E)^I \to M(E), \quad \mu_I \mapsto \frac{1}{|I|} \sum_{i \in I} \mu_i.$$

Therefore, for every $N \in \mathbb{N}$ the mapping $\varphi_E^N: E^N \to M_1(E)$, $(x_i)_{i=1,\dots,N} \to (1/N)\sum_{i=1}^N \mathscr{E}_{x_i}$ is continuous.

Definition 1.1. Let $\Delta = (\xi_i^N)_{i=1,\dots,N}^{N \in \mathbb{N}}$ be a triangular array of random elements in E and ξ a further random element in E.

(i) For every $N \in \mathbb{N}$ we denote with $\varphi_\Delta^N = (1/N)\sum_{i=1}^N \mathscr{E}_{\xi_i^N}$ the empirical measure belonging to $(\xi_i^N)_{i=1,\dots,N}$.

(ii) We say that Δ *is point convergent with limit* $\mathscr{L}\{\xi\}$ iff

$$\mathscr{L}\{\varphi_\Delta^N\} \to \mathscr{E}_{\mathscr{L}\{\xi\}} \quad \text{for } N \to \infty,$$

and write, $(\Delta, \mathscr{L}\{\xi\})$ (*resp.* (Δ, ξ)) *point convergent.*

In the sequel, a triangular array such as Δ of random elements in E will be referred to simply as an *array in E.* If such an array Δ is extended by a potential limit ξ to (Δ, ξ), this will be referred to as a *closed array in E.*

Lemma 1.2. *Let* $x \in E$, $\mathcal{N}$ *a subbase of neighborhoods of* x, $E \in \mathcal{N}$ *and* $(\mu_N)^{N \in \mathbb{N}}$ *a sequence of Borel probabilities on E. Then* $(\mu_N)^{N \in \mathbb{N}}$ *converges to* $\mathscr{E}_x$ *iff for any* $\Gamma \in \mathcal{N}$,

$$\mu_N(\Gamma) \to 1 \quad \textit{for } N \to \infty.$$

Proof. By definition $(\mu_N)^{N \in \mathbb{N}}$ converges weakly to $\mathscr{E}_x$ iff for any $G \in \mathscr{G}(E)$, $\liminf_{N \in \mathbb{N}} \mu_N(G) \geq \mathscr{E}_x(G)$. Since this is trivially the case if $x \notin G$, $(\mu_N)^{N \in \mathbb{N}}$ converges to $\mathscr{E}_x$ iff for any open neighborhood of x,

$$\mu_N(G) \to 1 \quad \text{for } N \to \infty. \tag{$*$}$$

Now if $(*)$ holds for some set, then it holds for any larger set. We must show then, that the system $\mathscr{H} = \{B \subset E\colon B$ is a neighborhood of x so that $(*)$ holds for $B\}$ is closed to finite intersections. To see this, let G, $H \in \mathscr{H}$. Obviously then $G \cup H \in \mathscr{H}$ and, as such

$$\mu_N(G \cup H) = \mu_N(G) + \mu_N(H) - \mu_N(G \cap H) \to 1 \quad \text{for } N \to \infty.$$

Therefore $G \cap H \in \mathscr{H}$. Since $\mathcal{N} \subset \mathscr{H}$, the lemma is immediate. □

The following lemma provides us with some necessary and sufficient criteria for the point convergence of an array.

Lemma 1.3. *Let* Δ *be an array in E and* μ *a Borel probability on E. Then*:

(i) (Δ, μ) *is point convergent iff for every* $F \in \mathscr{F}(E)$ (*resp.* $G \in \mathscr{G}(E)$) *and* $\varepsilon > 0$,

$$\lim_{N \to \infty} P(\varphi_\Delta^N(F) < \mu(F) + \varepsilon) = 1 \quad \left(\textit{resp.} \lim_{N \to \infty} P(\varphi_\Delta^N(G) > \mu(G) - \varepsilon) = 1)\right).$$

(ii) *If* (Δ, μ) *is point convergent, then for every* $f \in \mathscr{C}_b^\mu(E)$ *the sequence of random variables* $(\langle f, \varphi_\Delta^N \rangle)^{N \in \mathbb{N}}$ *converges in probability to the constant* $\langle f, \mu \rangle$.

(iii) *Let E be perfectly normal* (*see Kuratowski,* 1966a, *p.* 133) (*resp.* $E = D^d$), *and* $\mathscr{F} = \mathscr{C}_b(E)$ (*resp.* $\mathscr{F} = (\mathscr{C}_b(E))'$). *Then* (Δ, μ) *is point convergent iff for every* $f \in \mathscr{F}$ *the sequence of random variables* $(\langle f, \varphi_\Delta^N \rangle)^{N \in \mathbb{N}}$ *converges in probability to* $\langle f, \mu \rangle$.

(iv) *Let E be a separable metric space with metric m. Then* (Δ, μ) *is point convergent iff the sequence of random variables* $(\hat{m}(\varphi_\Delta^N, \mu))^{N \in \mathbb{N}}$ *converges in probability to zero.*

Proof. All four cases are shown through an application of Lemma 1.2.

(i) By definition of the weak topology, the family of sets

$$\{\{\lambda \in M_1(E): \lambda(F) < \mu(F) + \varepsilon\}: F \in \mathcal{I}(E), \varepsilon > 0\}$$

forms a neighborhood subbase for μ in $M_1(E)$. (The case with open sets $G \in \mathcal{G}(E)$ is completely analogous).

(ii) Let $f \in \mathcal{C}_b^{\mu}(E)$ and $\varepsilon > 0$. The set $\{\lambda \in M_1(E): |\langle f, \mu\rangle - \langle f, \lambda\rangle| < \varepsilon\}$ is a neighborhood of μ in $M_1(E)$ (see Topsøe, 1970, Theorem (8.1) vii)).

(iii) The family of sets

$$\{\{\lambda \in M_1(E): |\langle f, \lambda\rangle - \langle f, \mu\rangle| < \varepsilon\}: f \in \mathcal{F}, \varepsilon < 0\}$$

forms a neighborhood subbase for μ in $M_1(E)$ (see Topsøe, 1970, Theorem (8.1), and Kuratowski, 1966a, VI Theorem (1)) (resp. Jacod and Shiryaev, 1987, VI Theorem (1.14) and (1.24)-(1.27)).

(iv) This is shown as above using the fact that the metric $\hat{m}$ induces the topology on $M(E)$. □

Let X and Y be separable metric spaces. Then the space $C(X, Y)$ is, generally speaking, not a Suslin space (see Michael, 1961). It turns out though, that $C(X, Y)$ has sufficient countability properties to avoid any problems of measurability that otherwise might be encountered. We recall that a topological space X is called a *hereditarily Lindelöf space* iff to every subset $A \subset X$ and open cover of A there exists a countable subcover. In the following we will often construct new arrays as image of an existing array under a measurable mapping (see Theorem 1.8, Corollary 1.9). The product spaces on which the new arrays are defined often contain the space $\tilde{C} = C([0, \infty) \times D^d \times M_1(D^d), \mathbb{R}^{d \cdot m})$ where the (random) coefficients of the equations we consider take their values. The mappings used to generate the new arrays are usually only known to be coordinatewise measurable, (and not necessarily product measurable). The hereditary Lindelöf property of the product spaces demonstrated in the following lemma will insure the product measurability of these mappings, since the Borel sets of the product space is then equal to the product of the Borel sets of the coordinate spaces.

Lemma 1.4. *Let X, E and Y be separable metric spaces. Then, for any $N \in \mathbb{N}$ the space $((E \times C(X, Y)))^N$ is perfectly normal and hereditarily Lindelöf. Further, the evaluation mapping $e: C(X, Y) \times X \to Y$, $(f, x) \mapsto f(x)$ is Borel measurable.*

Proof. Let $\mathcal{N}_X$ (resp. $\mathcal{N}_Y$) be some countable base for the topology on X (resp. Y). Denote with C the set of continuous functions $f: X \to Y$. For two sets $B \subset X$, $D \subset Y$ set $[B, D] = \{f \in C: f(B) \subset D\}$. Finally let τ (resp. $\hat{\tau}$) denote the compact open topology, (resp. the topology generated by the family of sets $\mathcal{U} = \{[B^a, D^a]: B \in \mathcal{N}_X, D \in \mathcal{N}_Y\}$). Since $\mathcal{U}$ is countable, $(C, \hat{\tau})$ is second countable. We will show that the evaluation mapping $e: C \times Y \to Y$ is continuous when C is equipped with the topology $\hat{\tau}$. Let $G \neq \emptyset$ be some open set in Y and $(f, x) \in e^{-1}(G)$. Since Y is normal, there exists an open set $D \subset Y$ such that $f(x) \in D$, $D^{\mathrm{a}} \subset G$. Because $\mathcal{N}_Y$ is a base for the topology on Y, it is no restriction to assume that $D \in \mathcal{N}_Y$. By the continuity of f and normality of X one can find a $B \in \mathcal{N}_X$ with $x \in B^{\mathrm{a}} \subset f^{-1}(D)$. Thus $(f, x) \in [B^{\mathrm{a}}, D^{\mathrm{a}}] \times B = H$. If $(g, y) \in H$ then $g(y) \in G$. Therefore $H \subset e^{-1}(G)$. Since H is open in $(C, \hat{\tau}) \times X$, this proves the proposed continuity of e. It then follows from classical results that τ is coarser than $\hat{\tau}$ (see Kelley, 1955, p. 223). The idea for this proof is taken from Michael (1961).

The space $((E \times (C, \tau)))^N$ is regular (see Kuratowski, 1966b, p. 76). Since $((E \times (C, \hat{\tau})))^N$ is second countable and $\mathcal{G}((E \times (C, \tau))^N) \subset \mathcal{G}((E \times (C, \hat{\tau}))^N)$, it follows that $((E \times (C, \tau)))^N$ is hereditarily Lindelöf. A regular Lindelöf space is normal (see Engelking, 1968, p. 140). Further, every regular, hereditarily Lindelöf space Z is perfect. To see this consider any open set $G \subset Z$. Then there exists to $x \in G$ a set H_x with $x \in H_x^{\mathrm{o}} \subset H_x^{\mathrm{a}} \subset G$. By the hereditary Lindelöf property of Z there then exists a countable subset $I \subset G$ so that $G = \bigcup_{x \in I} H_x^{\mathrm{a}}$.

Finally, since $(C, \hat{\tau})$ is second countable, $\mathcal{B}((C, \hat{\tau}))$ is generated by $\mathcal{U}$. As $\mathcal{U} \subset \mathcal{B}((C, \tau))$, the measurability of e with respect to $\mathcal{B}((C, \tau))$ is immediate. □

The space D^d

Let $d \in \mathbb{N}$. We will be considering stochastic processes realized in the d-dimensional Skorokhod space $D^d = D^d[0, \infty)$ equipped with the Skorokhod topology. We collect here some facts about this topology that we require later.

Lemma 1.5. (i) *Let X be a random element in D^d. Then there exits a countable set $T \subset [0, \infty)$ so that $([0, \infty) \setminus T) \subset S(X)$ (see (12)).*

(ii) *Let $t \in [0, \infty)$. Then the functions $x \mapsto x(t)$, $x \mapsto x(\cdot \vee t)$ and $x \to x(\cdot \wedge t)$ are Borel measurable and continuous in every point $\alpha \in D^d$ with $\mathcal{J}\alpha(t) = 0$.*

(iii) *Let $x, y \in D^d$ and $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ be sequences in D^d so that $x_n \to x$, $y_n \to y$ for $n \to \infty$. Assume that for every $t > 0$ there is a sequence $(t_n)_{n \in \mathbb{N}}$ in $[0, \infty)$, $t_n \to t$ with $\mathcal{J}x_n(t_n) \to \mathcal{J}x(t)$ and $\mathcal{J}y_n(t_n) \to \mathcal{J}y(t)$ for $n \to \infty$. Then, $x_n + y_n \to x + y$ for $n \to \infty$.*

Proof. (i) See Jacod and Shiryaev (1987, VI (3.12)).

(ii) The measurability follows directly from the fact that the Borel sets of D^d are generated by the mappings $x \mapsto x(t)$, $t \in [0, \infty)$ (see Jacod and Shiryaev, 1987, VI (1.46)). The proposed continuity can then be easily derived using any of the usual characterizations of convergence in D^d (see Jacod and Shiryaev, 1987, VI (2.3), (2.5)).

(iii) See Jacod and Shiryaev (1987, VI (2.2)). □

Lemma 1.6. *Let $t_1, t_2 \in [0, \infty)$, $t_1 < t_2$ and $m \in \mathbb{N}$. Define $H(t_1, t_2) = \{(x, z) \in D^d \times D^m$: $x = x(\cdot \wedge t_1)$ and $\mathcal{J}z(t_1) = \mathcal{J}z(t_2) = 0\}$, $\tilde{C} = C([0, \infty) \times D^d \times M_1(D^d), \mathbb{R}^{d \cdot m})$ and* *finally*

$$\psi : D^d \times D^m \times \tilde{C} \times \mathbb{R}^{d\cdot m} \to D^d \times D^m \times \tilde{C},$$

$$(x, z, g, y) \mapsto ((x + y\cdot(z(t_2 \wedge (\cdot \vee t_1)) - z(t_1)), z, g).$$

Then we have:

(i) *The set $H(t_1, t_2)$ is measurable in $D^d \times D^m$.*

(ii) *The mapping ψ is Borel measurable and every $(x, z, g, y) \in H(t_1, t_2) \times \tilde{C} \times \mathbb{R}^{d\cdot m}$ is a continuity point of ψ.*

Proof. (i) Trivial.

(ii) This follows directly from Lemma 1.4 and 1.5 (ii), (iii). □

Lemma 1.7. *For $x, y \in D^d$, $k \in \mathbb{N}$ set*

$$h_k(x, y) = \sup_{t \in [0,k]} \|x(t) - y(t)\| \quad \text{and} \quad m_u(x, y) = \sum_{k \in \mathbb{N}} \frac{1}{2^k}(h_k(x, y) \wedge 1).$$

The metric m_u induces the topology of local uniform convergence on D^d. Further, there exists a complete bounded metric m_s on D^d which induces the Skorokhod topology on D^d so that for every $x, y \in D^d$, $m_s(x, y) \leq m_u(x, y)$.

Proof. See Jacod and Shiryaev (1987, VI (1.24)-(1.27)). □

Point convergence of transformed arrays

In the following we show that a point convergent array can be transformed in a variety of ways without losing the property of point convergence. Further, we demonstrate that if one can approximate an array of random elements 'uniformly' with a sequence of point convergent arrays, that the approximated array is itself point convergent.

Theorem 1.8. *Let X be a topological space with the property that X^N is hereditarily Lindelöf for any $N \in \mathbb{N}$. Further, let $f : E \to X$ be a Borel measurable mapping and $(\xi_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}} = \Delta$ a point convergent array in E with limit $\mu \in M_1(E)$. Finally, suppose that one of the following conditions is satisfied:*

(i) *X is perfectly normal and the set $C_f = \{y \in E : f \text{ is continuous in } y\}$ is Borel measurable with $\mu(C_f) = 1$.*

(ii) *There exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of closed sets in E with*

(a) *$f|K_n$ is continuous,*

(b) *$P(\varphi_\Delta^N(\bar{K}_n) > 1/n) < 1/n$ for every n, $N \in \mathbb{N}$.*

Then, the array $f(\Delta) = (f(\xi_i^N))_{i=1,\dots,N}^{N\in\mathbb{N}}$ in X is point convergent with limit $\tilde{f}\mu$.

Proof. By the hereditary Lindelöf property of X^N for $N \in \mathbb{N}$, all the random elements in the array $f(\Delta)$ are well defined.

(i) By Lemma 1.3 (iii) we must show that for every $g \in \mathscr{C}_b(X)$, the net $(\langle g, \varphi^N_{f(\Delta)}\rangle)_{N\in\mathbb{N}}$ converges in probability to $\langle g, \tilde{f}\mu\rangle$. Since the function $g\circ f \in \mathscr{C}^\mu_b(E)$ we have by Lemma 1.5 (ii) and definition of $f(\Delta)$,

$$\langle g, \varphi^N_{f(\Delta)}\rangle = \langle g, \tilde{f}\varphi^N_\Delta\rangle = \langle g\circ f, \varphi^N_\Delta\rangle \xrightarrow{\text{prob.}} \langle g\circ f, \mu\rangle \quad \text{for } N\to\infty.$$

By definition $\langle g\circ f, \mu\rangle = \langle g, \tilde{f}\mu\rangle$. This completes the proof of (i).

(ii) Let $F \in \mathscr{F}(X)$, $\varepsilon > 0$. For any $\delta > 0$ choose some $m \in \mathbb{N}$ with $1/m < \max\{\delta, \frac{1}{2}\varepsilon\}$. Since $f|K_m$ is continuous, the set $F_m = f^{-1}(F)\cap K_m$ is closed in E. Define

$$B = f^{-1}(F)\cap \bar{K}_m$$

We then have for $N \in \mathbb{N}$,

$$\begin{aligned}
&P(\varphi^N_{f(\Delta)}(F) < \tilde{f}\mu(F) + \varepsilon)\\
&\quad = P(\varphi^N_\Delta(F_m) + \varphi^N_\Delta(B) < \mu(F_m) + \mu(B) + \varepsilon)\\
&\quad \geqslant P(\varphi^N_\Delta(F_m) + \varphi^N_\Delta(B) < \mu(F_m) + \varepsilon)\\
&\quad \geqslant P(\{\varphi^N_\Delta(F_m) + 1/m < \mu(F_m) + \varepsilon\}\cap\{\varphi^N_\Delta(B) \leqslant 1/m\})\\
&\quad \geqslant P(\varphi^N_\Delta(F_m) < \mu(F_m) + \tfrac{1}{2}\varepsilon) - P(\varphi^N_\Delta(\bar{K}_m) > 1/m)\\
&\quad \geqslant P(\varphi^N_\Delta(F_m) < \mu(F_m) + \tfrac{1}{2}\varepsilon) - \delta.
\end{aligned}$$

By the point convergence of (Δ, μ) and Lemma 1.3(i), the last expression above converges for $N\to\infty$ to $1-\delta$. Since δ was arbitrarily chosen, the first expression above converges to 1 for $N\to\infty$. Apply Lemma 1.3(i) again to finish the proof. □

An application of Theorem 1.8 that we require in the sequel is contained in the following:

Corollary 1.9. *Let E be Polish and Y, S separable metric spaces. Further, let $\Delta = ((X^N_i, Z^N_i, g^N_i))^{N\in\mathbb{N}}_{i=1,\dots,N}$ be point convergent in $H = E\times S\times C(E, Y)$ with limit $\mathscr{L}\{(X^\infty, Z^\infty, g^\infty)\}$. Then, the array in $H\times Y$,*

$$\tilde{\Delta} = ((X^N_i, Z^N_i, g^N_i, g^N_i(X^N_i)))^{N\in\mathbb{N}}_{i=1,\dots,N}$$

is point convergent with limit $\mathscr{L}\{(X^\infty, Z^\infty, g^\infty, g^\infty(X^\infty))\}$.

Proof. Let $e: E\times C(E, Y)\to Y$, $(x, g)\mapsto g(x)$ denote the evaluation mapping. By Lemma 1.4, e is Borel measurable and the space H^N is hereditarily Lindelöf and perfectly normal for any $N\in\mathbb{N}$. The array $\Delta_X = (X^N_i)^{N\in\mathbb{N}}_{i=1,\dots,N}$ is by Theorem 1.8(i) together with Δ, point convergent. In particular, since $M(E)$ is Polish, the weakly convergent sequence

$$(\mathscr{L}\{\varphi^N_{\Delta_X}\})^{N\in\mathbb{N}}$$

is by Prochorov's theorem, tight. Thus, for every $n\in\mathbb{N}$ there exists a compact set $\tilde{K}_n \subset M(E)$ so that for every $N\in\mathbb{N}$,

$$\mathscr{L}\{\varphi^N_{\Delta_X}\}(\tilde{K}_n) \geqslant 1 - 1/n.$$

A second application of Prochorov's theorem to $\tilde{K}_n$, delivers the existence of a compact set $U_n \subset E$ with the property that for every $\mu \in \tilde{K}_n$,

$$\mu(U_n) \geq 1 - 1/n.$$

Define for every $n \in \mathbb{N}$ the closed set $K_n = U_n \times S \times C(E, Y) \times Y$. We then have for every n, $N \in \mathbb{N}$,

$$P(\varphi_{\tilde{\Delta}}^N(\bar{K}_n) > 1/n) = P(\varphi_{\Delta x}^N(\bar{U}_n) > 1/n) < 1/n.$$

Defining $\psi = (\mathrm{id}_H, e)$, it follows that $\tilde{\Delta} = \psi(\Delta)$ and $\mathscr{L}\{(X^\alpha, Z^\alpha, g^\alpha, g^\alpha(X^\alpha))\} = \tilde{\psi}\mu$. By classical results, (see Dugundji 1966, p. 259), the restriction of e to $U_n \times C(E, Y)$ is continuous for every $n \in \mathbb{N}$. As such, for every $n \in \mathbb{N}$, $\psi|_{K_n}$ is continuous. The proposition then follows from Theorem 1.8(ii). □

Our next theorem shows that a point convergent array of random elements can often be 'woven together' with a convergent sequence of random elements to form a new point convergent array.

Theorem 1.10. *Let X and E be topological spaces so that for any $N \in \mathbb{N}$, $(E \times X)^N$ is hereditarily Lindelöf. Further, let $\Delta = (\xi_i^N)_{i=1,\dots,N}^{N \in \mathbb{N}}$ be a point convergent array in E with limit μ, $x \in X$ and $(\eta^N)^{N \in \mathbb{N}}$ a sequence of random elments in X so that $\mathscr{L}\{\eta^N\} \to \mathscr{E}_x$ for $N \leftarrow \infty$. Then the array in $E \times X$,*

$$\tilde{\Delta} = ((\xi_i^N, \eta^N))_{i=1,\dots,N}^{N \in \mathbb{N}}$$

is point convergent with limit $\mu \otimes \mathscr{E}_x$.

Proof. Since $(E \times X)^N$ is hereditarily Lindelöf for every $N \in \mathbb{N}$, all of the random elements in $\tilde{\Delta}$ are well defined. We propose that for any $B \in \mathscr{G}(E \times X)$, $\delta > 0$ there exist $G_1 \in \mathscr{G}(E)$, $G_2 \in \mathscr{G}(X)$, $\varepsilon > 0$ so that

$$\{\lambda \in M_1(E \times X): \lambda(G_1 \times G_2) > \mu \otimes \mathscr{E}_x(G_1 \times G_2) - \varepsilon\}$$
$$\subset \{\lambda \in M_1(E \times X): \lambda(B) > \mu \otimes \mathscr{E}_x(B) - \delta\}.$$

If $\mu \otimes \mathscr{E}_x(B) = 0$, then this is trivial. Suppose that $\mu \otimes \mathscr{E}_x(B) \neq 0$. Since $E \times X$ is hereditarily Lindelöf there exist $A_i \in \mathscr{G}(E)$, $D_i \in \mathscr{G}(X)$ for $i \in \mathbb{N}$, with

$$B = \bigcup_{i \in \mathbb{N}} A_i \times D_i.$$

Set $I_x = \{i \in \mathbb{N}: x \in D_i\}$. Then

$$\mu \otimes \mathscr{E}_x(B) = \mu \otimes \mathscr{E}_x\left(\bigcup_{i \in I_x} A_i \times D_i\right).$$

Further, one can find a finite subset $J \subset I_x$ with

$$\mu\left(\bigcup_{i \in J} A_i\right) > \mu\left(\bigcup_{i \in I_x} A_i\right) - \tfrac{1}{2}\delta.$$

Define $G_1 = \bigcup_{i\in J} A_i$, $G_2 = \bigcap_{i\in J} D_i$. Then $G_1 \times G_2 \subset B$ and

$$\mu \otimes \mathscr{E}_x(G_1 \times G_2) = \mu(G_1) > \mu \otimes \mathscr{E}_x(B) - \tfrac{1}{2}\delta.$$

Set $\varepsilon = \frac{1}{2}\delta$.

Recall the result stated in Lemma 1.3(i). Assume that $\varepsilon > 0$, $G_1 \in \mathscr{G}(E)$, $G_2 \in \mathscr{G}(X)$ are given. We must show

$$P(\varphi_\Delta^N(G_1 \times G_2) > \mu \otimes \mathscr{E}_x(G_1 \times G_2) - \varepsilon) \to 1 \quad \text{for } N \to \infty.$$

It is no restriction to assume that $x \in G_2$. By definition,

$$\begin{aligned} &P(\varphi_\Delta^N(G_1 \times G_2) > \mu \otimes \mathscr{E}_x(G_1 \times G_2) - \varepsilon) \\ &\quad = P\left(\frac{1}{N}\sum_{i=1}^N \mathscr{E}_{(\xi_i^N, \eta^N)}(G_1 \times G_2) > \mu(G_1) - \varepsilon\right) \\ &\quad \geq P(\{\varphi_\Delta^N(G_1) > \mu(G_1) - \varepsilon\} \cap \{\eta^N \in G_2\}). \end{aligned}$$

The point convergence of (Δ, μ) implies

$$P(\varphi_\Delta^N(G_1) > \mu(G_1) - \varepsilon) \to 1 \quad \text{for } N \to \infty.$$

From the weak convergence of $(\eta^N)^{N\in\mathbb{N}}$ to x it follows from Lemma 1.2,

$$P(\eta^N \in G_2) \to 1 \quad \text{for } N \to \infty.$$

This proves the proposition. $\square$

An example of an array with the properties listed in Theorem 1.10 is given in the following:

Corollary 1.11. *Let X, Y and E be separable metric spaces and $((x_i^N, g_i^N))_{i=1,\dots,N}^{N\in\mathbb{N}}$ a point convergent array in $X \times C(E, Y)$ with limit $\mathscr{L}\{(x, g)\}$. Then the array*

$$\left(\left(x_i^N, g_i^N, \frac{1}{N}\sum_{j=1}^N \mathscr{E}_{x_j^N}\right)\right)_{i=1,\dots,N}^{N\in\mathbb{N}}$$

is point convergent in $X \times C(E, Y) \times M_1(X)$ with limit $\mathscr{L}\{(x, g)\} \otimes \mathscr{E}_{\mathscr{L}\{x\}}$.

Proof. The array $(x_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ is together with $((x_i^N, g_i^N))_{i=1,\dots,N}^{N\in\mathbb{N}}$ point convergent. Further, since $M_1(X)$ is together with X a separable metric space, it follows from Lemma 1.4 that the space $(X \times C(E, Y) \times M_1(X))^N$ is hereditarily Lindelöf for any $N \in \mathbb{N}$. $\square$

In the following theorem we will show that if one can approximate an array of random elements Δ^∞ 'uniformly' through a sequence of point convergent arrays then there exists a uniquely defined probability measure μ so that (Δ^∞, μ) is point convergent. We recall the definition of $\|\cdot\|_0$ given in (3)(ii).

Theorem 1.12 (Approximation Theorem). *Let E be a Polish space with complete metric d. Further, for every $n \in \mathbb{N}$, let Δ^n be a point convergent array of random elements in E with limit $\mathscr{L}\{\xi^n\}$ and Δ^∞ a further array of random elements in E. Finally, denoting with $\hat{d}$ the induced metric on $M(E)$ suppose that*

$$\sup_{N \in \mathbb{N}} \|\hat{d}(\varphi^N_{\Delta^n}, \varphi^N_{\Delta^\infty})\|_0 \to 0 \quad \textit{for } n \to \infty.$$

Then there exists a uniquely defined Borel probability μ on E so that $\mathscr{L}\{\xi^n\} \to \mu$ and (Δ^∞, μ) is point convergent.

Proof. Define for every n, $N \in \mathbb{N}$,

$$\eta^N_n = \hat{d}(\varphi^N_{\Delta^n}, \mathscr{L}\{\xi^n\}) \quad \text{and} \quad \psi^N_n = \hat{d}(\varphi^N_{\Delta^n}, \varphi^N_{\Delta^\infty}).$$

By the point convergence of $(\Delta^n, \mathscr{L}\{\xi^n\})$ and Lemma 1.3(iv) it follows for every $n \in \mathbb{N}$,

$$\|\eta^N_n\|_0 \to 0 \quad \text{for } N \to \infty.$$

Moreover, for any n, m, $N \in \mathbb{N}$,

$$\hat{d}(\mathscr{L}\{\xi^m\}, \mathscr{L}\{\xi^n\}) \leq \eta^N_n + \psi^N_n + \psi^N_m + \eta^N_m.$$

Hence, our hypothesis insures that $(\mathscr{L}\{\xi^n\})_{n \in \mathbb{N}}$ is a Cauchy net with respect to the complete metric $\hat{d}$. Let μ denote the limit of this sequence. For $N \in \mathbb{N}$ one has for any $n \in \mathbb{N}$,

$$\hat{d}(\varphi^N_{\Delta^\infty}, \mu) \leq \psi^N_n + \eta^N_n + \hat{d}(\mathscr{L}\{\xi^n\}, \mu).$$

Therefore, it follows from the preceding and Lemma 1.3(iv) that (Δ^∞, μ) is point convergent. □

2. Martingale theoretic preliminaries

In this section we present several concepts and results from general martingale theory that we require in the sequel. The first result is a standard stochastic Gronwall type Lemma. The second (Lemma 2.2) relates a type of L_2 convergence with the weak convergence of discrete random measures. We then introduce the solution concepts for stochastic differential equations we use, and conclude the paragraph with a simplified version of a stability theorem for stochastic differential equations due to J. Jacod and J. Memin.

In the following we will always assume that the filtered probability space $(\Omega, \mathscr{A}, F, P)$ is given, on which all processes that appear are defined and with respect to which all relevant concepts (stopping time, semimartingale, etc.) refer. Further, it is assumed that all 'processes' that appear are right continuous and have left hand limits. Finally, we set $\sup \emptyset = 0$.

Lemma 2.1. *Let S and R be stopping times, $S \leq R$ and K, ρ, $l \in [0,\infty)$. Further, let ϕ and A be adapted increasing processes with $\sup_{\omega \in \Omega} |A(R(\omega)) - A(S(\omega))| < l < \infty$. Finally, assume that for any stopping time U, $S \leq U \leq R$,*

$$E(\phi(U-)) \leq K + \rho E\left(\int_{[S,U)} \phi(h-)\,\mathrm{d}A(h)\right).$$

Then,

$$E(\phi(R-)) \leq 2K \sum_{j=0}^{[2\rho l]} (2\rho l)^j.$$

Proof. This lemma is a slightly modified version of Lemma (7.1) in Metivier and Pellaumail (1980). □

Lemma 2.2. *For every $n \in \mathbb{N}^x$, let $\Delta^n = ({}^nX_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ be an array of random elements in D^d and m_s the complete metric on D^d given in Lemma 1.7. Recalling the definition of the induced metric $\hat{m}_s$, we define for every $n \in \mathbb{N}$,*

$$\phi^n = \sup_{N\in\mathbb{N}} \|\hat{m}_s(\varphi_{\Delta^n}^N, \varphi_{\Delta^x}^N)\|_0.$$

Assume that for every ε, $T \in (0,\infty)$ there exists a set $K_\varepsilon^T \in \mathcal{A}$, so that $P(\overline{K_\varepsilon^T}) < \varepsilon$ and

$$\sup_{N\in\mathbb{N}} E\left(\left(\frac{1}{N}\sum_{i=1}^N \sup_{t\leq T} \|{}^nX_i^N(t) - {}^xX_i^N(t)\|^2\right)1_{K_\varepsilon^T}\right) \to 0 \quad \textit{for } n\to\infty.$$

Then $\phi^n \to 0$ for $n \to \infty$.

Proof. By the definition of the induced metric it follows for every n, $N \in \mathbb{N}$,

$$\hat{m}_s(\varphi_{\Delta^n}^N, \varphi_{\Delta^x}^N) \leq \frac{1}{N}\sum_{i=1}^N m_s({}^nX_i^N, {}^xX_i^N).$$

Recalling the definition of the metric m_u given in Lemma 1.7, it follows for every $l \in \mathbb{N}$, $x_1, \dots, x_N$, $y_1, \dots, y_N \in D^d$,

$$\begin{aligned}\hat{m}_s\left(\frac{1}{N}\sum_{i=1}^N \mathscr{E}_{x_i}, \frac{1}{N}\sum_{i=1}^N \mathscr{E}_{y_i}\right) &\leq \frac{1}{N}\sum_{i=1}^N m_s(x_i, y_i)\\ &\leq \frac{1}{N}\sum_{i=1}^N m_u(x_i, y_i)\\ &\leq \frac{1}{N}\sum_{i=1}^N \sup_{t\in[0,l]} \|x_i(t) - y_i(t)\| + \frac{1}{2^l}.\end{aligned}$$

Let $\varepsilon > 0$ be arbitrarily chosen. Choose some $l \in \mathbb{N}$ large enough so that $1/2^l < \varepsilon$.

Noting that $m_s \leq m_u \leq 1$, we have by hypothesis and the preceding for every $n \in \mathbb{N}$,

$$\phi^n \leq \sup_{N \in \mathbb{N}} E(\hat{m}_s(\varphi_\Delta^{N_n}, \varphi_\Delta^{N_\cdot}) 1_{K_\varepsilon^{l+1}}) + \varepsilon$$

$$\leq \sup_{N \in \mathbb{N}} E\left(\frac{1}{N} \sum_{i=1}^{N} \sup_{t \in [0,l]} \|{}^n X_i^N(t) - {}^x X_i^N(t)\| 1_{K_\varepsilon^{l+1}}\right) + 2\varepsilon$$

$$\leq \sup_{N \in \mathbb{N}} E\left(\left(\frac{1}{N} \sum_{i=1}^{N} \sup_{t \in [0,l+1)} \|{}^n X_i^N(t) - {}^x X_{i(t)}^N\|^2 1_{K_\varepsilon^{l+1}}\right)^{1/2}\right) + 2\varepsilon$$

(Cauchy-Schwarz)

$$\leq \sup_{N \in \mathbb{N}} \left(E\left(\frac{1}{N} \sum_{i=1}^{N} \sup_{t \in [0,l+1)} \|{}^n X_i^N(t) - {}^x X_i^N(t)\|^2 1_{K_\varepsilon^{l+1}}\right)\right)^{1/2} + 2\varepsilon$$

(Jensen).

Since ε was arbitrarily chosen, the proposition follows from our hypotheses by using the monotony and continuity of the square root function. □

Stochastic differential equations

Let $m \in \mathbb{N}$. We recall that an adapted, $\mathbb{R}^m$ valued process Z is a semimartingale iff Z admits a *control process* A. A positive, increasing adapted process A is called *a control process for* Z iff for every $l \in \mathbb{N}$, $\mathbb{R}^{l \cdot m}$ valued elementary predictable process X and stopping time τ, one has

$$E\left(\sup_{t<\tau} \left\| \int_{[0,t)} X \, \mathrm{d}Z \right\|^2\right) \leq E\left(J_l A(\tau-) \int_{[0,\tau)} \|X\|_{\mathrm{Op}}^2 \, \mathrm{d}A(s)\right),$$

where $J_l > 0$ is some constant that only depends on l, and $\| - \|_{\mathrm{Op}}$ denotes the operator norm on $\mathbb{R}^{l \cdot m}$ (see Metivier, 1982, p. 157).

We further recall that if Z is an $\mathbb{R}^m$ valued, locally integrable semimartingale, there exists a uniquely defined triple (V, C, ϑ) called the *local characteristics of* Z, consisting of a predictable $\mathbb{R}^m$ valued process V and a continuous $\mathbb{R}^{m \cdot m}$ valued process C, both with paths of locally bounded variation, and a predictable random measure ϑ on $\mathbb{R}^m \backslash \{0\}$ which can be used to characterize Z as the solution to a *martingale problem* (see Jacod and Shiryaev, 1987). Although we won't be making direct use of this characterization it is central to the concept of a 'weak solution' to a stochastic differential equation (see (2.1)) and to the stability Theorem 2.8 that we apply in the sequel. The following definitions and notation with regards to stochastic differential equations follow Jacod (1980), Jacod and Memin (1980, 1981).

Assume to be given: An m-dimensional semimartingale Z with $Z(0) = 0$ (*driving process*), an $\mathbb{R}^d$-valued F_0 measurable random variable K (*starting value*) and a coefficient $g : \bar{\Omega}^d \times [0, \infty) \to \mathbb{R}^{d \cdot m}$ which, viewed as a process on the canonical extension $(\bar{\Omega}^d, \bar{\mathscr{A}}^d, \bar{F}^d)$ (see (13)), is predictable. Such processes are referred to as *predictable functionals*. This is due to the fact that if X is an adapted process on the original space $(\Omega, \mathscr{A}, F^d, P)$, then one can define a predictable 'functional process' Y on $(\Omega, \mathscr{A}, \boldsymbol{F}^d, \boldsymbol{P})$ by setting for every $(\omega, t) \in \Omega \times [0, \infty)$, $Y(\omega, t) = g(\omega, t, X(\omega))$. Moreover, since $Y(\omega, t)$ only depends on the values of $X(\omega, s)$ for $s < t$, it is possible to define $Y(\omega, t)$ when $X(\omega, s)$ is only known for $s < t$ (see Metivier and Pellaumail, 1980, (6.4)). Consider the following equation:

$$X(\cdot, t) = K(\cdot) + \int_0^t g(\cdot, s, X(\cdot, \cdot))\, \mathrm{d}Z(\cdot, s) \tag{2.1}$$

(Doleans-Dade and Protter's equation). In the equations we consider, the law $\mathscr{L}\{X\}$ of the solution X will sometimes appear in the coefficient as well. This case can be treated though in a canonical fashion without leaving the framework given by equation (2.1). We now give precise meaning to what we mean by a 'solution to the equation given above.

Definition 2.3. *A strong solution* (*or solution process*) to (2.1) with respect to the *driving system* $(\Omega, \mathscr{A}, \boldsymbol{F}, \boldsymbol{P}, K, Z)$ is an $\mathbb{R}^d$ valued process X on Ω with the following properties:

(i) X is adapted to $\boldsymbol{F}$.

(ii) Define the process $g(X)$ by setting for every $(\omega, t) \in \Omega \times [0, \infty)$, $g(X)(\omega, t) = g(\omega, t, X(\omega))$. Then, $g(X)$ is integrable with respect to Z and for every $t \in [0, \infty)$ and $\boldsymbol{P}$ a.e. $\omega \in \Omega$,

$$X(\omega, t) = K(\omega) + \int_0^t g(\omega, s, X(\omega, \cdot))\, \mathrm{d}Z(\omega, x).$$

Let $\psi : \bar{\Omega}^d \to \Omega$, $(\omega, x) \mapsto \omega$ denote the projection of $\bar{\Omega}^d$ onto Ω. Then, a *weak solution* (*or good solution measure*) to (2.1) is a probability measure $\bar{\boldsymbol{P}}$ on $(\bar{\Omega}, \bar{\mathscr{A}}^d)$ which satisfies:

(i) The Ω marginal $\bar{\boldsymbol{P}}|_\Omega$ of $\bar{\boldsymbol{P}}$ (i.e. the image measure of $\bar{\boldsymbol{P}}$ under ψ) is equal to $\boldsymbol{P}$.

(ii) The process $\bar{Z} = Z \circ \psi$ is a semimartingale on $(\bar{\Omega}^d, \bar{\mathscr{A}}^d, \bar{\boldsymbol{F}}^d, \bar{\boldsymbol{P}})$.

(iii) If (B, C, ϑ) denotes the local characteristics of Z on $(\Omega, \mathscr{A}, \boldsymbol{F}, \boldsymbol{P})$ then $(B \circ \psi, C \circ \psi, \vartheta \circ \psi)$ are the local characteristics of $\bar{Z}$ on $(\bar{\Omega}^d, \bar{\mathscr{A}}^d, \bar{\boldsymbol{F}}^d, \bar{\boldsymbol{P}})$.

(iv) Define $\bar{K} = K \circ \psi$. The process $(X(t))_{t \geq 0}$ on $\bar{\Omega}^d$ given for every $((\omega, y), t) \in \bar{\Omega}^d \times [0, \infty)$ by setting $X((\omega, y), t) = y(t)$ is a solution process to (2.1) with respect to the driving system $(\bar{\Omega}^d, \bar{\mathscr{A}}^d, \bar{\boldsymbol{F}}^d, \bar{\boldsymbol{P}}, \bar{Z}, \bar{K})$.

We now present a simplified version of a stability theorem due to J. Jacod and J. Memin, dealing with the convergence of sequences of solution measures (weak solutions). Assume that for every $n \in \mathbb{N}^\infty$, a coefficient $g^n : \bar{\Omega}^d \times [0, \infty) \to \mathbb{R}^{d \cdot m}$ is given, which is a predictable functional on $(\bar{\Omega}^d, \bar{\mathscr{A}}^d, \bar{F}^d)$. Then define for every $n \in \mathbb{N}^\infty$ an equation

$$X^n(t) = K + \int_0^t g^n(s, X^n)\, \mathrm{d}Z(s). \tag{2.1n}$$

Weak solutions to equations such as (2.1n), $n \in \mathbb{N}^\infty$, are measures on the space $\bar{\Omega}^d = \Omega \times D^d$. Since $(\Omega, \mathcal{A})$ is an abstract measure space, one needs to introduce a convergence concept for measures on spaces such as $\bar{\Omega}^d$.

Definition 2.4. Denote with $B_{\mathrm{m,c}}(\bar{\Omega}^d)$ the set of all bounded $\bar{\mathcal{A}}^d$ measurable functions $V: \bar{\Omega}^d \to \mathbb{R}$ such that $V(\omega, \cdot)$ is continuous on D^d for every $\omega \in \Omega$. We then define $M_{\mathrm{m,c}}(\bar{\Omega}^d)$ to be the space of all probability measures on $(\bar{\Omega}^d, \bar{\mathcal{A}}^d)$ endowed with the topology induced by the mappings $\mu \mapsto \int V \,\mathrm{d}\mu$, for $V \in B_{\mathrm{m,c}}$. This topology is called the *weak-strong topology* on $M_{\mathrm{m,c}}(\bar{\Omega}^d)$.

We now list the hypotheses which are required for the stability theorem.

Hypothesis 2.5 (*Compactness*). For every $n \in \mathbb{N}$, there exists a unique weak solution $\bar{P}^n \in M_{\mathrm{m,c}}(\bar{\Omega}^d)$ to the equation (2.1n), and the sequence $(\bar{P}^n|_{D^d})_{n \in \mathbb{N}}$ converges weakly to a measure $\mu^\infty \in M_1(D^d)$.

Hypothesis 2.6 (*Continuity of* g^∞). For every $(\omega, t) \in \Omega \times [0, \infty)$ the function $g^\infty(\omega, t, \cdot): D^d \to \mathbb{R}^{d \cdot m}$ is continuous.

Hypothesis 2.7 (*Convergence of* $(g^n)_{n \in \mathbb{N}}$). For every $(\omega, t) \in \Omega \times [0, \infty)$ and every compact subset $K \subset D^d$, the sequence of functions $(g^n(\omega, t, \cdot))_{n \in \mathbb{N}}$ converges uniformly on K to $g^\infty(\omega, t, \cdot)$.

Theorem 2.8 (Stability Theorem). (i) *Under Hypothesis* 2.5, *the sequence* $(\bar{P}^n)_{n \in \mathbb{N}}$ *is relatively sequentially compact in* $M_{\mathrm{m,c}}(\bar{\Omega}^d)$.

(ii) *Assume that Hypotheses* 2.6 *and* 2.7 *hold as well. Then, all limit points of the sequence* $(\bar{P}^n)$ *are weak solutions to* (2.1∞). *Further, if* $\bar{P}^\infty$ *is such a limit point,* $\bar{P}^\infty|_{D^d} = \mu^\infty$.

Proof. Under Hypothesis 2.5, (i) follows from Theorem (2.8) of Jacod and Memin (1981). The result stated in (ii) can be derived using Theorem (3.24) of Jacod and Memin (1980), (hereafter simply [JM]). The conditions [JM](3.11), [JM](3.13) iii), [JM](3.20) and [JM](3.21) (resp. [JM](3.10)) are trivially satisfied in our situation. The condition [JM](3.18) is not required, since it is only used to demonstrate the relative compactness of the sequence $(\bar{P}^n)_{n \in \mathbb{N}}$ which we already have by (i), (see [JM] Lemma (3.56)). The condition [JM](3.15) follows from Hypothesis 2.6 and the condition [JM](3.19) corresponds to our Hypothesis 2.7. Finally, the fact that $\bar{P}^\infty|_{D^d = \mu^\infty}$ for any limit point $\bar{P}^\infty$ follows directly from Hypothesis 2.5.

Theorem (3.24) of [JM] is a very general stability theorem which also permits the starting values, the driving semimartingales and the basic filtered space to vary with $n \in \mathbb{N}^\infty$. The proof itself is, as such, very complex. A reader only interested in the proof to Hypothesis 2.8 is advised to consult Jacod and Memin (1981) and the proof of the (simpler) Theorem (1.8) therein. This contains all the steps needed to demonstrate the theorem (although the proposition itself is actually an existence theorem). □

3. Point convergence for solutions to stochastic differential equations with global interaction

We recall the systems of equations (N), $N\in\mathbb{N}$, and (∞) given in the introduction. In this paragraph we will show that under appropriate hypotheses on the array $\Delta=((K_i^N, Z_i^N, g_i^N))_{i=1,\dots,N}^{N\in\mathbb{N}}$ of starting values, driving processes and coefficients for every $N\in\mathbb{N}$ there exists a unique solution $(X_1^N,\dots,X_N^N)$ to (N), and the array of solutions $(X_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ is point convergent with limit $\mathscr{L}\{X\}$, where X is a unique solution to the equation (∞) induced by the limit $\mathscr{L}\{(K,Z,g)\}$ of the point convergent array Δ.

We require three conditions of the closed array $(\Delta,(K,Z,g))$. The first, (CM), specifies the martingale theoretic structure which is required so that the equations make sense. The second hypothesis, (CL), is a Lipschitz and growth condition that insures the existence of a unique solution to the equation (N) for every $N\in\mathbb{N}$. The final condition, (CP), is a point convergence property, sufficient then to prove the point convergence of the array of solutions to (N), $N\in\mathbb{N}$. In the following, for k, $l\in\mathbb{N}$, $x\in\mathbb{R}^{k\cdot l}$ we denote $\|x\|^2=\sum_{i=1}^{k\cdot l}\|x_i\|^2$.

Conditions 3.1. Let d, $m\in\mathbb{N}$.

(CM) We assume to be given:

(i) A closed array $((Z_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}, Z)$ of $\mathbb{R}^m$ valued semimartingales with $Z(0)=0=Z(0)_i^N$ for $i=1,\dots,N$, $N\in\mathbb{N}$.

(ii) A closed array $((K_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}, K)$ of $\mathbb{R}^d$ valued, F_0 measurable, square integrable random variables, and $I>0$ a constant so that for every $N\in\mathbb{N}$,

$$E\left(\frac{1}{N}\sum_{i=1}^{N}\|K_i^N\|^2\right)<I.$$

(iii) A closed array of coefficients $((g_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}, g)$,

$$g_i^N, g:\Omega\to C([0,\infty)\times D^d\times M_1(D^d),\mathbb{R}^{d\cdot m})$$

with the following properties:

(a) Define for every $N\in\mathbb{N}$ an $(\mathbb{R}^{d\cdot m})^N$ valued random functional on the canonical extension $(\bar\Omega^{d\cdot N},\bar{\mathscr{A}}^{d\cdot N},\bar F^{d\cdot N})$,

$$G^N:\Omega\times[0,\infty)\times D^{d\cdot N}\to(\mathbb{R}^{d\cdot m})^N$$

$$(\omega,t,(x_1,\dots,x_N))\to\left(g_i^N\left(\omega,t,x_i,\frac{1}{N}\sum_{j=1}^{N}\mathscr{E}_{x_j}\right)\right)_{i=1,\dots,N}.$$

Then, G^N is adapted, hence predictable.

(b) Define for every $\mu\in M_1(D^d)$ a random functional g^μ on the canonical extension $(\bar\Omega^d,\bar{\mathscr{A}}^d,\bar F^d)$,

$$g^\mu:\Omega\times[0,\infty)\times D^d\to\mathbb{R}^{d\cdot m}$$

$$(\omega,t,x)\mapsto g(\omega,t,x,\mu).$$

Then g^{μ} is adapted, hence predictable on this extension. Further, for any D^d valued random element Y, $(\omega, t, x) \in \bar{\Omega}^d$ we have

$$g(\omega, t, x, \mathscr{L}\{Y\}) = g(\omega, t, x, \mathscr{L}\{Y(\cdot \wedge t-)\}).$$

We further stipulate that for every $i = 1, \dots, N$, $N \in \mathbb{N}$, the coefficient g_i^N satisfies the following Lipschitz and growth conditions:

(CL) There exist constants $L, G > 0$ so that:

(i) For every $\omega \in \Omega$, t, $\delta \in [0, \infty)$, $x_1, y_1, \dots, x_N, y_N, x, y \in D^d$,

$$\left\| g_i^N\left(\omega, t, x, \frac{1}{N}\sum_{i=1}^N \mathscr{E}_{x_i}\right) - g_i^N\left(\omega, t+\delta, y, \frac{1}{N}\sum_{i=1}^N \mathscr{E}_{y_i}\right) \right\|^2$$

$$\leq L\left(\delta + \sup_{s \in [0,\infty)} \|x(s \wedge t-) - y(s \wedge (t+\delta)-)\|^2 + \frac{1}{N}\sum_{i=1}^N \sup_{s \in [0,\infty)} \|x_i(s \wedge t-) - y_i(s \wedge (t+\delta)-)\|^2\right).$$

(ii) For every $(\omega, t) \in \Omega \times [0, \infty)$: $\|g_i^N(\omega, t, 0, \mathscr{E}_0)\|^2 \leq G$.

Finally, there exists a $[0, \infty)$ valued increasing, continuous, adapted process A with $A(0) = 0$, so that:

(CP) (i) $\Delta = ((K_i^N, Z_i^N, g_i^N))_{i=1,\dots,N}^{N \in \mathbb{N}}$ is point convergent with limit $\mu = \mathscr{L}\{(K, Z, g)\}$.

(ii) For every $i = 1, \dots, N$, $N \in \mathbb{N}$, A is a control process for Z_i^N.

The condition (CL) can be interpreted as stating that there is a bound to the size of the reaction an agent will exhibit by a small variation of his environment and experiences. The condition (CP)(ii) may be seen as stipulating the existence of a common bound to the expected maximal growth of the processes Z_i^N, $i = 1, \dots, N$, $N \in \mathbb{N}$.

By (CM)(iii), for every $N \in \mathbb{N}$, G^N is a predictable functional on the $d \cdot N$-dimensional canonical extension. Further, as a consequence of (CL) the functional G^N satisfies classical growth and Lipschitz conditions, therefore we have,

Theorem 3.2. *For every $N \in \mathbb{N}$ there exists a unique strong solution $(X_1^N, \dots, X_N^N)$ to (N).*

Proof. See Metivier and Pellaumail (1980, Chapter III, Sections 6, 7). □

We now present the result which will permit us to demonstrate the point convergence of arrays of approximative solutions. Set $\tilde{C} = C([0, \infty) \times D^d \times M_1(D^d), \mathbb{R}^{d \cdot m})$.

Theorem 3.3. *Let $\Delta = ((K_i^N, Z_i^N, g_i^N))_{i=1,\dots,N}^{N \in \mathbb{N}}$ be a point convergent array in $\mathbb{R}^d \times D^m \times \tilde{C}$ with limit $\mathscr{L}\{(K, Z, g)\}$. Further let $(t_j)_{j \in \mathbb{N}}$ be a strictly increasing sequence in*

$(0,\infty)$ with $t_j \to \infty$ for $j \to \infty$ and $T = \{t_j : j \in \mathbb{N}\} \subset S(Z)$ (see (12)). Define for every $i = 1, \dots, N$, $N \in \mathbb{N}$, $t \in [0, \infty)$,

$$X_i^N(t) = \begin{cases} K_i^N & \text{for } t \in [0, t_1], \\ X_i^N(t_k) + g_i^N(t_k, X_i^N, (1/N)\sum_{i=1}^N \mathscr{E}_{X_i^N})(Z_i^N(t) - Z_i^N(t_k)) \\ \qquad \text{for } t \in (t_k, t_{k+1}],\ k \in \mathbb{N}, \end{cases}$$

and

$$X(t) = \begin{cases} K & \text{for } t \in [0, t_1], \\ X(t_k) + g(t_k, X, \mathscr{L}\{X\})(Z(t) - Z(t_k)) & \text{for } t \in (t_k, t_{k+1}],\ k \in \mathbb{N}. \end{cases}$$

Then $\tilde{\Delta} = (X_i^N)_{i=1,\dots,N}^{N \in \mathbb{N}}$ is point convergent with limit $\mathscr{L}\{X\}$.

Proof. We define recusively for every $k \in \mathbb{N}$ a closed array in $D^d \times D^m \times \tilde{C}$,

$$({}^k\Delta, ({}^kX, Z, g)) = (({}^kX_i^N, Z_i^N, g_i^N))_{i=1,\dots,N}^{N \in \mathbb{N}}, ({}^kX, Z, g)).$$

To define the first array $({}^1\Delta, ({}^1X, Z, g))$, interpret for every $i = 1, \dots, N$, $N \in \mathbb{N}$ the random element K_i^N in $\mathbb{R}^d$ as a (constant) process in D^d, then set ${}^1X_i^N = K_i^N$. One defines ${}^1X = K$ analogously.

If $({}^k\Delta, ({}^kX, Z, g))$ is already defined, then set for every $i = 1, \dots, N$, $N \in \mathbb{N}$,

$${}^{k+1}X_i^N = {}^kX_i^N + g_i^N\left(t_k, {}^kX_i^N, \frac{1}{N}\sum_{i=1}^N \mathscr{E}_{{}^kX_i^N}\right)(Z_i^N((t_k \vee \cdot) \wedge t_{k+1}) - Z_i^N(t_k)),$$

and

$${}^{k+1}X = {}^kX + g(t_k, {}^kX, \mathscr{L}\{{}^kX\})(Z((t_k \vee \cdot) \wedge t_{k+1}) - Z(t_k)).$$

For $s, t \in [0, \infty)$ recall the set $H(s, t)$ defined in connection with Lemma 1.6. We will show using induction that for every $k \in \mathbb{N}$, the array ${}^k\Delta$ is point convergent with limit $\mathscr{L}\{({}^kX, Z, g)\}$ and that $\mathscr{L}\{({}^kX, Z)\}(H(t_k, t_{k+1})) = 1$. For $k = 1$ this is part of the hypothesis. Assume that the proposition has been shown for $k \in \mathbb{N}$. By Corollary 1.11 the array

$$\left(\left(({}^kX_i^N, Z_i^N, g_i^N), \frac{1}{N}\sum_{i=1}^N \mathscr{E}_{{}^kX_i^N}\right)\right)_{i=1,\dots,N}^{N \in \mathbb{N}}$$

is point convergent with limit $\mathscr{L}\{({}^kX, Z, g)\} \otimes \mathscr{E}_{\mathscr{L}\{{}^kX\}}$.
By Corollary 1.9, the array

$$\bar{\Delta} = \left({}^kX_i^N, Z_i^N, g_i^N, g_i^N\left(t_k, {}^kX_i^N, \frac{1}{N}\sum_{i=1}^N \mathscr{E}_{{}^kX_i^N}\right)\right)_{i=1,\dots,N}^{N \in \mathbb{N}}$$

is point convergent with limit $\mathscr{L}\{\eta\}$, where

$$\eta = ({}^kX, Z, g, g(t_k, {}^kX, \mathscr{L}\{{}^kX\})).$$

Define then the mapping

$$f_{k+1}: D^d \times D^m \times \tilde{C} \times \mathbb{R}^{d\cdot m} \to D^d \times D^m \times \tilde{C}$$

$$(x, z, g, y) \mapsto (x + y\cdot(z((t_k \vee \cdot) \wedge t_{k+1}) - z(t_k)), z, g).$$

By induction and Lemma 1.6, the mapping f_{k+1} and the closed array $(\bar{\Delta}, \eta)$ satisfy the conditions of Theorem 1.8(i). Since ${}^{k+1}\Delta = f_{k+1}(\bar{\Delta})$ and (recalling (4)(ii)), $\tilde{f}_{k+1}\mathscr{L}\{\eta\} = \mathscr{L}\{({}^{k+1}X, Z, g)\}$, the point convergence of $({}^{k+1}\Delta, ({}^{k+1}X, Z, g))$ then follows from this theorem. The fact that $\mathscr{L}\{({}^{k+1}X, Z)\}(H(t_{k+1}, t_{k+2})) = 1$ follows from our hypotheses and the definition of ${}^{k+1}X$.

To verify that the array $(X_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ is point convergent with limit $\mathscr{L}\{X\}$, it is adequate by Lemma 1.3(iii) to consider the array restricted to bounded intervals of $[0, \infty)$. For any bounded interval $I \subset [0, \infty)$ there exists a $j \in \mathbb{N}$ so that $t_j > \sup\{t\colon t \in I\}$. Since for every $i = 1, \dots, N$, $N \in \mathbb{N}$, ${}^jX_i^N$ and X_i^N (resp. ${}^jX$ and X) coincide on I, the proposition is immediate. □

Definition 3.4 (*Approximating arrays*). For every $n \in \mathbb{N}$ let $({}^nt_k)_{k\in\mathbb{Z}_+}$ be an increasing sequence in $S(Z)$ (see (12), Lemma 1.5(i)) with ${}^nt_0 = 0$, and $1/(2n) < |{}^nt_{k-1} - {}^nt_k| < 1/n$ for every n, $k \in \mathbb{N}$. We then define to $(\Delta, \mathscr{L}\{(K, Z, g)\})$ and $({}^nt_k)_{k\in\mathbb{Z}_+}$ a point convergent array $\Delta^n = ({}^nX_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}$ with limit $\mathscr{L}\{{}^nX\}$, as in Theorem 3.3. Further define for every $n \in \mathbb{N}$, $i \in 1, \dots, N$, $N \in \mathbb{N}$ a predictable functional on the $d\cdot N$ dimensional canonical extension by setting

$${}^ng_i^N : \Omega \times [0, \infty) \times D^{d\cdot N} \to \mathbb{R}^{d\cdot m}$$

$$(\omega, t, (x_1, \dots, x_N)) \mapsto \begin{cases} 0 & \text{for } t \in [0, {}^nt_1], \\ g_i^N(\omega, {}^nt_k, x_i, (1/N)\sum_{j=1}^N \mathscr{E}_{x_j}) & \text{for } t \in ({}^nt_k, {}^nt_{k+1}],\ k \in \mathbb{N}. \end{cases}$$

Let ${}^n\mu = \mathscr{L}\{{}^nX\}$ for every $n \in \mathbb{N}$. We then define

$${}^ng : \Omega \times [0, \infty) \times D^d \to \mathbb{R}^{d\cdot m}$$

$$(\omega, t, x) \mapsto \begin{cases} 0 & \text{for } t \in [0, {}^nt_1], \\ g(\omega, {}^nt_k, x, {}^n\mu) & \text{for } t \in ({}^nt_k, {}^nt_{k+1}],\ k \in \mathbb{N}. \end{cases}$$

For every N, $n \in \mathbb{N}$, $({}^nX_1^N, \dots, {}^nX_N^N)$ is a solution of the equation

$${}^nX_i^N(t) = K_i^N + \int_0^t {}^ng_i^N(s, {}^nX_i^N, \varphi_{\Delta^n}^N)\, \mathrm{d}Z_i^N(s), \quad i = 1, \dots, N. \qquad (n, N)$$

And by (CM)(iii)(b) for every $n \in \mathbb{N}$, ${}^nX$ is a solution of

$${}^nX(t) = K + \int_0^t g(s, {}^nX, \mathscr{L}\{{}^nX\})\, \mathrm{d}Z(s). \qquad (n, \infty)$$

In the following we denote

$$\Delta^\infty = (X_i^N)_{i=1,\dots,N}^{N\in\mathbb{N}}.$$

To complete the third step of our program we need some technical results.

Definition 3.5. Let T, $\delta \in [0, \infty)$, $\mathcal{H} \in \mathcal{A}$.

(i) Set

$$\eta_\delta(\mathcal{H}, T) = \sup\{|A(\omega, s) - A(\omega, t)|: \omega \in \mathcal{H}, s, t \in [0, T], |s - t| \leq \delta\}.$$

We now define a number of stopping times.

(ii) $\hat{T}(\mathcal{H}) = \inf\{t \in [0, \infty]: A(t) > \eta_T(\mathcal{H}, T)\} \wedge T.$

(iii) For every $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$ set

$${}^n\tau_k(\mathcal{H}, T) = \inf\{t \in [{}^nt_k, \infty]: A(t) - A({}^nt_k) > \eta_{1/n}(\mathcal{H}, T)\} \wedge \hat{T}(\mathcal{H}) \wedge {}^nt_{k+1}.$$

(iv) For every $n \in \mathbb{N}$ set ${}^nk = \min\{k \in \mathbb{N}: {}^nt_k > T\}$. Then define the adapted process

$${}^nV = \sum_{k=0}^{{}^nk-1} 1_{[{}^nt_k, {}^n\tau_k(\mathcal{H}, T))}$$

and stopping time

$${}^nT = \inf\{t \in [0, \infty): {}^nV(t) = 0\}.$$

In the following, the indexes $\mathcal{H}$ and T will be dropped whenever the reference is clear.

Lemma 3.6. *For every T, $\varepsilon > 0$ there exists a set $\mathcal{H}_\varepsilon^T \in \mathcal{A}$ with:*

(i) $P(\overline{\mathcal{H}_\varepsilon^T}) < \varepsilon$.

(ii) $\eta_T(\mathcal{H}_\varepsilon^T, T) < \infty$ *and* $\eta_{1/n}(\mathcal{H}_\varepsilon^T, T) \to 0$ *for* $n \to \infty$.

Proof. The process $A|_{[0,T]}$ is realized in the Polish space $C[0, T]$ $(= C([0, T], \mathbb{R}))$. Therefore, for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset C[0, T]$ so that $\mathcal{L}\{A|_{[0,T]}\}(\overline{K_\varepsilon}) < \varepsilon$. Set

$$\mathcal{H}_\varepsilon^T = (A|_{[0,T]})^{-1}(K_\varepsilon).$$

By the Arzela–Ascoli theorem, the set K_ε is uniformly bounded and equicontinuous. The result is then immediate. □

In the following two lemmata we will assume that T, $\varepsilon \in [0, \infty)$ are arbitrarily chosen, but fixed and $\mathcal{H} = \mathcal{H}_\varepsilon^T \in \mathcal{A}$ is the set given in Lemma 3.6. Further, any of the symbols η_δ, ${}^n\tau_k$, $\hat{T}$, ${}^nT$, etc., which appear, will be assumed to refer to T and $\mathcal{H}$. Finally, for every $s \in [0, \infty)$, $n \in \mathbb{N}$, we write

$${}^ns = \begin{cases} 0 & \text{if } s \in [0, {}^nt_1], \\ {}^nt_k & \text{if } s \in ({}^nt_k, {}^nt_{k+1}] \text{ for some } k \in \mathbb{N}. \end{cases}$$

Lemma 3.7. *There exists a constant W so that for every n, $N \in \mathbb{N}$,*

$$E\left(\frac{1}{N}\sum_{i=1}^N \int_{[0,{}^nT)} \sup_{h \leq s} \|X_i^N(h \wedge {}^ns-) - X_i^N(h-)\|^2 \, dA(s)\right) < W\eta_{1/n}.$$

Proof. Since all norms on $\mathbb{R}^{d\cdot m}$ are equivalent, there exists a $\hat{C}>0$ so that $\|g\|^2_{\text{Op}} \le \hat{C}\|g\|^2$ for any $g\in\mathbb{R}^{d\cdot m}$. Let J_d be the constant belonging to the control process A and d, (described in the section on stochastic differential equations), then set $C=\hat{C}J_d$.

(1) Let U be some stopping time, $U\le\hat{T}$. Then, for every $N\in\mathbb{N}$,

$$E\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{h<U}\|X_i^N(h)-K_i^N\|^2\right)$$

$$=E\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{h<U}\left\|\int_{[0,h)} g_i^N(t,X_i^N,\varphi_{\Delta x}^N)\,\mathrm{d}Z_i^N(t)\right\|^2\right)$$

$$\le E\left(\frac{1}{N}\sum_{i=1}^{N}J_dA(U)\int_{[0,U)}\|g_i^N(t,X_i^N,\varphi_{\Delta x}^N)\|^2_{\text{Op}}\,\mathrm{d}A(t)\right)\quad((\text{CP})(\text{ii}))$$

$$\le 2CE\left(\frac{1}{N}\sum_{i=1}^{N}A(U)\int_{[0,U)}\|g_i^N(t,X_i^N,\varphi_{\Delta x}^N)-g_i^N(t,0,\mathscr{E}_0)\|^2\,\mathrm{d}A(t)\right)$$

$$+2CE\left(\frac{1}{N}\sum_{i=1}^{N}A(U)\int_{[0,U)}\|g_i^N(t,0,\mathscr{E}_0)\|^2\,\mathrm{d}A(t)\right)$$

$$\le 4CL\eta_TE\left(\frac{1}{N}\sum_{i=1}^{N}\int_{[0,U)}\sup_{h<t}\|X_i^N(h)\|^2\,\mathrm{d}A(t)\right)+2CG\eta_T^2$$

(2) For any $t\in[0,\infty)$ define

$$\phi^N(t)=\frac{1}{N}\sum_{i=1}^{N}\sup_{h<t}\|X_i^N(h)\|^2.$$

Recalling (CM) (ii), we have for any stopping time $U\le\hat{T}$,

$$E(\phi^N(U-))\le 2I+2E\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{h<U}\|X_i^N(h)-K_i^N\|^2\right)$$

$$\le 8CL_{\eta_T}E\left(\int_{[0,U)}\phi^N(t-)\,\mathrm{d}A(t)\right)+2I+4CG\eta_T^2$$

Setting $K=2I+4CG\eta_T^2$, $\rho=8CL\eta_T$, it follows from Lemma 2.1 that there exists a constant $W_1\ge 0$, such that

$$E(\phi^N(\hat{T}-))\le W_1.$$

Now, using the second and third inequalities in (1), it follows for any $N\in\mathbb{N}$,

$$E\left(\frac{1}{N}\sum_{i=1}^{N}J_dA(\hat{T})\int_{[0,\hat{T})}\|g_i^N(t,X_i^N,\varphi_{\Delta x}^N)\|^2_{\text{Op}}\,\mathrm{d}A(t)\right)$$

$$\le 4CL\eta_T^2W_1+2CG\eta_T^2=W.$$

(3) Finally, let $n \in \mathbb{N}$ be arbitrarily chosen but fixed. Then,

$$E\left(\frac{1}{N}\sum_{i=1}^{N}\int_{[0,{}^nT)}\sup_{h<s}\|X_i^N(h\wedge {}^ns-)-X_i^N(h-)\|^2\,\mathrm{d}A(s)\right)$$

$$\leqslant E\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{{}^nk-1}\int_{[{}^nt_k,{}^n\tau_k)}\sup_{{}^nt_k\leqslant h<s}\|X_i^N(h-)-X_i^N({}^nt_k-)\|^2\,\mathrm{d}A(s)\right)$$

(Def. ${}^nT$)

$$\leqslant \sum_{k=0}^{{}^nk-1}E\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{{}^nt_k\leqslant h<{}^n\tau_k}\|X_i^N(h-)-X_i^N({}^nt_k-)\|^2(A({}^n\tau_k)-A({}^nt_k))\right)$$

$$\leqslant \eta_{1/n}\sum_{k=0}^{{}^nk-1}E\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{{}^nt_k\leqslant s<{}^n\tau_k}\left\|\int_{[{}^nt_k,s)}g_i^N(t,X_i^N,\varphi_{\Delta x}^N)\,\mathrm{d}Z_i^N(t)\right\|^2\right)$$

(Def. ${}^n\tau_k$)

$$\leqslant \eta_{1/n}E\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{k=0}^{{}^nk-1}J_dA({}^n\tau_k)\int_{[{}^nt_k,{}^n\tau_k)}\|g_i^N(t,X_i^N,\varphi_{\Delta x}^N)\|_{\mathrm{Op}}^2\,\mathrm{d}A(t)\right)$$

((CP)(ii))

$$\leqslant \eta_{1/n}E\left(\frac{1}{N}\sum_{i=1}^{N}J_dA(\hat{T})\int_{[0,\hat{T})}\|g_i^N(t,X_i^N,\varphi_{\Delta x}^N)\|_{\mathrm{Op}}^2\,\mathrm{d}A(t)\right)$$

$\leqslant \eta_{1/n}W$ ((2)). $\square$

Lemma 3.8. *For every $n \in \mathbb{N}$, define*

$${}^n\beta=\sup_{N\in\mathbb{N}}E\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{[0,T)}\|{}^nX_i^N(t)-X_i^N(t)\|^2 1_{\mathscr{H}}\right).$$

Then we have ${}^n\beta\to 0$ for $n\to\infty$.

Proof. By the definition of the stopping time ${}^nT$, we have for every $n\in\mathbb{N}$, $1_{\mathscr{H}\times[0,T)}\leqslant 1_{[0,{}^nT)}$. Therefore, for every n, $N\in\mathbb{N}$,

$$\frac{1}{N}\sum_{i=1}^{N}\sup\sup_{h<T}\|{}^nX_i^N(h)-X_i^N(h)\|^2 1_{\mathscr{H}}\leqslant\frac{1}{N}\sum_{i=1}^{N}\sup_{h<{}^nT}\|{}^nX_i^N(h)-X_i^N(h)\|^2. \quad (*)$$

Now let $n\in\mathbb{N}$ be arbitrarily chosen but fixed and U any stopping time, $U\leqslant{}^nT$. Then, for every $N\in\mathbb{N}$,

$$E\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{h<U}\|{}^nX_i^N(h)-X_i^N(h)\|^2\right)$$

$$=E\left(\frac{1}{N}\sum_{i=1}^{N}\sup_{h<U}\left\|\int_{[0,h)}({}^ng_i^N(s,{}^nX_i^N,\varphi_{\Delta^n}^N)-g_i^N(s,X_i^N,\varphi_{\Delta x}^N))\,\mathrm{d}Z_i^N(s)\right\|^2\right)$$

$$\leqslant E\left(\frac{1}{N}\sum_{i=1}^{N}J_dA({}^nT)\int_{[0,U)}\|{}^ng_i^N(s,{}^nX_i^N,\varphi_{\Delta^n}^N)-g_i^N(s,X_i^N,\varphi_{\Delta x}^N)\|_{\mathrm{Op}}^2\,\mathrm{d}A(s)\right)$$

((CP))

$$\leq 2C\eta_T E\left(\frac{1}{N}\sum_{i=1}^{N}\int_{[0,U)}(\|{}^ng_i^N(s,{}^nX_i^N,\varphi_{\Delta^n}^N)-{}^ng_i^N(s,X_i^N,\varphi_{\Delta^x}^N)\|^2\right.$$

$$\left.+\|{}^ng_i^N(s,X_i^N,\varphi_{\Delta^x}^N)-g_i^N(s,X_i^N,\varphi_{\Delta^x}^N)\|^2)\,\mathrm{d}A(s)\right)$$

$$\leq \underbrace{4C\eta_T L}_{=H}E\left(\int_{[0,U)}\frac{1}{N}\sum_{i=1}^{N}\sup_{t<s}\|{}^nX_i^N(t)-X_i^N(t)\|^2\,\mathrm{d}A(s)\right)$$

$$+2C\eta_T E\left(\frac{1}{N}\sum_{i=1}^{N}\int_{[0,U)}\|g_i^N({}^ns,X_i^N,\varphi_{\Delta^x}^N)-g_i^N(s,X_i^N,\varphi_{\Delta^x}^N)\|^2\,\mathrm{d}A(s)\right)$$

((CL))

$$\leq HE(\cdots)+HE\left(\frac{1}{N}\sum_{i=1}^{N}\int_{[0,U)}\left(\frac{1}{n}+\sup_{h<s}\|X_i^N(h\wedge{}^ns-)-X_i^N(h-)\|^2\right)\mathrm{d}A(s)\right)$$

((CL))

$$\leq HE\left(\int_{[0,U)}\frac{1}{N}\sum_{i=1}^{N}\sup_{h<s}\|{}^nX_i^N(h)-X_i^N(h)\|^2\,\mathrm{d}A(s)\right)+H\eta_T/n+HW\eta_{1/n}$$

(Lemma 3.7).

Define then for $t\in[0,\infty)$,

$$\phi_n^N(t)=\frac{1}{N}\sum_{i=1}^{N}\sup_{s<t}\|{}^nX_i^N(s)-X_i^N(s)\|^2.$$

Further, for $n\in\mathbb{N}$ set

$$K(n)=H(\eta_T/n+W\eta_{1/n}).$$

Finally, define $\rho=H$ and $l=\eta_T$. Then, by Lemma 2.1,

$$E(\phi_n^N({}^nT-))\leq 2K(n)\sum_{i=0}^{[2\rho l]}(2\rho l)^i$$

By Lemma 3.6, $\eta_{1/n}\to 0$ for $n\to\infty$. Therefore,

$$K(n)\to 0\quad\text{for } n\to\infty.$$

The proposition then follows directly from (*). □

This result completes the third step of our program which we summarize in the following theorem. Recall the sequence of measures $({}^n\mu)_{n\in\mathbb{N}}$ given in Definition 3.4.

Theorem 3.9. *There exists a uniquely defined Borel probability μ_∞ on D^d so that ${}^n\mu \to \mu_\infty$ for $n \to \infty$ and so that $(\Delta^\infty, \mu_\infty)$ is point convergent.*

Proof. This follows directly from Lemma 2.2, 3.8 and Theorem 1.12, 3.3. □

We now can complete the final step of our program. That is the subject of the following:

Theorem 3.10. (i) *There exists a weak solution $\bar{P}$ to* (∞).

(ii) *The D^d marginal of $\bar{P}$ is uniquely defined and equal to μ_∞.*

Proof. Recall the processes ${}^nX$, $n \in \mathbb{N}$, introduced in Definition 3.4. Define for every $n \in \mathbb{N}$ a mapping

$$ {}^nh : \Omega \to \bar{\Omega}^d, \quad \omega \mapsto (\omega, {}^nX(\omega)) $$

and the induced measure ${}^nP = {}^n\tilde{h}P$, on the canonical extension $(\bar{\Omega}^d, \bar{\mathscr{A}}^d)$. For every $n \in \mathbb{N}$, ${}^nP$ is the unique weak solution to (n, ∞). We want to apply the stability Theorem (2.8) to the sequence $({}^nP)_{n \in \mathbb{N}}$. For this purpose we introduce the predictable random functional

$$ \tilde{g} : \Omega \times [0, \infty) \times D^d \to \mathbb{R}^{d \cdot m}, \quad (\omega, t, x) \mapsto g(\omega, t, x, \mu_\infty) $$

and equation

$$ X(t) = K + \int_0^t \tilde{g}(s, X)\, \mathrm{d}Z \tag{$\tilde{\infty}$} $$

We are now in the situation described by the stability theorem. A measure $\bar{P}$ on $(\bar{\Omega}^d, \bar{\mathscr{A}}^d)$ with $\bar{P}|_{D^d} = \mu_\infty$ is a solution to $(\tilde{\infty})$ iff it is a solution to (∞). To apply Theorem 2.8, three conditions must be fulfilled. The continuity condition (Hypothesis 2.6) on $\tilde{g}$ follows from (CM) (iii). The compactness property (Hypothesis 2.5) of $({}^n\mu)_{n \in N}$ is a consequence of Theorem 3.9. What remains to be demonstrated, is that the convergence condition (Hypothesis 2.7) is fulfilled.

Let $\emptyset \neq K \subset D^d$ be compact and $(\omega, t) \in \Omega \times [0, \infty)$. We must show that the sequence of functions $({}^ng(\omega, t, \cdot)|_K)_{n \in \mathbb{N}}$ converges uniformly to $\tilde{g}(\omega, t, \cdot)|_K$. For $t \in [0, \infty)$ and $n \in \mathbb{N}$ it follows from the definition of ${}^nt$ that $|{}^nt - t| < 1/n$. Therefore, ${}^nt \to t$ for $n \to \infty$. Thus the set

$$ \tilde{K} = (\{{}^nt : n \in \mathbb{N}\} \cup \{t\}) \times K \times (\{{}^n\mu : n \in \mathbb{N}\} \cup \{\mu_\infty\}) $$

is compact in $[0, \infty) \times D^d \times M(D^d)$. It follows that the function

$$ h : \tilde{K} \to \mathbb{R}^{d \cdot m}, \quad (s, x, \lambda) \mapsto g(\omega, s, x, \lambda) $$

is uniformly continuous. Therefore

$$ \begin{aligned} &\sup_{x \in K} \|{}^ng(\omega, t, x) - \tilde{g}(\omega, t, x)\|^2 \\ &\quad = \sup_{x \in K} \|{}^ng(\omega, t, x) - g(\omega, t, x, \mu_\infty)\|^2 \\ &\quad = \sup_{x \in K} \|h({}^nt, x, {}^n\mu) - h(t, x, \mu_\infty)\|^2 \to 0 \quad \text{for } n \to \infty. \end{aligned} $$

All the conditions of Theorem 2.8 have been fulfilled. An application of this theorem then completes the proof. □

References

H. Babovsky, A convergence proof for Nanbu's Boltzmann simulation scheme, to appear in: J. Mech. Theory Appl.

A. Borde-Boussion, Stochastic demographic model: Age of a population, Stochastic Process. Appl. 35 (1990) 279-291.

A. Bose, A law of large numbers for the scaled age distribution of linear birth and death processes, Canad. J. Statist. 14 (1986) 233-244.

W. Brown and K. Hepp, The Vlasov dynamic and it's fluctuations in the 1/ N limit of interacting particles, Comm. Math. Phys. 56 (1977) 101-113.

D.A. Dawson, Critical dynamics and fluctuations for a mean field model of cooperative behavior, J. Statist. Phys. 31 (1983) 29-85.

D.A. Dawson, Asymptotic analysis of multilevel stochastic systems, preprint (1988).

D.A. Dawson and J. Gärtner, Large deviations from the McKean-Vlasov limit for weakly interacting diffusions, Stochastics 20 (1987) 247-308.

R.L. Dobrushin, Vlasov equations, Funct. Anal. Appl. 13 (1979) 115-123.

J. Dugundji, Topology (Allyn and Bacon, Boston, 1966).

R. Engelking, Outline of General Topology (North-Holland, Amsterdam, 1968).

B. Engquist and T. Hou, particle method approximation of oscillatory solutions to hyperbolic differential equations, SIAM J. Numer. Anal. 26 (1989) 289-319.

R. Ellis, Entropy, Large Deviations and Statistical Mechanics (Springer, Berlin, 1985).

W. Finnoff, Derivation of deterministic macroeconomic models as limits of stochastic microeconomic models, Ifo-Ökonometri Arbeitspapier Nr. 7, Ifo Inst. für Wirtschaftsforschung München (Munich, 1989).

W. Finnoff, Laws of large numbers for systems of stochastic processes with interaction, Dissertation, Univ. München (Munich, 1990).

J. Gärtner, On the McKean-Vlasov limit for interacting diffusions, Math. Nach. 137 (1988) 197-248.

C. Graham, The martingale problem with sticky reflection conditions and a system of particles interacting at the boundary, Ann. Inst. H. Poincaré 24 (1988) 45-72.

C. Graham and M. Metivier, System of interacting particles and nonlinear diffusion reflecting in a domain with a sticky boundary, preprint (1988).

J.M. Grandmont and P. Malgrange, Nonlinear economic dynamics, J. Econom. Theory 40 (1986) 3-12.

D.F. Griffiths and A.R. Mitchell, Stable periodic bifurcations of an explicit discretization of a nonlinear partial differential equation in reaction diffusion, IMA J. Numer. Anal. 8 (1988) 435-454.

J. Grunbaum, Propagation of chaos for the Boltzmann equation, Arch. Rational Mech. Ann. 42 (1971) 323-345.

J. Jacod and J. Memin, Weak and strong solutions to stochastic differential equations, Existence and Stability. Lecture Notes in Math. No. 851 (Springer, Berlin, 1980) pp. 169-201.

J. Jacod and J. Memin, Existence of weak solutions to stochastic differential equations driven by semimartingales, Stochastics 4 (1981) 317-337.

J. Jacod and A.N. Shiryaev, Limit Theorems for Stochastic Processes (Springer, Berlin, 1987).

J.L. Kelley, General Topology (Van Nostrand, Toronto, 1955).

K. Kuratowski, Topology, Vol. I (Academic Press, New York, 1966a).

K. Kuratowski, Topology, Vol. II (Academic Press, New York, 1966b).

C. Lécot, A direct simulation Monte Carlo scheme and uniformly distributed sequences for solving the Boltzmann equation, to appear in: Computing.

C. Leonard, Une loi des grandes nombres pour des system de diffusion avec interaction et à coefficients non bornés, Ann. Inst. H. Poincaré 22 (1986) 237-262.

T. Maruyama, Stochastic Problems in Population Genetics. Lecture Notes in Biomath. No. 17 (Springer, Berlin, 1977).

D. McFadden, Econometric models of probabalistic choice, in: C.F. Manski and D. McFadden, eds., Structural Analysis of Discrete Data with Economic Applications (MIT Press, Cambridge, MA, 1981) pp. 198-272.
H.P. McKean, Propagation of chaos for a class of nonlinear parabolic equations, in: Lecture Ser. in Differential Equations No. 7 (Catholic Univ. of America, Washington, DC, 1967) pp. 41-57.
S. Meleard and S. Roelly-Coppoletta, Systeme de particules et measures-martingales: Un theorem de propagation due chaos, preprint (1988).
M. Metivier, Semimartingales (de Gruyter, New York, 1982).
M. Metivier and J. Pellaumail, Stochastic Integration (Academic Press, London, 1980).
E. Michael, On a theorem of Rudin and Klee, Proc. Amer. Math. Soc. 12 (1961) 921-926.
H. Murata, Propagation of chaos for Boltzmann-like equations of noncutoff-type in the plane, Hiroshima Math. J. 7 (1977) 479-515.
M. Nagasawa, Segregation of population in an environment, J. Math. Biol. 9 (1980) 213-235.
M. Nagasawa and H. Tanaka, Propagation of chaos for diffusing particles of two types with a singular mean field interaction, Z. Wahrsch. Verw. Gebiete 71 (1987a) 213-235.
M. Nagasawa and H. Tanaka, Diffusion with interactions and collisions between coloured particles and propagation of chaos, Z. Wahrsch. Verw. Gebiete 74 (1987b) 161-198.
M. Nagasawa and H. Tanaka, On the propagation of chaos for diffusion processes with drift coefficients not of average form, Tokyo J. Math. 10 (1987c) 403-418.
G. Nappo and E. Orlandi, Limit laws for a coagulation model of interacting random particles, Ann. Inst. H. Poincaré 24 (3) (1988) 319-344.
K. Oelschläger, Grenzwertsätze für zwei Modelle von zufällige Zeitentwicklungen für Systeme schwach miteinander wechselwirkende Teilchen, Dissertation, Univ. Heidelberg (Heidelberg, 1982).
K. Oelschläger, A martingale approach to the law of large numbers for weakly interacting stochastic processes, Ann. Probab. 12 (1984) 458-479.
M. Scheutzow, periodic behavior of the stochastic Brusselator in the mean field limit, Probab. Theory Rel. Fields 72 (1986) 425-429.
A. Schief, Topologische Eigenshaften Masswertiger Abbildungen und Stochastische Anwendungen, Dissertation, Univ. München (Munich, 1986).
T. Seidman, A new algorithmic model for the transient semiconductor problem, in R. Bank, R. Bulirsch and K. Merten, eds., Mathematical Modelling and Simulation of Electrical Circuits and Semiconductor Devices (Birkhäuser, Berlin, 1988) pp. 141-152.
T. Shiga, An interacting system in population genetics, J. Math. Kyoto Univ. 19 (1980) 213-242.
H. Spohn, Kinetic equations from Hamiltonian dynamics: Markovian limits, Rev. Modern Phys. 53 (1980) 569-615.
A.S. Sznitman, Equations de type Boltzmann spaticielement homogenés, Z. Wahrsch. Verw. Gebiete 66 (1984a) 559-593.
A.S. Sznitman, Nonlinear reflecting diffusion processes and the propagation of chaos and fluctuations associated, J. Funct. Anal. 56 (1984b) 311-336.
A.S. Sznitman, A propagation of chaos result for Burger's equation, Probab. Theory Rel. Fields 71 (1986) 581-613.
Y. Tamura, Free energy and the convergence of distributions of diffusion processes of McKean type, J. Fac. Sci. Univ. Tokyo 34 (1987) 443-484.
H. Tanaka, Limit theorems for certain processes with interaction, in: K. Itô, ed., Stochastic Analysis (North-Holland, New York, 1984) pp. 469-488.
F. Topsøe, Topology and Measure, Lecture Notes in Math. No. 133 (Springer, Berlin, 1970).
F.J.S. Wong, Limit theorems for age and density dependent stochastic population models, J. Math. Biol. 2 (1975) 373-400.