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The Application of Centre Manifolds to Amplitude Expansions. II. Infinite Dimensional Problems

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The theory of centre manifolds for infinite dimensional systems is described, with emphasis on the practical computational aspects of applying the theory to near-critical problems, and in particular to computation of the centre manifold. The calculations are illustrated by detailed analyses of two specific problems.

1. INTRODUCTION

This is the second of two papers dealing with the theory of centre manifolds and its application to critical and near-critical problems. In the first[1] we described in general terms the type of results one may expect to obtain using invariant manifold theory, and also their relationships to methods of analysis using amplitude expansions based upon multiple time scales. In summarizing the main abstract results and illustrating their application to concrete problems, we restricted attention to systems of ordinary differential equations. In this paper we present corresponding results for infinite dimensional problems, and analyse in detail two non-trivial problems arising from partial differential equations.

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[1] Carr and Muncaster [3], hereafter referred to as [I].

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2. General Theory

Let Z be a Banach space with norm $\| \|$, and consider the abstract equation

$$\dot{u} = Cu + F(u), \qquad u(0) \in Z. \tag{2.1}$$

Let C be the infinitesimal generator of a strongly continuous semigroup $S(t)$ on Z, and let F be of class C^2 from Z to Z with $F(0) = 0$ and $DF(0) = 0$ (D denotes a Frechet derivative). These conditions ensure that (2.1), considered as an ordinary differential equation on the Banach space Z, has a unique solution in some maximal time interval.

We are interested in cases in which $Z = X \oplus Y$, where X is a finite-dimensional C-invariant subspace, Y is a closed subspace, and if $U(t)$ is the restriction of $S(t)$ to Y, then Y is $U(t)$-invariant for $t \geqslant 0$. Let P: $Z \to X$ be the projection on X along Y, and set $A \equiv PC$, $B \equiv (I - P)C$, $f(x, y) = PF(x + y)$, and $g(x, y) = (I - P) F(x + y)$. Then (2.1) becomes

$$\begin{aligned} \dot{x} &= Ax + f(x, y), \qquad x(0) \in X, \\ \dot{y} &= By + g(x, y), \qquad y(0) \in Y. \end{aligned} \tag{2.2}$$

This is the *canonical* form of (2.1) corresponding to such a decomposition of Z.

Definition 1. *A set $M \subset Z$ is an* invariant *manifold for* (2.2) *if for any solution* $(x(t), y(t))$, $(x(0), y(0)) \in M$ *implies that for some* $T > 0$, $(x(t), y(t)) \in M$ *for all* $t \in [0, T]$.

The main results concerning the existence, asymptotic status, and approximation of centre manifolds are the following:

Theorem 1 (Existence). *Let the real parts of the eigenvalues of A be zero, and assume there are positive constants a and b for which* $\|U(t)\| \leqslant ae^{-bt}$. *Then there exists an invariant manifold* $y = h(x)$, $|x| < \delta$, *for* (2.2), *where h has a Lipschitz continuous derivative,* $h(0) = 0$, *and* $Dh(0) = 0$. *If D^2F is uniformly continuous on a neighbourhood of the origin, then D^2h exists and is uniformly continuous.*

We call this manifold a *centre* manifold for (2.2). The flow on it is governed by the following ordinary differential equation on X:

$$\dot{u} = Au + f(u, h(u)), \qquad u(0) \in X, \tag{2.3}$$

and in terms of it $(x, y) = (u, h(u))$ is a solution of (2.2).

THEOREM 2 (ASYMPTOTIC STATUS). (i) *Let the zero solution of* (2.3) *be stable* (*asymptotically stable*) (*unstable*). *Then the zero solution of* (2.2) *is stable* (*asymptotically stable*) (*unstable*). (ii) *Let the zero solution of* (2.3) *be stable. Then for each solution* $(x(t), y(t))$ *of* (2.2) *with* $\|(x(0), y(0))\|$ *sufficiently small, there exists a solution* $u(t)$ *of* (2.3) *such that as* $t \to \infty$,

$$\begin{aligned} x(t) &= u(t) + O(e^{-\gamma t}), \\ y(t) &= h(u(t)) + O(e^{-\gamma t}), \end{aligned} \tag{2.4}$$

where γ *is a positive constant.*

Define an operator N by

$$N\phi(x) = D\phi(x)[Ax + f(x, \phi(x))] - B\phi(x) - g(x, \phi(x)), \tag{2.5}$$

the domain of N being the set of all functions ϕ of class C^1 from a neighbourhood of the origin in X into Y such that for each x in this neighbourhood $\phi(x)$ lies in the domain of B. Carr and Al-amood [2b, Remark 3] have proved that $h(x)$ does lie in the domain of B for all x near zero, and so by invariance h is a solution of the equations $Nh = 0$, $h(0) = 0$, and $Dh(0) = 0$. These are the only conditions h must satisfy since, given any h satisfying them and any solution u of (2.3) for this h, $(x, y) = (u, h(u))$ is a solution of (2.2), and the set of all such solutions u forms an invariant manifold.

THEOREM 3 (APPROXIMATION). *Let* ϕ *lie in the domain of* N *and satisfy* $\phi(0) = 0$ *and* $D\phi(0) = 0$. *If for some* $q > 1$, $N\phi(x) = O(|x|^q)$ *as* $x \to 0$, *then* $h(x) = \phi(x) + O(|x|^q)$ *as* $x \to 0$.

These three theorems parallel those for ordinary differential equations described in [I]. Remarks 1–4 in that paper apply here also, and should be noted in applications, particularly of Theorem 2, to specific problems.

Because the real parts of the eigenvalues of A are zero, the system (2.2) is *critical. Near-critical* systems are ones involving a parameter $\varepsilon \in \mathbb{R}^p$ such that for $\varepsilon \neq 0$, $A = A(\varepsilon)$ has fewer eigenvalues with zero real part than $A(0)$. Systems of this type are very common in applications, and they can be reduced to the critical case by enlarging the space Z so as to include ε as an additional dependent variable. This is done by adjoining to (2.1) the equation $\dot{\varepsilon} = 0$ and analysing this new system on $\hat{Z} \equiv Z \times \mathbb{R}^p$. Then $\hat{Z} = \hat{X} \oplus Y$, where $\hat{X} \equiv X \times \mathbb{R}^p$, and the new canonical form consists of (2.2) and $\dot{\varepsilon} = 0$. To it Theorems 1–3 may be applied directly. This technique is described in greater detail in Section 3 of [I]. For near-critical systems the following modified version of Theorem 3 is often more useful:

THEOREM 4 (APPROXIMATION). *Let ϕ lie in the domain of N and satisfy $\phi(0,0)=0$, $D_x\phi(0,0)=0$, and $D_\varepsilon\phi(0,0)=0$. If for some $q>1$ and $r\geqslant 1$, $N\phi(x,\varepsilon)=O(|x|^q(1+|\varepsilon|^r))$ as $(x,\varepsilon)\to 0$, then $h(x,\varepsilon)=\phi(x,\varepsilon)+O(|x|^q(1+|\varepsilon|^r))$ as $(x,\varepsilon)\to 0$.*

Proofs of these four main theorems, as well as numerous applications, may be found in the article of Carr [1] (cf. also Carr and Al-amood [2a, b]). Certain generalizations and specializations have also been considered in the literature. Proof of the existence of a centre manifold for equations which generate a nonlinear semigroup $T(t,x)$ such that $x\to T(t,x)$ is smooth may be found in Marsden and McCracken [8]. Theorems 1, 2(i), and 3 have been proved for a class of semilinear parabolic equations by Henry [7, Chapter 6]. A presentation of centre manifold theory with particular emphasis on Hopf bifurcation can be found in the work of Hassard *et al.* [6].

3. EXAMPLE: A SEMILINEAR WAVE EQUATION

Consider the semilinear wave equation

$$\begin{aligned} w_{tt}+w_t-w_{ss}+\alpha f(w)&=0,\\ w(0,t)=w(\pi,t)&=0, \end{aligned} \tag{3.1}$$

for a real-valued function $w=w(s,t)$, $(s,t)\in(0,\pi)\times(0,\infty)$. The function f is of class C^3 with $f(z)=z+az^3+O(z^4)$ as $z\to 0$, and α and $a\neq 0$ are real constants.

First we formulate (3.1) as an ordinary differential equation on a Hilbert space. Set $Q_\alpha\equiv d^2/ds^2-\alpha$, $D(Q_\alpha)=H^2(0,\pi)\cap H_0^1(0,\pi)$. Then Q_0 is positive and self-adjoint, and $D(Q_0^{1/2})=H_0^1(0,\pi)$. Let Z be the Hilbert space $H_0^1(0,\pi)\times L^2(0,\pi)$. Then (3.1) can be written in the form

$$\dot{u}=Cu+F(u),\qquad u(0)\in Z, \tag{3.2}$$

with $Cu=C(u^1,u^2)^{\mathrm{T}}\equiv(u^2,Q_\alpha u^1-u^2)^{\mathrm{T}}$, $F(u)\equiv(0,g(u^1))^{\mathrm{T}}$, and $g(z)\equiv -a(f(z)-z)$. Since C is the sum of a skew self-adjoint operator and a bounded operator, it generates a strongly continuous semigroup on Z. Also each $u^1\in H_0^1(0,\pi)$ is continuous, so $F\colon Z\to Z$. Moreover F is of class C^3 on Z with $F(0)=0$ and $DF(0)=0$, so (3.2) has a unique solution on a maximal time interval.

To obtain a decomposition $Z=X\oplus Y$, we look first at the spectrum of C. By straightforward calculation this spectrum consists only of eigenvalues, which in terms of $\varepsilon\equiv -1-\alpha$ can be written $\lambda_1(\varepsilon)=[-1+(1+4\varepsilon)^{1/2}]/2$, $\lambda_2(\varepsilon)=-1-\lambda_1(\varepsilon)$, and $\lambda_{n+1}^{\pm}(\varepsilon)=[-1\pm(1+4(1+\varepsilon-n^2))^{1/2}]/2$ for $n\geqslant 2$.

Corresponding to each eigenvalue there is a one-dimensional eigenspace. For $\lambda_1(\varepsilon)$ it is spanned by $e_{1\varepsilon}(s)=(1,\lambda_1(\varepsilon))^{\mathrm{T}}\sin s$, for $\lambda_2(\varepsilon)$ by $e_{2\varepsilon}(s)=(1,\lambda_2(\varepsilon))^{\mathrm{T}}\sin s$, and for $\lambda_{n+1}^{\pm}(\varepsilon)$ by $(1,\lambda_{n+1}^{\pm}(\varepsilon))^{\mathrm{T}}\sin ns$, $n\geqslant 2$.

Since $\lambda_1(0)=0$, (3.2) is near-critical in a neighbourhood of $\varepsilon=0$. We will choose our decomposition of Z so that we can use centre manifold theory to analyse small solutions of (3.2) near $\varepsilon=0$. Set $X_\varepsilon=\operatorname{span}\{e_{1\varepsilon}\}$, $Y_{1\varepsilon}=\operatorname{span}\{e_{2\varepsilon}\}$, $Y_{2\varepsilon}=(X_\varepsilon\oplus Y_{1\varepsilon})^{\perp}$. Then $Z=X_\varepsilon\oplus Y_\varepsilon$, where $Y_\varepsilon=Y_{1\varepsilon}\oplus Y_{2\varepsilon}$. The projection P_ε of Z onto X_ε along Y_ε is

$$P_\varepsilon\begin{pmatrix}u^1\\u^2\end{pmatrix}=\frac{(\overline{u^2}-\lambda_2(\varepsilon)\,\overline{u^1})}{(\lambda_1(\varepsilon)-\lambda_2(\varepsilon))}e_{1\varepsilon}, \tag{3.3}$$

an overbar denoting the weighted average

$$\bar{w}\equiv\frac{2}{\pi}\int_0^\pi w(\sigma)\sin\sigma d\sigma. \tag{3.4}$$

Write $u\in Z$ in the form $xe_{1\varepsilon}+y$, $x\in\mathbb{R}$ and $y\in Y_\varepsilon$. In terms of x and y the canonical form of (3.2) is

$$\begin{aligned}\dot{x}&=\lambda_1(\varepsilon)x+\langle e_{1\varepsilon},P_\varepsilon F(xe_{1\varepsilon}+y)\rangle/\langle e_{1\varepsilon},e_{1\varepsilon}\rangle,\\ \dot{y}&=B_\varepsilon y+(I-P_\varepsilon)\,F(xe_{1\varepsilon}+y),\\ \dot{\varepsilon}&=0,\end{aligned} \tag{3.5}$$

where $B_\varepsilon\equiv(I-P_\varepsilon)C$ and we have adjoined to (3.2) already the equation $\dot{\varepsilon}=0$. $\langle\cdot,\cdot\rangle$ denotes the inner product in Z.

By Theorem 1, (3.5) has a centre manifold $y=h(x,\varepsilon)$, $|x|<\delta_1$, $|\varepsilon|<\delta_2$, and the flow on this manifold is governed by the system of ordinary differential equations

$$\begin{aligned}\dot{x}&=\lambda_1(\varepsilon)x+\langle e_{1\varepsilon},P_\varepsilon F(xe_{1\varepsilon}+h(x,\varepsilon))\rangle/\langle e_{1\varepsilon},e_{1\varepsilon}\rangle,\\ \dot{\varepsilon}&=0.\end{aligned} \tag{3.6}$$

To approximate h, set

$$\begin{aligned}N\phi(x,\varepsilon)=D\phi(x,\varepsilon)[\lambda_1(\varepsilon)x+\langle e_{1\varepsilon},P_\varepsilon F(xe_{1\varepsilon}+\phi(x,\varepsilon))\rangle/\langle e_{1\varepsilon},e_{1\varepsilon}\rangle]\\ -B_\varepsilon\phi(x,\varepsilon)-(I-P_\varepsilon)\,F(xe_{1\varepsilon}+\phi(x,\varepsilon)).\end{aligned} \tag{3.7}$$

Using (3.3) and the definition of F we see that

$$\begin{aligned}(I-P_\varepsilon)\,F(u)&=-\overline{g(u^1)}\sin s/(\lambda_1(\varepsilon)-\lambda_2(\varepsilon))\\ &=g(u^1)-\lambda_1(\varepsilon)\,\overline{g(u^1)}\sin s/(\lambda_1(\varepsilon)-\lambda_2(\varepsilon)),\end{aligned} \tag{3.8}$$

and since $g(z)$ is at least cubic in z it follows from (3.7) and (3.8) that

$$N(0)(x, \varepsilon) = -(I - P_\varepsilon) F(xe_{1\varepsilon}) = O(|x|^3(1 + |\varepsilon|)) \tag{3.9}$$

as $(x, \varepsilon) \to 0$. Thus, by Theorem 4 we find that $h(x, \varepsilon) = O(|x|^3(1 + |\varepsilon|))$ as $(x, \varepsilon) \to 0$. Therefore the second term on the right-hand side of $(3.6)_1$ becomes

$$\begin{aligned} \frac{\langle e_{1\varepsilon}, P_\varepsilon F(xe_{1\varepsilon} + h)\rangle}{\langle e_{1\varepsilon}, e_{1\varepsilon}\rangle} &= \overline{g(x \sin s + O(|x|^3(1 + |\varepsilon|)))}/(\lambda_1(\varepsilon) - \lambda_2(\varepsilon)) \\ &= ax^3 \overline{\sin^3 s} + O(|x|^3(|x| + |\varepsilon|)). \end{aligned} \tag{3.10}$$

Since $\overline{\sin^3 s} = \frac{3}{4}$, (3.6) reduces finally to

$$\begin{aligned} \dot{x} &= \lambda_1(\varepsilon)x + \tfrac{3}{4}ax^3 + O(|x|^3(|x| + |\varepsilon|)), \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{3.11}$$

In analysing the behaviour of small solutions of this system we consider separately the cases $a < 0$ and $a > 0$. If $a < 0$ the zero solution of (3.11) is stable, so the representation of solutions given by (2.4) of Theorem 2 applies here. Hence if ε is small and negative the zero solution of (3.1) is asymptotically stable. If ε is small and positive, the unstable manifold of the origin consists in two stable orbits connecting the origin to two small equilibria.

Suppose now that $a > 0$. When (3.11) is considered as a dynamical system on $\{(x, \varepsilon) \mid \varepsilon < 0\}$, the zero solution is asymptotically stable. Hence by Theorem 2 (cf. Carr and Al-amood [2a, Remark 5]), if $\varepsilon < 0$ the zero solution of (3.1) is asymptotically stable. Finally, if $\varepsilon > 0$ the zero solution of (3.11), and hence also of (3.1), is unstable.

4. Example: A Semilinear Parabolic System

The system

$$\begin{aligned} v_t &= d_1 v_{ss} + v[-(v - \varepsilon_1)(v - 1) - \alpha w], \\ w_t &= d_2 w_{ss} + w[-\varepsilon_2 - \beta w + \alpha v], \\ v_s(0, t) &= v_s(\pi, t) = w_s(0, t) = w_s(\pi, t) = 0, \end{aligned} \tag{4.1}$$

for two scalar functions $v = v(s, t)$ and $w = w(s, t)$, $(s, t) \in (0, \pi) \times (0, \infty)$, has been studied by Conway and Smoller [5] as a model for the evolution in time of two populations undergoing both interaction and spatial diffusion. The constants $d_1 > 0$, $d_2 > 0$, ε_1, ε_2, α, and β represent measurable quantities of the two populations such as rates of spatial diffusion, birth, death, and predation. We will use centre manifold theory to examine small solutions of (4.1) in the near-critical case in which ε_1 and ε_2 are small.

Set $Q_i = d_i d^2/ds^2 - \varepsilon_i$, $i = 1, 2$, with $D(Q_1) = D(Q_2) = \{v \in H_0^3(0, \pi) \mid v_s(0) = v_s(\pi) = 0\}$. Then Q_1 and Q_2 are self-adjoint, and when $\varepsilon = (\varepsilon_1, \varepsilon_2) = 0$, both are positive. We analyse (4.1) on the Hilbert space $Z = H_0^1(0, \pi) \times H_0^1(0, \pi)$. Setting

$$\begin{aligned} C &\equiv \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \\ F(u) = F\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} &\equiv \begin{pmatrix} u^1[(1+\varepsilon_1)u^1 - \alpha u^2 - (u^1)^2] \\ u^2[-\beta u^2 + \alpha u^1] \end{pmatrix}, \end{aligned} \tag{4.2}$$

we may write (4.1) as the ordinary differential equation

$$\dot{u} = Cu + F(u), \qquad u(0) \in Z. \tag{4.3}$$

By standard theory of elliptic systems, C generates a strongly continuous semigroup on Z. Since each $u \in H_0^1(0, \pi) \times H_0^1(0, \pi)$ is continuous and F has polynomial components, F: $Z \to Z$. Also F is C^∞ on Z and $F(0) = 0$ and $DF(0) = 0$, so (4.3) has a unique solution on a maximal time interval.

The spectrum of C consists only of eigenvalues $\lambda_1(\varepsilon) = -\varepsilon_1$, $\lambda_2(\varepsilon) = -\varepsilon_2$, $\lambda_{n1}(\varepsilon) = -\varepsilon_1 - d_1 n^2$, and $\lambda_{n2}(\varepsilon) = -\varepsilon_2 - d_2 n^2$, $n \geqslant 1$. The corresponding eigenspaces, all one-dimensional, are spanned by $e_1(s) = (1, 0)^T$, $e_2(s) = (0, 1)^T$, $e_{n1}(s) = (\cos ns, 0)^T$, and $e_{n2}(s) = (0, \cos ns)^T$, respectively. Thus $\lambda_1(0) = \lambda_2(0) = 0$, so for ε near zero (4.1) is near critical.

Set $X = \text{span}\{e_1, e_2\}$ and $Y = X^\perp$, so that $Z = X \oplus Y$. The projection P of Z onto X along Y is simply the orthogonal projection

$$Pu = \overline{u^1}\, e_1 + \overline{u^2}\, e_2, \tag{4.4}$$

the overbar denoting an average:

$$\bar{v} = \frac{1}{\pi} \int_0^\pi v(\sigma)\, d\sigma. \tag{4.5}$$

For each $u \in Z$ we write $u = x_1 e_1 + x_2 e_2 + y$ with $x_1, x_2 \in \mathbb{R}$ and $y \in Y$. In terms of x_1, x_2, and y the canonical form of (4.3) is

$$\begin{aligned} \dot{x}_1 &= \lambda_1(\varepsilon) x_1 + \overline{F^1(x_1 e_1 + x_2 e_2 + y)}, \\ \dot{x}_2 &= \lambda_2(\varepsilon) x_2 + \overline{F^2(x_1 e_1 + x_2 e_2 + y)}, \\ \dot{y} &= By + (I - P)\, F(x_1 e_1 + x_2 e_2 + y), \\ \dot{\varepsilon} &= 0, \end{aligned} \tag{4.6}$$

where $B \equiv (I - P) C$.

By Theorem 1, (4.6) has a centre manifold $y = h(x, \varepsilon)$, $|x| < \delta_1$, $|\varepsilon| < \delta_2$, and the flow on this manifold is governed by the system obtained by substituting $y = h(x, \varepsilon)$ into $(4.6)_{1,2,4}$. To simplify this system we first approximate h. Set

$$N\phi(x, \varepsilon) = D_{x_1}\phi(x, \varepsilon)[\lambda_1(\varepsilon)x_1 + \overline{F^1(x_1e_1 + x_2e_2 + \phi(x, \varepsilon))}]$$
$$+ D_{x_2}\phi(x, \varepsilon)[\lambda_2(\varepsilon)x_2 + \overline{F^2(x_1e_1 + x_2e_2 + \phi(x, \varepsilon))}]$$
$$- B\phi(x, \varepsilon) - (I - P)\,F(x_1e_1 + x_2e_2 + \phi(x, \varepsilon)) \tag{4.7}$$

for functions ϕ of class C^1 lying in the domain of B. We note that

$$(I - P)\,F(x_1e_1 + x_2e_2) = 0 \tag{4.8}$$

for all $x_1, x_2 \in \mathbb{R}$. This follows from the facts that e_1 and e_2 are constant vectors and $\bar{v} = v$ for any constant function v. However, (4.8) implies that the system of equations $Nh = 0$, $h(0, 0) = 0$, $D_x h(0, 0) = D_\varepsilon h(0, 0) = 0$, which governs h has the exact solution $h(x, \varepsilon) \equiv 0$. Moreover, Theorem 3 shows that any other solution of this system satisfies $h(x, \varepsilon) = O((|x| + |\varepsilon|)^n)$ for all $n > 0$ as $(x, \varepsilon) \to 0$.

If we set $y = h(x, \varepsilon) \equiv 0$ in $(4.6)_{1,2,4}$ and use the definition of F, we obtain the following equations for the flow on the centre manifold:

$$\begin{aligned} \dot{x}_1 &= x_1[-(x_1 - \varepsilon_1)(x_1 - 1) - \alpha x_2], \\ \dot{x}_2 &= x_2[-\varepsilon_2 - \beta x_2 + \alpha x_1], \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{4.9}$$

These, apart from the added equation $\dot{\varepsilon} = 0$, are precisely what we would obtain from (4.1) by setting $d_1 = d_2 = 0$ and ignoring the boundary conditions, and so the behaviour of small solutions of (4.1) when ε is near zero is essentially unaffected by diffusion. The replacement of a system of partial differential equations with a system of ordinary differential equations by treating all spatial derivatives as negligible is often termed the "lumped parameter assumption," and Conway *et al.* [4] have examined the validity of this assumption for certain semilinear parabolic systems by use of energy estimates. The analysis here justifies the lumped parameter assumption for (4.1) since the corresponding system of ordinary differential equations, namely, (4.9), governs the flow on a centre manifold for (4.1).

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