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Complexity of conditional colorability of graphs☆

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ABSTRACT

For positive integers k and r, a conditional (k, r)-coloring of a graph G is a proper k-coloring of the vertices of G such that every vertex v of degree $d(v)$ in G is adjacent to vertices with at least $\min\{r, d(v)\}$ different colors. The smallest integer k for which a graph G has a conditional (k, r)-coloring is called the rth-order conditional chromatic number, and is denoted by $\chi_r(G)$. It is easy to see that conditional coloring is a generalization of traditional vertex coloring (the case $r = 1$). In this work, we consider the complexity of the conditional colorability of graphs. Our main result is that conditional $(3, 2)$-colorability remains *NP*-complete when restricted to planar bipartite graphs with maximum degree at most 3 and arbitrarily high girth. This differs considerably from the well-known result that traditional 3-colorability is polynomially solvable for graphs with maximum degree at most 3. On the other hand we show that $(3, 2)$-colorability is polynomially solvable for graphs with bounded tree-width. We also prove that some other well-known complexity results for traditional coloring still hold for conditional coloring.

1. Introduction

We follow the terminology and notation of [1] and consider simple connected graphs only. For a vertex v in a graph G, the *neighborhood* of v in G is $N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$, and the degree of v is $d(v) = |N_G(v)|$. Vertices in $N_G(v)$ are called *neighbors of* v. With $\delta(G)$ and $\Delta(G)$ we denote the minimum degree and maximum degree of a graph G, respectively. With P_n we denote the path on n vertices. An edge e is said to be *subdivided* when it is deleted and replaced by a path P_3 of length 2 connecting its ends, the internal vertex of this path being a newly added vertex.

For a positive integer k, a *proper* k-coloring of a graph G is a surjective mapping $c : V(G) \to \{1, 2, \ldots, k\}$ with the property that if u and v are neighbors in G, then $c(u) \neq c(v)$. The smallest k such that G has a proper k-coloring is the *chromatic number* of G, denoted by $\chi(G)$. For a subset S of $V(G)$, we use $c(S)$ to denote $\{c(u) | u \in S\}$.

In the following we will consider a generalization of traditional coloring. For integers $k > 0$ and $r > 0$, a *proper* (k, r)-coloring of a graph G is a surjective mapping $c : V(G) \to \{1, 2, \ldots, k\}$ such that both of the following two conditions hold:

(C1) if $u, v \in V(G)$ are neighbors in G, then $c(u) \neq c(v)$; and
(C2) for any $v \in V(G)$, $|c(N_G(v))| \geq \min\{d(v), r\}$.

For a given integer $r > 0$, the smallest integer $k > 0$ such that G has a proper (k, r)-coloring is called the *rth-order conditional chromatic number* of G, and is denoted by $\chi_r(G)$.

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By the definition of $\chi_r(G)$, it follows immediately that $\chi(G) = \chi_1(G)$, and so conditional coloring is a generalization of traditional graph coloring. The conditional chromatic number has a very different behavior from the traditional chromatic number. For example, for when $r = 2$, Lai et al. [8] showed that for many graphs G, $\chi_2(G-v) > \chi_2(G)$ for at least one vertex v of G, and there are graphs G for which $\chi_r(G) - \chi(G)$ may be very large.

From [7] we know that if G is a graph with $\Delta(G) \leq 2$, then for any r there exists a simple polynomial time algorithm that gives a (k, r)-coloring of G. In [9], Lai, Montgomery and Poon obtained the following upper bound on $\chi_2(G)$: if $\Delta(G) \geq 3$, then $\chi_2(G) \leq \Delta(G) + 1$. The proof of this result is very long compared with the almost trivial proof of a similar result for traditional coloring. In [7], Lai, Lin, Montgomery, Shui and Fan got many new and interesting results on conditional coloring. In the present work, we are going to investigate the complexity of deciding whether a given graph is (k, r)-colorable. We first give a simple proof that for any $k \geq 3$ and $r \geq 2$ it is *NP*-complete to decide whether a given graph is (k, r)-colorable. Then we give the main theorem in the work that conditional $(3, 2)$-colorability remains *NP*-complete when restricted to planar bipartite graphs with maximum degree at most 3 and arbitrarily high girth. This differs considerably from the well-known result that traditional 3-colorability is polynomially solvable for graphs with maximum degree at most 3. On the other hand we know from [7] that every tree is $(3, 2)$-colorable, and we will show that $(3, 2)$-colorability is polynomially solvable for graphs with bounded tree-width. We also prove that some other well-known complexity results for traditional coloring still hold for conditional coloring.

2. The complexity of conditional coloring

In this section, we shall analyze the complexity of (k, r)-colorability of graphs. We refer the reader to [4] for terminology, notation and basic results on complexity not given here.

If a connected graph G has only one vertex, then clearly $\chi_r(G) = 1$; if a connected graph G has only two vertices, then $\chi_r(G) = 2$. For any other connected graph G, we have $\chi_r(G) \geq 3$ for $r \geq 2$. But the following theorem shows that for any integers r and k with $2 \leq r < k$, the problem of deciding whether a given graph is (k, r)-colorable is *NP*-complete. Formulated as a decision problem, the (k, r)-colorability problem, denoted by (k, r)-Col, is defined as follows:

Input: A graph $G = (V, E)$ and two integers r and k with $k > r \geq 2$.

Question: Is $\chi_r(G) \leq k$?

Theorem 2.1. *For every fixed integers k and r with $2 \leq r < k$, (k, r)-Col is* NP*-complete.*

Proof. First, it is easy to see that the problem (k, r)-Col is in *NP*.

Secondly, it is known that the traditional k-colorability problem is *NP*-complete. So, in order to complete the *NP*-completeness proof, it is sufficient to reduce the traditional k-colorable problem to (k, r)-Col. We want to relate any instance G of the k-colorability problem to a graph G', such that G is k-colorable if and only if G' is (k, r)-colorable.

For each vertex v in $V(G)$, we add a new complete graph K_r and add new edges such that v and K_r form a complete graph of order $r + 1$. The resulting graph is denoted by G'. So, G' has $(r + 1)|V(G)|$ vertices, and every vertex in G' is contained in a K_{r+1}. It is easy to see that G is k-colorable if and only if G' is (k, r)-colorable. ∎

If we compare 3-colorability and $(3, 2)$-colorability, there are several interesting phenomena to note. First of all, it is well known within traditional coloring that all graphs with maximum degree 3 are 3-colorable except for K_4 (by Brook's Theorem). So the 3-colorability problem is trivial for this class of graphs. In contrast, the next theorem tells us that the problem $(3, 2)$-Col remains *NP*-complete even when restricted to planar bipartite graphs with maximum degree 3 and arbitrarily large girth. The girth condition can be interpreted as that these graphs come "as close to trees" as possible. On the other hand, from [7] we know that any tree is $(3, 2)$-colorable, so for trees the related decision problem is trivial. Moreover, we will use Monadic Second-Order Logic (MSOL) to show that the problem $(3, 2)$-Col is polynomially solvable for graphs with bounded tree-width. The latter class includes, e.g., all outerplanar graphs.

We begin with the *NP*-completeness result on $(3, 2)$-colorability.

Theorem 2.2. *The problem $(3, 2)$-Col remains* NP*-complete for planar bipartite graphs with maximum degree at most 3 and arbitrarily high girth.*

Proof. First note that the problem $(3, 2)$-Col restricted to planar bipartite graphs with maximum degree at most 3 and girth at least g for any integer g is obviously in *NP*.

Now we want to reduce a variant of the 3-colorability problem to the problem $(3, 2)$-Col for planar bipartite graphs with maximum degree at most 3 and girth at least g. We use the following restricted version 3-Col* of planar 3-colorability, in which one is given a planar graph G with $\delta(G) \geq 3$, and the question is whether G is 3-colorable. To verify that 3-Col* is *NP*-complete, we first note the following easy observation: if a graph G has a vertex v with degree less than 3, then $G - v$ is 3-colorable if and only if G is 3-colorable. One way is trivial, and for the other way we can always use a color for v that has not been used for its at most two neighbors. So, starting with a planar instance graph G for 3-colorability, one can recursively delete vertices with degree less than 3 in the current graph. If this process terminates with a trivial graph (on one vertex), then G is clearly 3-colorable; otherwise, we end up with a planar graph G^* with $\delta(G^*) \geq 3$, and such that G^* is 3-colorable if and only if G is 3-colorable. This clearly shows that 3-Col* is an *NP*-complete problem.

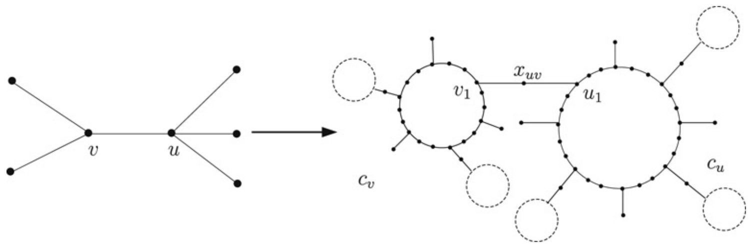

Fig. 1. The local transformation in the proof of Theorem 2.2.

To complete the proof that (3, 2)-Col is *NP*-complete for our restricted graph class, we start with a planar instance graph G for 3-Col*. We replace each vertex v of G by a cycle C_v of length $6d(v)$, with an outgoing "half-edge" at each of the positions 1, 4, 7, 10, 13, . . . along the cycle, representing twice the $d(v)$ edges in G incident with v, and all other vertices on C_v having degree 2. For every edge $uv \in E(G)$ we identify one of the half-edges at positions 1, 7, 13, . . . from each of the C_u and C_v, and we glue the two half-edges together at a newly added vertex x_{uv} with degree 2. So x_{uv} is the internal vertex of a P_2 representing the edge uv of G. For each of the half-edges at positions 4, 10, . . . we put a newly added vertex with degree 1 at the other end. We denote the resulting graph by $\tilde{G}$. The local transformation is shown in Fig. 1.

The new graph $\tilde{G}$ is clearly bipartite, since every edge of G is replaced by a P_2 while the segments of the cycles representing the vertices of G are obviously of even length equal to 6. One easily checks that $\tilde{G}$ has maximum degree 3 and girth at least 18. We can clearly push up the girth arbitrarily further by extending the segments and adding more half-edges ending in newly added vertices with degree 1, as long as the size of the new graph is polynomial in the size of G. This can all be done without destroying the planarity of the graph.

We complete the *NP*-completeness proof by showing that G is 3-colorable if and only if $\tilde{G}$ is (3, 2)-colorable.

Suppose first that $\tilde{G}$ is (3, 2)-colorable. Observe that any (3, 2)-coloring forces the same colors at the two ends of a P_3 with internal vertices with degree 2. This implies that in any (3, 2)-coloring of $\tilde{G}$ all vertices at positions 1, 4, 7, 10, 13, . . . of a C_v have the same color. We use this color on vertex v in G. Since every edge uv of G is represented in $\tilde{G}$ by a P_2 with an internal vertex with degree 2, any (3, 2)-coloring forces different colors at the ends of this P_2. So the restriction of any (3, 2)-coloring of $\tilde{G}$ to G does indeed yield a 3-coloring of G.

For the converse, assume we have a particular 3-coloring of G. For each vertex $v_i \in V(G)$ we assign the color of v_i to all vertices at positions 1, 4, 7, . . . of the C_{v_i} representing v_i in $\tilde{G}$. The middle vertex of the P_2 representing the edge v_iv_j of G in $\tilde{G}$ can receive the color which is not used for v_i or v_j in G. For each neighbor in the C_{v_i} of a vertex at position 1, 7, 13, . . . we use the third color (that is not used at v_i or the middle vertex adjacent to it). This fixes the other colors in the C_{v_i}. We can extend this to a (3, 2)-coloring by assigning suitable colors to the vertices with degree 1. ■

Alternatively, we could have modified the method given in [5] to prove the *NP*-completeness of 3-colorability, by reducing the 3-SAT problem to the problem (3, 2)-Col for graphs with maximum degree at most 3. Pushing up the girth and guaranteeing bipartiteness can then be accomplished by using arguments similar to those in the above proof. It is more tricky to cope with the planarity, though.

From [7] we know that if $\Delta(G) = 1$ or 2, $\chi_2(G)$ can be determined in polynomial time. From [9] we also know that if $\Delta(G) = 3$, then $\chi_2(G) = 3$ or 4. So by Theorem 2.2 we can get the following result.

Corollary 2.3. *Within the class of graphs with $\Delta(G) = 3$, it is* NP*-hard to determine whether $\chi_2(G) = 3$ or $\chi_2(G) = 4$.*

3. Tree-like graphs

We have seen in the previous section that (3, 2)-Col remains *NP*-complete when restricted to graphs with high girth, so that they are "locally" the same as trees. In [7] there is a simple proof of the fact that all trees are (3, 2)-colorable, implying that the corresponding decision problem is trivial for trees. Moreover, we are going to use MSOL, that is, that fragment of second-order logic where quantified relation symbols must have arity 1, to show that (3, 2)-Col is polynomially solvable for graphs with bounded tree-width.

To start with, it is well known that the following sentence, which expresses that a graph (whose edges are given by the binary relation E) can be 3-colored, is a sentence of monadic second-order logic:

$$\exists R \exists W \exists B \{\forall x\, ((R(x) \vee W(x) \vee B(x)) \wedge \neg(R(x) \wedge W(x)) \wedge \neg(R(x) \wedge B(x)) \wedge \neg(W(x) \wedge B(x))) \wedge \forall x \forall y\, (E(x, y) \Rightarrow (\neg(R(x) \wedge R(y)) \wedge \neg(W(x) \wedge W(y)) \wedge \neg(B(x) \wedge B(y))))\}$$

(the quantified unary relation symbols are R, W and B, and should be read as sets of 'red', 'white' and 'blue' vertices, respectively). Thus, in particular, there exist *NP*-complete problems that can be defined in monadic second-order logic.

A seminal result of Courcelle [3] is that on any class of graphs of bounded tree-width, every problem definable in MSOL can be solved in time linear in the number of vertices of the graph. Moreover, Courcelle's result holds not just when graphs are given in terms of their edge relation, as in the example above, but also when the domain of a structure encoding a graph G consists of the disjoint union of the set of vertices and the set of edges, as well as unary relations V and E for distinguishing the vertices and the edges, respectively, and also a binary incidence relation I which denotes when a particular vertex is incident with a particular edge (thus, $I \subseteq V \times E$). The reader is referred to [3] for more details as regards MSOL on graphs and also for the definition of tree-width which is not required here. For the proof of our claim that the (3, 2)-Col problem is solvable in linear time for graphs with bounded tree-width, it is sufficient to show the following.

Theorem 3.1. *The problem* (3, 2)*-Col can be defined in MSOL.*

Proof. We already have the sentence that expresses 3-colorability in MSOL. Let $V(v)$ denote that $v \in V$, and let $E(u, v)$ denote that $uv \in E$. (To be precise, instead of $uv \in E$, we should write $\exists e : e \in E \wedge (u, e) \in I \wedge (v, e) \in I$).

If we can write a formula of MSOL that says

a vertex with degree larger than 1 has at least two neighbors with different colors,

then we have proven the proposition by combining this for all vertices of V with the expression for 3-colorability.

It is very simple to find the proper expression for this as a logical implication $A \Rightarrow B$, where A expresses that a vertex v has at least two neighbors, whereas B expresses that such a v has at least two neighbors with different colors.

For A consider the following:

$$\exists V(u)\exists V(w)\,(u \neq w \wedge E(u, v) \wedge E(w, v))\,.$$

For B consider the following:

$$\exists V(x)\exists V(y)\,\{E(x, v) \wedge E(y, v) \wedge ((B(x) \wedge \neg B(y)) \vee (R(x) \wedge \neg R(y)) \vee (W(x) \wedge \neg W(y)))\}\,. \quad \blacksquare$$

4. Further *NP*-completeness results

Next we will briefly show how we can establish further *NP*-completeness results if we consider special classes of graphs, e.g., hamiltonian graphs, planar graphs, and claw-free graphs.

Theorem 4.1. *The problem* (3, 2)*-Col remains* NP*-complete when restricted to hamiltonian graphs with* $\Delta(G) \leq 6$.

Proof. A known result is that to determine whether a hamiltonian graph with maximum degree at most 4 is 3-colorable is *NP*-complete. Now, suppose we are given a hamiltonian graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and, without loss of generality, let $v_1v_2\ldots v_nv_1$ be a hamiltonian cycle of G. We construct a new hamiltonian graph G' as follows: For each v_i we add two new vertices x_{i1} and x_{i2} and three new edges v_ix_{i1}, $x_{i1}x_{i2}$ and $x_{i2}v_i$ (a triangle). Then, add new edges $x_{12}x_{21}$, $x_{32}x_{41}$, $x_{52}x_{61}$, …, $x_{(n-1)2}x_{n1}$ for n even; add new edges $x_{12}x_{21}$, $x_{32}x_{41}$, $x_{52}x_{61}$, …, $x_{(n-2)2}x_{(n-1)1}$ and add a new vertex u and three edges $x_{n1}u$, $x_{n2}u$ and $x_{(n-1)2}u$ for n odd.

It is easy to check that G' is a hamiltonian graph with $\Delta(G') \leq 6$. Now first suppose that G' is (3, 2)-colorable. Then restricting a (3, 2)-coloring to the vertices $v_1, v_2, \ldots, v_n$, one easily sees that it is a proper 3-coloring of G. For the converse, suppose G is 3-colorable. Then we color the vertices $v_1, v_2, \ldots, v_n$ in G' with the same color that they are colored with in G. Since there are 3 colors, the remaining vertices of G' can easily be colored properly, in such a way that we obtain a 3-coloring of G'. Since every vertex in G' is contained in a triangle, G' is 3-colorable means G' is (3, 2)-colorable. So we have shown that G is 3-colorable if and only if G' is (3, 2)-colorable. This gives the result. $\blacksquare$

Next we consider planar graphs. From [2] we know that the 3-colorability problem for planar hamiltonian graphs is *NP*-complete. With the help of this result we can prove the following theorem.

Theorem 4.2. *The problem* (3, 2)*-Col is* NP*-complete for planar hamiltonian graphs.*

Proof. Suppose we are given a planar hamiltonian graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ in which, without loss of generality, $v_1v_2\ldots v_nv_1$ is a hamiltonian cycle C_n of G. For each edge v_iv_{i+1} of C_n, we do a local transformation to get a new graph G' as follows: For each edge v_iv_{i+1} in C_n, we add four new vertices x_{i1}, x_{i2}, $y_{(i+1)1}$, $y_{(i+1)2}$ and seven new edges $x_{i1}v_i$, $x_{i2}v_i$, $x_{i1}x_{i2}$, $y_{(i+1)1}v_{i+1}$, $y_{(i+1)2}v_{i+1}$, $y_{(i+1)1}y_{(i+1)2}$, $x_{i2}y_{(i+1)1}$, and so two triangles with an edge connecting them. Then G' has $5|V(G)|$ vertices, and every vertex is in a triangle, and moreover, each "two triangles with an edge joining them" can be drawn in the local space of v_iv_{i+1} without crossing the boundary of any face, so that the new graph G' remains planar. It is easy to see that it is also hamiltonian. By the same reasoning as in the proof of Theorem 4.1, it is easy to see that G is 3-colorable if and only if G' is (3, 2)-colorable. This completes the proof. $\blacksquare$

In a similar way, we can easily show that (3, 2)-Col remains *NP*-complete for claw-free graphs. The details are omitted here.

To conclude the work, we point out that there are polynomial algorithms for solving the (k, r)-Col problem for some other special classes of graphs. From the proof of [9], one can design a polynomial algorithm to color the graph G by $\Delta(G) + 1$ colors when $\Delta(G) \geq 3$. For some classes of perfect graphs, such as triangulated (or chordal) graphs and comparability graphs, there are polynomial algorithms for coloring the graph G by $\chi(G)$ colors for traditional coloring in [6]. For these classes of graphs, one can also design polynomial algorithms to obtain colorings corresponding to the conditional coloring number, with minor adaptations of the original algorithms in [6]. The details are omitted.

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