# A Note on the Limiting Distribution of the Zeros of Padé Approximants for Some Classes of Functions

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**Abstract**—In the present paper, we deal with functions $f(z) := \sum_{n=0}^{\infty} a_n z^n$ whose coefficients satisfy a special smoothness condition. Theorems concerning the asymptotic behaviour as $n \to \infty$, $m$-fixed, of the normalized in an appropriate way Padé approximants $\pi_{n,m}$ are provided. As a consequence, results concerning the limiting distribution of the zeros are deduced.

## 1. INTRODUCTION

Let

$$f(z) := \sum_{j=0}^{\infty} a_j z^j \qquad (1)$$

be a function with $a_j \neq 0$ for all nonnegative integers $j (j \in \mathbb{N})$ large enough. We set

$$\eta_j := a_{j+1} \cdot \frac{a_{j-1}}{a_j^2}, \qquad j = j_0, j_1, \ldots . \qquad$$

The basic assumption throughout the present work is that

$$\eta_j \to 1, \qquad \text{as } j \to \infty. \qquad (2)$$

This kind of asymptotic behaviour of the Maclaurin coefficients has been introduced and studied by D. Lubinsky in [1]. More precisely, he considers a large class of functions for which the number 1 in (2) is replaced by a number $\eta$, $\eta \neq \infty$. In [1], theorems resulting from this smoothness condition with respect to Toeplitz determinants and the uniform convergence of the row in the table of classical Padé approximants are proved. Therefore, in what follows, condition (2) will be called "Lubinsky's smoothness condition for $\eta = 1$."

Further, we assume that the numbers $\eta_j$ tend to 1 in a prescribed "smooth way," namely: there exist complex numbers $\{c_i\}_{i=1}^{\infty}$ with $c_1 \neq 0$ such that for each positive integer $N$, $N > 1$, the representation

$$\eta_n = 1 + \frac{c_1}{n} + \sum_{i=2}^{N} \frac{c_i}{n^i} + o\left(n^{-N}\right). \qquad (3)$$

holds.

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Important functions to which the presented considerations in this paper may be applied are the exponential function (see [2])

$$f(z) = \exp z = \sum_{j=0}^{\infty} \frac{z^j}{j!},$$

and the Mittag-Lefler function of order $\lambda$, $\lambda > 0$ (see [3])

$$f(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma\left(1 + \dfrac{j}{\lambda}\right)}, \qquad \lambda > 0.$$

Now let $m$ be a fixed positive integer. In our further considerations, we shall assume that $f$ is holomorphic at the zero (in a neighborhood) and is not a rational function having less or equal than $m$ finite poles (multiplicities included) in $C$ (we write $f \notin \mathcal{R}_m$).

For each $n$, $n \in N$, let $\pi_{n,m}(= \pi_{n,m}(f))$ be the Padé approximant to the function $f$ of order $(n, m)$. Recall that $\pi_{n,m} = p/q$, $\deg p \leq n$, $\deg q \leq m$, $q \not\equiv 0$, where the polynomials $p$ and $q$ are determined by the condition $(f \cdot q - p)(z) = O(z^{n+m+1})$. For each pair $(n, m)$, the function $\pi_{n,m}$ always exists and is uniquely determined (see, for example, [4]). We set

$$\pi_{n,m} = \frac{P_{n,m}}{Q_{n,m}},$$

where $Q_{n,m}(0) = 1$ and both polynomials $P_{n,m}$ and $Q_{n,m}$ do not have a common divisor.

Let $D(n, m) = \det\{a_{n-j+k}\}_{j,k=1}^m$ be the Toeplitz determinant formed from the Maclaurin coefficients of the function $f$. From the nonrationality of $f$, it follows that the sequence $\Lambda$ of those positive integers $n$ for which $D(n, m) \cdot D(n, m+1) \neq 0$, is infinite (see, [4,5]) and the equality $\pi_{n,m} \equiv \pi_{k(n),m}$, where $k(n) := \max\{k, k \leq n, k \in \Lambda\}$ is valid. Without losing the generality, we shall assume that $\Lambda \equiv N$. In this case, there holds (see [5])

$$Q_{n,m}(z) = 1 + \cdots + z^m \cdot (-1)^m \frac{D(n+1, m)}{D(n, m)},$$

and

$$P_{n,m}(z) = z^m \cdot \frac{D(n, m+1)}{D(n, m)} + \cdots + d_{n,m}.$$

Denote by $R_{n,m}(u)$ the numerator of the rational Padé function associated with $f$ and normalized as follows:

$$R_{n,m}(u) := \frac{\pi_{n,m}\left(\dfrac{ua_n}{a_{n+1}}\right)}{\left(\dfrac{ua_n}{a_{n+1}}\right)^n \cdot \dfrac{D(n, m+1)}{D(n, m)}}.$$

In [6], theorems concerning the asymptotic behaviour as $n \to \infty$ of the sequence $R_{n,m}(u)$ in the case when the numbers $\eta_n$ satisfy Lubinsky's smoothness condition for an arbitrary number $\eta$, $\eta \neq \infty$ are proved. In the present paper, we confine ourselves to the case when (3) holds. Of basic importance for the forthcoming considerations is that (see [1])

$$Q_{n,m}\left(\frac{ua_n}{a_{n+1}}\right) \to (1 - u)^m, \qquad \text{as } n \to \infty.$$

THEOREM 1. *Let $m \in N$ be fixed and $f \notin \mathcal{R}_m$. Assume that $a_j \neq 0$ for $j$ large enough; assume, further that $\eta_n$ admits the expansion (3) with $\eta = 1$, $c_1 \neq O$ and $|\eta_n| \leq 1$ for all $n \in N$ sufficiently large.*

*Then*

$$R_{n,m}(u) \to \frac{u^{m+1}}{(u-1)^{m+1}}, \qquad \text{as } n \to \infty$$

*uniformly inside in $\{u, |u| > 1\}$.*

As usual, "uniformly inside" means a uniform convergence on compact subsets in the metric of Chebyshev.

From Theorem 1, we have the following corollary.

COROLLARY 1. *With the assumptions of Theorem 1, for each fixed $m \in \mathbb{N}$ and any positive $\varepsilon$, the Padé approximant $\pi_{n,m}(z)$ has no zeros in $|z| > |a_n/a_{n+1}| \cdot (1 + \varepsilon)$ for $n$ sufficiently large.*

The next result provides more precise information concerning the behaviour of the zeros of the sequence of the normalized Padé approximant $R_{n,m}(u)$ as $n \to \infty$ for the special case when the first coefficient $c_1$ in (3) is a real negative number.

THEOREM 2. (See [6]). *If $c_1 < 0$, then $u = 1$ is a limit point of zeros of $\{R_{n,m}(u)\}_{n=1}^{\infty}$.*

Set $\mathcal{A}_n(\varepsilon) := \{z, (1 - \varepsilon)|a_n/a_{n+1}| \leq |z| \leq (1 + \varepsilon)|a_n/a_{n+1}|\}$.

Combining Theorem 1 and Theorem 2, we come to the following corollary.

COROLLARY 2. *In the conditions of Theorem 2, for each fixed $m \in \mathbb{N}$, any $\varepsilon$, $0 < \varepsilon < 1$ and $n$ large enough the Padé approximant $\pi_{n,m}(z)$ has at least one zero in the annulus $\mathcal{A}_n(\varepsilon)$.*

For $n \in \mathbb{N}$, we denote by $P_n$ the set of the zeros of $R_{n,m}$. Set $P_n := \{\xi_{n,k}\}_{k=1}^n$ with the normalization $|1 - \xi_{n,k}| \leq |1 - \xi_{n,k+1}|$, $k = 1, \ldots, n - 1$. From Theorem 2, we have

$$\mathrm{dist}(P_n, 1) \to 0, \qquad \text{as } n \to \infty.$$

For any positive $\varepsilon$, denote by $\iota_n(\varepsilon)$ the number of the zeros of $\xi_{n,k}$ that lie in the disk of radius $\varepsilon$ and are centered at $u = 1$. In the present paper, we prove the following theorem.

THEOREM 3. *In the conditions of Theorem 2, for any $\varepsilon$ small enough, we have*

$$\liminf_{n \to \infty} \frac{\iota_n(\varepsilon)}{n} > 0. \tag{4}$$

From here, we have the following corollary.

COROLLARY 3. *In the conditions of Theorem 2, for each fixed $m \in \mathbb{N}$, any $\varepsilon$, $0 < \varepsilon < 1$ and $n$ large enough, the Padé approximant $\pi_{n,m}$ has at least $k_n$ zeros in the annulus $\mathcal{A}_n(\varepsilon)$, where the numbers $\iota_n$ fulfill, as $n \to \infty$, condition (4).*

The structure of the paper is as follows. First, for the sake of perfection, we state the general idea of the proof of Theorem 2; then we provide the proof of Theorem 3.

The basis of all the forthcoming considerations is the following lemma.

LEMMA. (See [6]). *In the conditions of Theorem 2, for any $n$, it is valid:*

$$R_{n,m}(u) = 1 + \sum_{j=1}^{n} b_{n,j} A_{n,m,j} u^{-j}, \tag{5}$$

*with*

$$b_{n,j} := \prod_{l=1}^{j} \eta_{n-j+l}^l. \tag{6}$$

*The asymptotic behaviour of $A_{n,j,m}$ is as follows: for $j < n/3m$*

$$A_{n,m,j} = \prod_{l=1}^{m} \frac{(j + l)}{m!} + \mathcal{N}_m(j, n), \qquad \text{as } n \to \infty,$$

*and*

$$|n \cdot \mathcal{N}_m(j, n)| < C_1(m) \cdot j^{m+1}, \qquad \text{as } n \to \infty;$$

*for $j \geq n/3m$*

$$|A_{n,m,j}| \leq c_1(m) j^{m+1},$$

*with $C_1(m)$ a positive constant not depending on $n$.*

In what follows, we shall denote by $C(\ldots)$ positive constants that do not depend on $n$.

## 2. PROOFS OF THE RESULTS

PROOF OF THEOREM 2. Recall that $m$ is fixed and $n \to \infty$.

Arguing in the same way as in [6], we shall establish that for every $\delta$ small enough there exists a positive integer $n_\delta$ such that for any $n > n_\delta$ the inequality

$$\operatorname{Re} R_{n,m}\left(e^{-2\delta}\right) \geq C(0) \cdot e^{n \cdot \alpha(\delta)} \tag{7}$$

is valid, where

$$\alpha(\delta) := \frac{\delta^2}{2d_1},$$

and $C(0)$ is a positive constant.

For convenience, we shall use the notation $c_1 := -2d_1$. In the conditions of Theorem 2, $d_1 > 0$. In [6], the validity of the following inequalities for each $n$ large enough ($n > n_0$) was established:

$$|\eta_n| \leq 1 - \frac{d_1}{n}, \tag{8}$$

and

$$|\eta_n| \leq |\eta_{n+1}|. \tag{9}$$

Both of the previous inequalities lead to

$$|b_{n,j}| \leq (|\eta_n|)^{j(j+1)/2} \leq \left(1 - \frac{d_1}{n}\right)^{j(j+1)/2}. \tag{10}$$

Let $\varepsilon$ be a fixed positive number, $\varepsilon < 1$.

In our further considerations, we assume that for $n > n_0$ the following inequalities are fulfilled:

$$\left(\frac{n}{d_1}\right)\left|\log\left(1 - \frac{d_1}{n}\right)\right| \geq 1 - \varepsilon, \tag{11}$$

and

$$|\operatorname{Im} \eta_n| \leq C(1) \cdot \operatorname{Re} \frac{\eta_n}{n^2} \tag{12}$$

for a suitable positive constant $C(1)$. Without loss of generality, we may assume that $C(1) > 1$.

In accordance with the lemma, we may also write

$$|\mathcal{N}_{1,m}(j,n)| \leq C(1) \cdot \frac{j^{m+1}}{n}, \tag{13}$$

for $j < n/3m$ and

$$|A_{n,j,m}| \leq C(1)j^{m+1}, \tag{14}$$

otherwise. Select a positive number $\delta_0$ such that

$$0 < \frac{6C(1)\, m!\delta_0}{d_1\, (1 - \varepsilon)} < \frac{1}{3}, \tag{15}$$

and set

$$d(\varepsilon) := d_1(1 - \varepsilon).$$

In what follows, we shall assume that each $n > n_0$ satisfies the inequality

$$\operatorname{Re} \eta_n \geq 1 - \frac{(2d_1 + \delta_0)}{n} > 0. \tag{16}$$

Let $\delta$ be a positive number such that $\delta < \delta_0$. Set $\mathcal{D}(\varepsilon, \delta) := 1 - 6\delta/d(\varepsilon)$. Obviously, there is an integer $n_\delta$, $n_\delta > n_0$, such that for any $n \geq n_\delta$ the inequalities

$$\operatorname{Re} \eta_n \geq 1 - \frac{(2d_1 + \delta)}{n}, \tag{17}$$

and

$$n \cdot \mathcal{D}(\varepsilon, \delta) \left| \log \left( 1 - \frac{d_1}{n\mathcal{D}(\varepsilon, \delta)} \right) \right| \leq 2 \tag{18}$$

are fulfilled. Set $j_1(\delta) := 6\delta/d(\varepsilon)$. In accordance with (10) and (11), we may write for $j > j_1(\delta) \cdot n$ that

$$|b_{n,j}| \leq e^{-3j\delta},$$

which, in view of (13),(14) and of the choice of $\delta$, implies the inequality

$$\left\| \sum_{j=j_1(\delta)n}^{n} b_{n,j} A_{n,j,m} u^{-j} \right\|_{|u|=e^{2\delta}} \leq C(\delta_0) e^{-\delta j_1(\delta) n}. \tag{19}$$

Consider the product $\prod_{l=1}^{j} (\operatorname{Re} \eta_{n-j+l})^l$. Applying (17), for $j + 1 \leq j_1(\delta) \cdot n$, we obtain

$$\prod_{l=1}^{j} (\operatorname{Re} \eta_{n-j+l})^l \geq \left( 1 - \frac{d_1}{n\mathcal{D}(\varepsilon, \delta)} \right)^{j(j+1)/2}. \tag{20}$$

For the same number $j$ we obtain by (8),(9),(12), and by the choice of $\delta$ the inequalities

$$\left| \frac{b_{n,j}}{\prod\limits_{l=1}^{j} (\operatorname{Re} \eta_{n-j+l})^l} - 1 \right| \leq C(2) \cdot \delta^2, \tag{21}$$

where $C(2) \leq 18C(1)/d_{(\varepsilon)}^2$. The choice of $\delta$ ensures that $C(2) \cdot \delta_0^2 < 1/2$.

Further, from inequality (21), we obtain

$$\left( 1 - C(2)\delta^2 \right) \prod_{l=1}^{j} (\operatorname{Re} \eta_{n-j+l})^l \leq \operatorname{Re} b_{n,j} \leq \left( 1 + C(2)\delta^2 \right) \prod_{l=1}^{j} (\operatorname{Re} \eta_{n-j+l})^l,$$

and

$$|\operatorname{Im} b_{n,j}| \leq C(2)\delta^2 \prod_{l=1}^{j} (\operatorname{Re} \eta_{n-j+l})^l.$$

Using (13),(16),(17), and the last inequalities, we obtain

$$\operatorname{Re} A_{n,j,m} \cdot \operatorname{Re} b_{n,j} - \operatorname{Im} A_{n,j,m} \cdot \operatorname{Im} b_{n,j} \geq Q_{\delta_0}(j) \cdot \prod_{l=1}^{j} (\operatorname{Re} \eta_{n-j+l})^l,$$

with

$$Q_{\delta_0}(j) := \left( 1 - C(2)\delta_0^2 \right) \cdot \left( \prod_{l=1}^{m} \frac{(j+l)}{m!} - C(1)j^m \cdot \frac{n\delta_0}{d(\varepsilon)} \right) - C(2)\delta_0^2 C(1)j^m \cdot \frac{6\delta_0}{d(\varepsilon)}.$$

As we see, $Q_{\delta_0}$ is a polynomial of degree exactly $m$ and all its coefficients are positive. Further, in view of (18),(20), and to the choice of $\delta_0$, we may write

$$\operatorname{Re} A_{n,j,m} \cdot \operatorname{Re} b_{n,j} - \operatorname{Im} A_{n,j,m} \cdot \operatorname{Im} b_{n,j} > 0. \tag{22}$$

Recall that the last inequality is valid for $n > n_\delta$ and for any $j$ with $j + 1 < j_1(\delta) \cdot n$. Now set $j_2(\delta) := \delta \cdot \mathcal{D}(\varepsilon, \delta)$ and consider $R_{n,m,\delta}(e^{2\delta}) := \sum_{j=0}^{j_2(\delta)(n+1)-1} b_{n,j} A_{n,j,m}(e^{2\delta j})$. In view of (20), for $j < j_2(\delta)(n+1) - 1$ and for $n$ large enough, we may write

$$\prod_{l=1}^{j} (\operatorname{Re} \eta_{n-j+l})^l > e^{-\delta j}.$$

Now, combining (19),(21),(22), and the last result, we obtain

$$\operatorname{Re} R_{n,m}\left(e^{-2\delta}\right) > \sum_{j=0}^{j_2(\delta)(n+1)-1} Q_{\delta_0}(j) e^{\delta j} - C(\delta_0) e^{-6(\delta)^2 n/d(\epsilon,\delta)}.$$

Inequality (7) results from here.

Now, it easily follows that the point $u = 1$ attracts, as $n \to \infty$, at least one zero of the sequence $R_{n,m}$. Before presenting the proof, we set $w = 1/u$ and $R_{n,m}(1/w) := R_{n,m}(w)$. In accordance with (4), it is valid that

$$R_{n,m}(w) = 1 + \sum_{j=1}^{n} b_{n,j} A_{n,m,j} w^j. \tag{5'}$$

Also, in view of (7), we have

$$\operatorname{Re} R_{n,m}(w)_{w=(e^{2\delta})} \geq C(0) \cdot e^{n \cdot \alpha(\delta)}. \tag{7'}$$

We introduce the notation $U_a(r)$; that is a disk of radius $r$, centered at the point $a$; further, we set $\Gamma_a(r) := \partial U_a(r)$.

We prove Theorem 2 by arguing the contrary. Suppose that $w = 1$ is not a limit point of zeros of the sequence $\{R_{n,m}(w)\}$, as $n \to \infty$; then there is a disk $U_1(e^{-\rho})$ such that $R_{n,m}(w) \neq 0$ there. Set $\tau := \log(1 + e^{-\rho})$. Set $V := U_0(1) \bigcup U_1(e^{-\rho})$ and let $X_n(w)$, be the regular branch of $(R_{n,m})^{1/n}$ determined by the condition $X_n(0) = 1$. Now select a positive number $r$ with $r < 1 - e^{-\rho}$. Obviously, the sequence $\{X_n(w)\}$ is uniformly bounded on $U_0(r)$, and hence, by the well-known result of Bernstein-Walsh, inside $V$, as well. On the other hand, Theorem 1 ensures that $X_n(w) \to 1$, uniformly inside $U_0(1)$. Thus, by the theorem of uniqueness for holomorphic functions, $X_n(w) \to 1$ uniformly inside $V$. Combining this result and ($'$) (namely that for any $\delta < \delta_0$, the inequality $X_n(e_\delta) \geq \exp(n(\delta^2)/8d_1)$ holds for $n$ large enough), we come to a contradiction.

This contradiction proves Theorem 2.

PROOF OF THEOREM 3. Preserving the notations of Theorem 2, denote now by $\xi_{n,k}$, $k = 1, \ldots, \iota_n$ the zeros of $R_{n,m}(w)$ in $U_1(e^{-\rho})$. By Theorem 2, $\iota_n \geq 1$. We shall show that

$$\liminf_{n \to \infty} \frac{\iota_n}{n} > 0. \tag{4'}$$

Select a positive number $\theta$ such that $r < 1 - e^{-\rho} \cdot e^\theta$. Set $\tau(\theta) := \log(1 + e^{-\rho+\theta})$. Without loss of generality, we may assume that the number $\tau/2$ satisfies inequality (15).

Suppose, to the contrary, that there is an infinite sequence $\Lambda$, $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n \to \infty, n \in \Lambda} \frac{\iota_n}{n} = 0. \tag{23}$$

Set

$$q_n(w) := \prod_{k=1}^{\iota_n} \left(1 - \frac{w}{\xi_{n,k}}\right),$$

and

$$\chi_n(w) := \left\{\frac{R_{n,m}}{q_n(w)}\right\}^{1/n},$$

with $\chi_n(0) = 1$.

Consider the sequence $\{\chi_n\}_{n \in \Lambda}$.

For $q_n(w)$, we have

$$\min_{w \in \Gamma_0(e^{\tau(0)})} |q_n(w)| \geq \left\{\frac{e^{-\rho}(e^{\theta} - 1)}{(1 + e^{-\rho})}\right\}^{\iota_n}.$$

On the other hand, applying the well-known Bernstein-Walsh lemma to $R_{n,m}(w)$, we get

$$\|R_{n,m}(w)\|_{\Gamma_0(\tau(\theta))} \leq \|R_{n,m}(w)\|_{U_0(r)} \cdot \frac{(1 + e^{-\rho+\theta})^n}{r^n}.$$

Combining both inequalities and (23), and applying Theorem 1, we easily obtain that the sequence $\{\chi_n\}_{n \in \Lambda}$ is uniformly bounded inside $V$ (recall that according to the geometric construction and to the choice of $\theta$, we have $V \subset U_0(\tau(\theta))$). Further, for $w \in U_0(r)$, we obviously have

$$\left\{\frac{(1 + e^{-\rho} + r)}{(1 - e^{\rho})}\right\}^{\iota_n} \geq |q_n(w)| \geq \left\{\frac{(1 - e^{-\rho} - r)}{(1 + e^{\rho})}\right\}^{\iota_n}.$$

Therefore, in view of Theorem 1 and (23), we may write

$$\chi_n \to 1, \qquad \text{as } n \to \infty, n \in \Lambda$$

on the disk $U_0(r)$. Then

$$\chi_n \to 1, \qquad \text{as } n \to \infty, n \in \Lambda \qquad (24)$$

uniformly inside the domain $V$.

Select a positive number $\varepsilon_0$ such that

$$\varepsilon_0 < \frac{e^{-\rho}}{4}.$$

Set $\Omega(\varepsilon_0) := \bigcup_{n \in \Lambda} \bigcup_{k=1}^{\iota_n} \{w, |w - \xi_{n,k}| < \varepsilon_0/\iota_n \cdot n^2\}$. Obviously,

$$mes_1(\Omega(\varepsilon_0)) < \varepsilon_0 < \frac{e^{-\rho}}{4}. \qquad (25)$$

Further, for $w \in U_1(e^{-\rho}) - \Omega(\varepsilon_0)$, we have

$$\left\{\frac{2e^{-\rho}}{(1 - e^{-\rho})}\right\}^{\iota_n} \geq |q_n(w)|. \qquad (26)$$

The choice of $\varepsilon_0$ and (25) imply the existence of a positive number $\delta$, $\delta < \tau$ such that $e^{\delta} \in U_1(e^{-\rho}) - \Omega(\varepsilon_0)$. Applying (7) to those numbers $\delta$, using (26) and (23), we conclude that $\chi(e^{\delta}) > e^{\delta^2/8d_1}$. This inequality contradicts (24).

Consequently, (4′) holds and Theorem 3 is valid.

# REFERENCES

1. D.S. Lubinsky, Uniform convergence of rows of Padé table for functions with smooth Maclaurin coefficients, *Constr. Approx.* **3**, 307–330, (1987).
2. E.B. Saff and R.S. Varga, On zeros and poles of Padé approximants to $e^z$, *Numer. Math.* **25**, 1–14, (1975).
3. A. Edrei, E.B. Saff and R.S. Varga, Zeros of sections of power series, In *Lecture Notes in Mathematics*, Vol. 1002, Springer-Verlag, Berlin, (1983).
4. O. Perron, *Die Lehre von den Kettenbruechen*, 3rd ed., Chelsea Publ., New York, (1957).
5. G.A. Baker, *Essentials of Padé Approximants*, Academic Press, New York, (1975).
6. R.K. Kovacheva and E.B. Saff, Zeros of Padé approximants for entire functions with smooth Maclaurin coefficients, *J. of Approx. Th.*, (1993) (to appear).