Forced oscillation of super-half-linear impulsive differential equations

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Abstract

By using a Picone type formula in comparison with oscillatory unforced half-linear equations, we derive new oscillation criteria for second order forced super-half-linear impulsive differential equations having fixed moments of impulse actions. In the superlinear case, the effect of a damping term is also considered.

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1. Introduction

Recently, the present authors [1] developed Sturmian comparison theory for half-linear equations of the form

\[(k(t)\varphi_\alpha(x'))' + p(t)\varphi_\alpha(x) = 0, \quad t \neq \theta_i,\]
\[\Delta(k(t)\varphi_\alpha(x')) + p_i\varphi_\alpha(x) = 0, \quad t = \theta_i, \quad (i \in \mathbb{N})\] (1.1)

where \(\varphi_\alpha(s) := |s|^{\alpha-1}s \ (\alpha > 0); \ \Delta z(t) := z(t^+) - z(t^-), \ z(t^\pm) = \lim_{\tau \to t^\pm} z(\tau); \ t \in J \equiv [t_0, \infty) \) for some fixed \(t_0 \in \mathbb{R}; \ \{\theta_i\} \) is a strictly increasing unbounded sequence of real numbers with \(\theta_1 > t_0; \ \{p_i\} \) is a sequence of real numbers; \(k, p \in \text{PLC}(t_0, \infty), k(t) > 0 \) for all \(t \geq t_0, \) where

\[\text{PLC}(J) := \{h : J \to \mathbb{R} \text{ is continuous on each interval } (\theta_i, \theta_{i+1}), \ h(\theta_i^+) \text{ exist}, \ h(\theta_i) := h(\theta_i^-) \text{ for } i \in \mathbb{N}\}\]

The space PLC\(^1\)\((J)\) is defined in the usual way.

Several oscillation criteria for (1.1) were also derived in [1]. In this work assuming the oscillation of a solution of (1.1) we aim to establish new oscillation criteria for forced super-half-linear equations of the form

\[(m(t)\varphi_\alpha(y'))' + q(t)\varphi_\beta(y) = f(t), \quad t \neq \theta_i,\]
\[\Delta(m(t)\varphi_\alpha(y')) + q_i\varphi_\beta(y) = f_i, \quad t = \theta_i, \quad (i \in \mathbb{N})\] (1.2)

where \(\beta \geq \alpha; m, q, f \in \text{PLC}(J), m(t) > 0 \) for all \(t \in J; \ \{q_i\} \) and \(\{f_i\} \) are sequences of real numbers.

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In case $\alpha = 1$, we also provide similar results for forced superlinear impulsive equations with damping terms of the form
\[
(m(t)y')' + s(t)y' + q(t)\varphi_\beta(y) = f(t), \quad t \neq \theta_i,
\]
\[
\Delta(m(t)y') + q_i\varphi_\beta(y) = f_i, \quad t = \theta_i, \quad (\beta \geq 1)
\]
(1.3)
where additionally we assume that $s \in \text{PLC}(J)$.

By a solution of (1.2), we mean a continuous function $y(t)$ defined on $J$ such that $y'$, $(m\varphi_\alpha(y'))' \in \text{PLC}(J)$ and (1.2) is fulfilled for all $t \in J$, and $\sup\{|y(t)|, t \geq t_j\} > 0$ for some $t_j \geq t_0$. It is tacitly assumed that such solutions exist. The meaning of a solution of (1.3) is understood similarly. As usual, a solution of (1.2) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. The equation is called oscillatory if every solution is oscillatory.

When the impulses are dropped, (1.2) reduces to
\[
(m(t)\varphi_\alpha(y'))' + q(t)\varphi_\beta(y) = f(t)
\]
(1.4)
for which there exist several oscillation criteria in the literature, see [2–18] and the references cited therein. In 1993, by using a Leighton comparison type argument, El-Sayed [9] was able to remove the sign condition $q(t) \geq 0$ previously imposed, and to establish sufficient conditions for oscillation of (1.4) when $\alpha = \beta = 1$, and $m(t) \equiv 1$; see also Wong [11] for a similar result. The oscillation of (1.4) when $\alpha = 1$, $\beta > 1$ can be found in [12,13]. Recently, Jaroš et al. [14] extended some of the results in [9,11–13] to (1.4) for $\beta \geq \alpha$ by deriving and using a Picone type formula for forced super-half-linear equations, see also [15,16] for some related works about (1.4).

As far as the oscillation theory of impulsive differential equations is concerned, there is very little known. To the best of our knowledge, the first article in this direction is [19]. For more results, see [1,18–24] and the references cited therein. It should be noted that almost all impulsive differential equations considered in the literature so far have been free of forcing terms. In this paper, we aim to establish oscillation criteria for forced super-half-linear impulsive differential equations of the form (1.2) and for forced superlinear impulsive equations with damping of the form (1.3). As special cases, we recover some results in [1,9,11,12,14,18]. Moreover, the oscillation criteria derived for (1.3) are new even if the impulses are absent, i.e., for
\[
(m(t)y')' + s(t)y' + q(t)\varphi_\beta(y) = f(t), \quad \beta \geq 1.
\]
(1.5)
Throughout the article, we use the convention that $0^0 = 1$.

2. Oscillation of forced super-half-linear equations

For convenience we define the differential operators
\[
L_\alpha[u] = (k(t)\varphi_\alpha(u'))' + p(t)\varphi_\alpha(u), \quad t \neq \theta_i,
\]
(2.1)
\[
L_\alpha[u] = \Delta(k(t)\varphi_\alpha(u')) + p_i\varphi_\alpha(u), \quad t = \theta_i.
\]
(2.2)
\[
L_{\alpha\beta}[u] = (m(t)\varphi_\alpha(u'))' + q(t)\varphi_\beta(u), \quad t \neq \theta_i,
\]
(2.3)
\[
L_{\alpha\beta}[u] = \Delta(m(t)\varphi_\alpha(u')) + q_i\varphi_\beta(u), \quad t = \theta_i.
\]
(2.4)
Let $J_0 \subseteq J$ be a nondegenerate interval. Denote by
\[
D_{L_\alpha}(J_0) = \{u : u \in C[t_0, \infty), k\varphi_\alpha(u') \in \text{PLC}^1(J_0)\}
\]
and
\[
D_{L_{\alpha\beta}}(J_0) = \{u : u \in C[t_0, \infty), m\varphi_\alpha(u') \in \text{PLC}^1(J_0)\}
\]
the domains of $L_\alpha$ and $L_{\alpha\beta}$, respectively.

In view of (2.1)–(2.4) we may rewrite (1.1) and (1.2) as
\[
L_\alpha[x] = 0, \quad t \neq \theta_i; \quad L_\alpha[x] = 0, \quad t = \theta_i
\]
and

\[ L_{\alpha\beta}[y] = f(t), \quad t \neq \theta_i; \quad I_{\alpha\beta}[y] = f_i, \quad t = \theta_i, \]

respectively.

For \( x \in D_{L_{\alpha}}(J_0) \) and \( y \in D_{L_{\alpha\beta}}(J_0) \) with \( y(t) \neq 0 \) for any \( t \in J_0 \), we may define

\[ w(t) := \frac{x(t)}{\varphi_\alpha(y(t))} \left[ \varphi_\alpha(y(t))k(t)\varphi_\alpha(x'(t)) - \varphi_\alpha(x(t))m(t)\varphi_\alpha(y'(t)) \right], \quad t \in J_0. \]

Suppressing the variable \( t \) for brevity, we see that

\[ w' = (k - m)|x'|^{\alpha+1} + (q|y|^{\beta-\alpha} - p)|x|^{\alpha+1} + mA_\alpha(x', xy'/y) \]
\[ + \frac{x}{\varphi_\alpha(y)} \left( \varphi_\alpha(y)L_\alpha[x] - \varphi_\alpha(x)L_{\alpha\beta}[y] \right), \quad t \neq \theta_i; \]
\[ \Delta w = (q_i|y|^{\beta-\alpha} - p_i)|x|^{\alpha+1} + \frac{x}{\varphi_\alpha(y)} \left( \varphi_\alpha(y)L_\alpha[x] - \varphi_\alpha(x)L_{\alpha\beta}[y] \right), \quad t = \theta_i \] (2.5)

where

\[ A_\alpha(u, v) := |u|^{\alpha+1} + \alpha|v|^{\alpha+1} - (\alpha + 1)u \varphi_\alpha(v). \]

It follows from the Hardy, Littlewood and Polya lemma [26] that \( A_\alpha(u, v) \geq 0 \) for all \( u, v \in \mathbb{R} \), with the equality holding if and only if \( u = v \).

Integrating (2.5) from \( a \) to \( b \) ([a, b] \subseteq J_0) and using (2.6), we easily obtain the following Picone type formula.

**Lemma 2.1 (Picone’s Formula).** Let \( x \in D_{L_{\alpha}}(J_0) \) and \( y \in D_{L_{\alpha\beta}}(J_0) \) with \( y(t) \neq 0 \) for any \( t \in J_0 \). If \([a, b] \subseteq J_0\), then

\[ \frac{x}{\varphi_\alpha(y)} \left[ \varphi_\alpha(y)k\varphi_\alpha(x') - \varphi_\alpha(x)m\varphi_\alpha(y') \right]_{t=a}^{t=b} \]
\[ = \int_a^b \left\{ (k - m)|x'|^{\alpha+1} + (q|y|^{\beta-\alpha} - p)|x|^{\alpha+1} + mA_\alpha(x', xy'/y) \right\} dt \]
\[ + \frac{x}{\varphi_\alpha(y)} \left( \varphi_\alpha(y)L_\alpha[x] - \varphi_\alpha(x)L_{\alpha\beta}[y] \right) \]
\[ + \sum_{a \leq \theta_i < b} \left\{ (q_i|y|^{\beta-\alpha} - p_i)|x|^{\alpha+1} + \frac{x}{\varphi_\alpha(y)} \left( \varphi_\alpha(y)L_\alpha[x] - \varphi_\alpha(x)L_{\alpha\beta}[y] \right) \right\}. \] (2.7)

The following theorem extends Theorem 2 in [14] to impulsive differential equations.

**Theorem 2.1.** Let \( x(t) \) be an oscillatory solution of (1.1) with zeros at \( \{t_n\} \), \( \lim_{n \to \infty} t_n = \infty \). Suppose that for any given \( t_\ast \geq t_0 \), there exist intervals \( I_1 = [t_1, t_{m_1}], I_2 = [t_{m_2}, t_{m_2}] \subseteq [t_\ast, \infty) \), such that

(i) \( q(t) \geq 0 \quad \forall t \in \{I_1 \cup I_2\} \setminus \{\theta_i\} \) and \( q_i \geq 0 \quad \forall i \in \mathbb{N} \) for which \( \theta_i \in I_1 \cup I_2 \); (ii) \( f(t) \bigoplus \mathbb{N} = 0 \), \( f_i \bigoplus \mathbb{N} = 0 \), \( \theta_i \bigoplus \mathbb{N} = 0 \), \( \theta_i \bigoplus \mathbb{N} = 0 \) \( \forall i \in \mathbb{N} \).

If

\[ \int_{t_{m_j}}^{t_{m_j}} \left\{ (k(t) - m(t))|x'(t)|^{\alpha+1} + (q(t) - p(t))|x(t)|^{\alpha+1} \right\} dt + \sum_{t_{m_j} \leq \theta_i < t_{m_j}} (\tilde{q}_i - p_i)|x(\theta_i)|^{\alpha+1} \geq 0, \] (2.8)

for each \( j = 1, 2 \), where

\[ \tilde{q}(t) = \beta \alpha^{-\alpha/\beta}(\beta - \alpha)^{(\alpha-\beta)/\beta} q(t)^{\alpha/\beta} f(t)^{(\beta-\alpha)/\beta}, \]
\[ \tilde{q}_i = \beta \alpha^{-\alpha/\beta}(\beta - \alpha)^{(\alpha-\beta)/\beta} q_i^{\alpha/\beta} f_i^{(\beta-\alpha)/\beta}, \]

then (1.2) is oscillatory.
Suppose for the sake of a contradiction that \( y(t) \) is a nonoscillatory solution of (1.2). We may assume that \( y(t) > 0 \) when \( t \in [t^*, \infty) \) for some \( t^* \geq t_0 \). By our assumption, there exist \( I_1, I_2 \subset [t^*, \infty) \) for which (i) and (ii) hold.

Let \( a = t_1 \) and \( b = t_{n_1} \). Since \( x(t) \) and \( y(t) \) are solutions of (1.1) and (1.2) respectively, we have \( L_\alpha[x] \equiv I_\alpha[x] \equiv 0 \), \( L_\beta[y] \equiv f(t) \) and \( I_\alpha[y] \equiv f_I \). In view of (ii), and employing (2.7), we see that

\[
\int_{t_{n_1}}^{t_{n_1+1}} \left( k-m \right) |x'|^{\alpha+1} + \left( q \right) |y|^{\beta-\alpha} - p + \frac{|f|}{|y|^{\alpha}} \right) |x|^{\alpha+1} + m A_\alpha(x', xy'/y) \, dt \]

\[
+ \sum_{t_{n_1} \leq t < t_{n_1+1}} \left( q_i |y|^{\beta-\alpha} - p_i + \frac{|f_i|}{|y|^{\alpha}} \right) |x|^{\alpha+1} = 0. \tag{2.9}
\]

Define a function \( G(u) : (0, \infty) \rightarrow (0, \infty) \) by

\[
G(u) := \lambda_1 u^{\beta-\alpha} + \frac{\lambda_2}{u^{\alpha}}, \quad \lambda_{1,2} \geq 0, \beta \geq \alpha > 0.
\]

It is not difficult to see that

\[
\min_{u \in (0, \infty)} G(u) = \beta \alpha - \alpha/\beta (\beta - \alpha)^{(\alpha-\beta)/\beta} \alpha_1^{\alpha/\beta} \alpha_2^{(\beta-\alpha)/\beta}. \tag{2.10}
\]

Taking (i) into account and using (2.10) in (2.9) yields

\[
\int_{t_{n_1}}^{t_{n_1+1}} \left( k-m \right) |x'|^{\alpha+1} + \left( \tilde{g} - p \right) |x|^{\alpha+1} + m A_\alpha(x', xy'/y) \, dt + \sum_{t_{n_1} \leq t < t_{n_1+1}} \left( \tilde{g}_i - p_i \right) |x|^{\alpha+1} \leq 0, \tag{2.11}
\]

which, in view of (2.8), results in \( A_\alpha(x', xy'/y) = 0 \), and hence \( x' = xy'/y \) for all \( t \in I_1 \). Then we see that \( x(t) \) must be a constant multiple of \( y(t) \) on \( I_1 \), which contradicts the positivity of \( y(t) \) on \( I_1 \). If \( y(t) \) is eventually negative, then by taking \( a = t_{n_2} \) and \( b = t_{m_2} \) and repeating the above arguments, we get a similar contradiction. The proof is complete. \( \square \)

**Remark 2.1.** If we set \( f(t) \equiv f_I \equiv 0 \) and \( \beta = \alpha > 0 \), then we recover Theorem 2.2 in [1].

**Example 2.1.** The generalized sine function \( S(t) \) defined by Elbert [25] is the solution of

\[
(\varphi_\alpha(x'))' + \alpha \varphi_\alpha(x) = 0, \quad \alpha > 0
\]

satisfying \( x(0) = 0 \), \( x'(0) = 1 \). The function \( S(t) \) is \( 2\pi_\alpha \) periodic and has zeros at \( i\pi_\alpha, i \in \mathbb{Z} \), where

\[
\pi_\alpha = \frac{2\pi}{\alpha + 1} \left( \sin \frac{\pi}{\alpha + 1} \right).
\]

The generalized cosine function \( C(t) \) is a solution of (2.12) satisfying \( x(0) = 1 \), \( x'(0) = 0 \). The relations \( S'(t) = C(t) \) and \( |S(t)|^{\alpha+1} + |C(t)|^{\alpha+1} = 1 \) hold. It is easy to see that \( x(t) = S(\lambda t) \) is a solution of

\[
(\varphi_\alpha(x'))' + \alpha (\lambda)^{\alpha+1} \varphi_\alpha(x) = 0, \quad \alpha > 0.
\]

We may take \( k(t) = 1 \), \( p(t) = \alpha (\lambda)^{\alpha+1} \) with \( \lambda = \pi_\alpha, \pi_i = 0 \), and consider the forced super-half-linear impulsive differential equation

\[
(\varphi_\alpha(y'))' + \gamma \sin(\pi t) \varphi_\beta(y) = \sin(2\pi t), \quad t \notin \mathcal{H},
\]

\[
\Delta(\varphi_\alpha(y')) + |S(\pi_\alpha t)|^{-\beta(\alpha+1)} \varphi_\beta(y) = \tan(\pi t), \quad t \in \mathcal{H}, \tag{2.13}
\]

where \( \mathcal{H} = \{t : t = 2i + 1/4, t = 2i + 3/4, i \in \mathbb{N}\} \), and \( \gamma \) is a positive real number.

Letting \( I_1 = [2n + 1/2, 2n + 1] \) and \( I_2 = [2n, 2n + 1/2], n \in \mathbb{N} \), we see that (i) and (ii) are satisfied, and that the conditions in (2.8) hold if

\[
\int_m^{m+1} \left[ \gamma^{\alpha/\beta} |\sin(\pi t)|^{\alpha/\beta} |\sin(2\pi t)|^{(\beta-\alpha)/\beta} - A \right] |S(\pi_\alpha t)|^{\alpha+1} \, dt \geq -1
\]
for \( m = 2n \) and \( m = 2n + 1/2 \) (\( n \) is sufficiently large), where
\[
A = \beta^{-1}a^{1+\alpha/\beta}(\beta-a)^{(\beta-\alpha)}/\beta(\pi a)^{\alpha+1}
\]

It is clear that if \( \gamma \) is sufficiently large, then the above condition holds, and hence by Theorem 2.1, (2.13) is oscillatory. For instance, when \( \alpha = 1 \) and \( \beta \geq 1 \), we see that (2.13) is oscillatory if
\[
\gamma \geq 2 \left[ \frac{(2\beta-1)(4\beta-1)\pi}{4\beta^2} \left\{ \frac{1}{4\beta} (\beta-1)^{\frac{1}{\beta}} \pi^2 - 1 \right\} \right]^\beta.
\]

3. Oscillation of forced superlinear equations with damping

In this section we consider (1.3) in connection with linear impulsive equations
\[
\mathcal{L}[x] \equiv (k(t)x')' + r(t)x' + p(t)x = 0, \quad t \neq \theta_i,
\]
\[
\mathcal{I}[x] \equiv \Delta(k(t)x') + p_i x = 0, \quad t = \theta_i,
\]
where \( r \in \text{PLC}(J) \).

Let
\[
\mathcal{L}_\beta[y] \equiv (m(t)y')' + s(t)y' + q(t)\varphi_\beta(y), \quad t \neq \theta_i,
\]
\[
\mathcal{I}_\beta[y] \equiv \Delta(m(t)y') + q_\varphi_\beta(y), \quad t = \theta_i.
\]

With \( J_0 \subseteq J \) as before, we denote by \( D_{\mathcal{L}_\beta}(J_0) \) and \( D_{\mathcal{L}}(J_0) \) the domains of \( \mathcal{L}_\beta \) and \( \mathcal{L} \), respectively.

It turns out that the condition
\[
k(t) \neq m(t) \quad \text{whenever} \quad r(t) \neq s(t)
\]
(H)
is crucial for developing a Picone type formula, see [18,27]. The following lemma provides an extension of Theorem 2.1 in [18].

**Lemma 3.1 (Picone’s Formula).** Suppose that (H) holds. Let \( x \in D_{\mathcal{L}}(J_0) \) and \( y \in D_{\mathcal{L}_\beta}(J_0) \) with \( y(t) \neq 0 \) for any \( t \in J_0 \). If \( [a, b] \subseteq J_0 \), then
\[
\int_a^b \left[ \frac{y k x' - x m y'}{y} \right] |_{t=a}^{t=b} = \int_a^b \left\{ (k-m) \left[ \left( \frac{s-r}{2(k-m)} \right) x' + \left( \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right) x \right] \right. + \frac{m}{y^2} \left( x y' - x y' - \frac{s}{2m} x y \right) + \frac{x}{y} (y \mathcal{L}[x] - x \mathcal{L}_\beta[y]) \right\} \, dt
\]
\[
+ \sum_{a \leq \theta < b} \left[ (q_\varphi_\beta(y)^{\beta-1} - p_i) x^2 + \frac{x}{y} (y \mathcal{I}[x] - x \mathcal{I}_\beta[y]) \right].
\]

**Proof.** Define
\[
w(t) = \frac{x(t)}{y(t)} \left[ y(t)k(t)x'(t) - x(t)m(t)y'(t) \right], \quad t \in J_0.
\]

It is not difficult to verify that
\[
w' = (k-m)(x')^2 + (q|y|^{\beta-1} - p)x^2 + m \left( x' - x y' \right)^2 + x^2 \frac{xy'}{y} - r x x'
\]
\[
+ \frac{x}{y} (y \mathcal{L}[x] - x \mathcal{L}_\beta[y]), \quad t \neq \theta_i;
\]
\[
\Delta w = x \mathcal{L}[x] - p_i x - \frac{x^2}{y} [\mathcal{I}_\beta[y] - q_\varphi_\beta(y)], \quad t = \theta_i.
\]

Proceeding as in [18] we obtain from (3.3) and (3.4) that
\[ w' = (k - m) \left\{ x' + \frac{(s - r)x}{2(k - m)} \right\}^2 + \left\{ q|y|^{\beta - 1} - p - \frac{(s - r)^2}{4(k - m)} - \frac{s^2}{4m} \right\} x^2 + \frac{m}{y^2} \left( x'y - xy' - \frac{s}{2m}xy \right)^2 + \frac{x}{y} (y\mathcal{L}[x] - x\mathcal{L}_\beta[y]), \quad t \neq \theta_i \]
\[ \Delta w = \{ q_i |y|^{\beta_i - 1} - p_i \} x^2 + \frac{x}{y} (y\mathcal{L}[x] - x\mathcal{L}_\beta[y]), \quad t = \theta_i. \] (3.5)

Integrating (3.5) from \( a \) to \( b \) and using (3.6), we easily obtain (3.2). \( \Box \)

**Theorem 3.1.** Let \( x(t) \) be an oscillatory solution of (3.1) with zeroes at \( \{ t_n \} \), \( \lim_{n \to \infty} t_n = \infty \). Suppose that for any given \( t_s \geq t_0 \), there exist intervals \( I_1 = [t_{n_1}, t_{m_1}] \), \( I_2 = [t_{n_2}, t_{m_2}] \subset [t_s, \infty) \) for which (i) and (ii) are satisfied. If (H) holds and
\[ \int_{t_{n_j}}^{t_{m_j}} \left\{ \widetilde{Q}(t) - p(t) - \frac{(s(t) - r(t))^2}{4(k(t) - m(t))} - \frac{s^2(t)}{4m(t)} \right\} x^2(t) + (k(t) - m(t)) \left\{ x'(t) + \frac{s(t) - r(t)}{2(k(t) - m(t))} x(t) \right\} \] \[ + \sum_{t_{n_j} \leq \theta_i < t_{m_j}} (\widetilde{Q}_i - p_i) x^2(\theta_i) \geq 0 \] (3.7)
for \( j = 1, 2 \), where
\[ \widetilde{Q}(t) = \beta(\beta - 1)^{(1 - \beta)/\beta} \left| q(t) \right|^{1/\beta} \left| f(t) \right|^{(\beta - 1)/\beta}; \]
\[ \widetilde{Q}_i = \beta(\beta - 1)^{(1 - \beta)/\beta} \left| q_i \right|^{1/\beta} \left| f_i \right|^{(\beta - 1)/\beta}, \]
then (1.3) is oscillatory.

**Proof.** The proof is similar to that of Theorem 2.1.

**Remark 3.1.** If \( f(t) \equiv f_i \equiv 0 \) and \( \beta = 1 \), then Theorem 3.1 coincides with [18, Theorem 2.3]. If we ignore the impulses and the damping term, then we recover [13, Theorem 2].

If (H) fails but \( r, s \in \text{PLC}^1(J_0) \), then we make use of “device of Picard” [27, p. 12]. The method relies on modifying \( w \) as
\[ w = \frac{x}{y} (ykx' - xmy') + x^2(r - s)/2, \quad t \in J_0 \]
to get
\[ w' \geq \left\{ \widetilde{Q} - p - \frac{1}{2}(s' - r') - \frac{s^2}{4m} \right\} x^2 + (k - m)(x')^2 + \frac{m}{y^2} \left( x'y - xy' - \frac{s}{2m}xy \right)^2, \quad t \neq \theta_i \]
\[ \Delta w \geq \left\{ \widetilde{Q}_i - p_i - \frac{1}{2}(s_i - r_i) \right\} x^2, \quad t = \theta_i. \]

Hence, we have the following alternative to Theorem 3.1.

**Theorem 3.2.** Let \( x(t) \) be an oscillatory solution of (3.1) with zeroes at \( \{ t_n \} \), \( \lim_{n \to \infty} t_n = \infty \). Suppose that for any given \( t_s \geq t_0 \), there exist intervals \( I_1 = [t_{n_1}, t_{m_1}] \), \( I_2 = [t_{n_2}, t_{m_2}] \subset [t_s, \infty) \) for which (i) and (ii) are satisfied. If \( r, s \in \text{PLC}^1(I_j) \) for \( j = 1, 2 \) and if
\[ \int_{t_{n_j}}^{t_{m_j}} \left\{ \widetilde{Q}(t) - p(t) - \frac{s'(t) - r'(t)}{2} - \frac{s^2(t)}{4m(t)} \right\} x^2(t) + (k(t) - m(t)) \left\{ x'(t) \right\} \] \[ + \sum_{t_{n_j} \leq \theta_i < t_{m_j}} \left( \widetilde{Q}_i - p_i - \frac{\Delta s(\theta_i) - \Delta r(\theta_i)}{2} \right) x^2(\theta_i) \geq 0, \] (3.8)
for \( j = 1, 2 \), then (1.2) is oscillatory.
Remark 3.2. Theorem 3.2 generalizes [2, pp. 358], [20, Corollary 1], [13, Theorem 2], [27, pp. 12].

Example 3.1. Let $\mu$ be a fixed real number. Define $\lambda = (e^\mu + 1)^{-1}$ and $v = 2(1 + \coth \mu)$. It is known [18] that $x(t) = x(t_i)$, where

$$x_i(t) = (-1)^i e^{\mu(t_i-t)} + \{e^{\mu(t_i-t)}(t_i-t + \lambda) - 1\}, \quad t \in (i + \lambda - 1, i + \lambda], (i \in \mathbb{N})$$

is an oscillatory solution (with zeros at $t_i = i, i \in \mathbb{N}$) of

$$x'' - 2\mu x' + \mu^2 x = 0, \quad t \neq i + \lambda,$$

$$\Delta x' + \nu x = 0, \quad t = i + \lambda, (i \in \mathbb{N}).$$

Furthermore, it can be shown that

$$|x(t)| \leq e \quad \text{for all} \ t \geq 0 \quad \text{and} \quad |x(\lambda + i)| = e \quad \text{for all} \ i \in \mathbb{N}. \quad (3.9)$$

Comparing with (3.1) we have $\kappa(t) = 1$, $r(t) = -2\mu$, $\rho(t) = \mu^2$, $p_i = v$, and $\theta_i = i + \lambda$.

Consider the forced superlinear impulsive differential equation

$$y'' + \frac{2}{t} y' + |\sin \pi t| |y| = \sin \pi t, \quad t \neq i + \lambda, (\beta = 2)$$

$$\Delta y' + 4i^2 y = (-1)^i, \quad t = i + \lambda, (i \in \mathbb{N}). \quad (3.10)$$

Letting $I_1 = [2n+1, 2n+2)$ and $I_2 = [2n+2, 2n+3], n \in \mathbb{N}$, we see that all conditions of Theorem 3.2 are satisfied. In particular, the integrals in (3.8) simplify to

$$\int_m^{m+1} \left(2 |\sin \pi t| - \mu^2\right) x^2(t) \, dt + (4m - v) x^2(m + \lambda), \quad m = 2n, m = 2n + 1,$$

which are nonnegative for $n$ sufficiently large. Indeed by using (3.9) we get

$$\int_m^{m+1} \left(2 |\sin \pi t| - \mu^2\right) x^2(t) \, dt + (4m - v) x^2(m + \lambda) \geq 2 \int_m^{m+1} |\sin \pi t| x^2(t) \, dt + (4m - \mu^2 - v) e^2 \geq 0.$$

Thus, (3.10) is oscillatory.

References