Discrete Applied Mathematics 18 (1987) 305–308
North-Holland

NOTE

ON A CONJECTURE OF FOULDS AND ROBINSON ABOUT DELTAHEDRA

Qi NING
Mathematics Department, Shandong University, Jinan, China

Received 16 June 1986
Revised 25 November 1986

In a paper by Foulds and Robinson in Discrete Appl. Math. 1 (1979) 75–87, it is proved that any two deltahedra of the same order can be transformed into each other by a sequence of two kinds of operations defined in that paper. They also conjectured that one kind of the operations is redundant. In the present paper we prove that their conjecture is true.

In [2], Foulds and Robinson studied the problem of how to transform one deltahedron into another. They defined two kinds of operations, and proved that any two deltahedra of the same order can be transformed into each other by a finite sequence of these operations. They conjectured that one kind of the operations was redundant. In this note we prove that the conjecture is true.

Definitions of the operations

We will follow the terminology in [2] and the graph-theoretical terminology in [1]. The graphs in this paper are undirected and have no loops or multiple edges. A cycle does not allow repeated vertices. A triangle is a cycle consisting of three edges. If e is an edge of a trangle t, we say that t is incident with e. Similarly, if v is a vertex on t, then we say that t is incident with v.

Definition. A *deltahedron* is an ordered triple (V,E,T) such that the following axioms are satisfied:

$\Delta 1$. (V,E) is a connected graph with more than one vertex.

$\Delta 2$. T is a set of triangles in the graph (V,E).

$\Delta 3$. If $\{u,v\}\in E$, then there are exactly two members of T incident with $\{u,v\}$.

$\Delta 4$. If $u\in V$, then the vertices adjacent to u may be labelled $w_1, w_2, \ldots, w_k$, so that the triangles incident with u are $\{u,w_1,w_2\}, \{u,w_2,w_3\}, \ldots, \{u,w_k,w_1\}$.

$\Delta 5$. Each cycle C of the graph (V,E) determines a partition of T into two disjoint non-empty sets, called *caps*, such that

(a) the two triangles incident with an edge of C are in different caps, and

(b) the triangles incident with a common edge not in C are in the same cap.

Let $\Delta=(V,E,T)$ be a deltahedron. $|V|$ is called the order of Δ. It is easy to see that a deltahedron is of at least order 4 and the only deltahedron of order 4 is the tetrahedron. The following theorems are proved in [2].

Theorem 1. *If $\Delta=(V,E,T)$ is a deltahedron such that $u\notin V$ and $\{a,b,c\}\in T$, and if we insert u into $\{a,b,c\}$, that is, set*

$$V'=V\cup\{u\},$$

$$E'=E\cup\{\{a,u\},\{b,u\},\{c,u\}\},$$

$$T'=(T\setminus\{\{a,b,c\}\})\cup\{\{a,b,u\},\{b,c,u\},\{a,c,u\}\},$$

then $\Delta'=(V',E',T')$ is a deltahedron.

Theorem 2. *Let $\Delta=(V,E,T)$ be a deltahedron other than tetrahedron with a vertex u of valency 3, incident with edges $\{u,a\},\{u,b\},\{u,c\}$. If we remove u, that is, set*

$$V'=V\setminus\{u\},$$

$$E'=E\setminus\{\{u,a\},\{u,b\},\{u,c\}\}$$

$$T'=(T\setminus\{\{u,a,b\},\{u,b,c\},\{u,a,c\}\})\cup\{\{a,b,c\}\},$$

then $\Delta'=(V',E',T')$ is a deltahedron.

Let Δ be a deltahedron, u be a vertex of valency 3 and t be a triangle in T. If we first remove u and insert u into triangle t, then we get a deltahedron Δ' of the same order as the old one. The new deltahedron is said to be obtained by applying a β-operation, and is denoted by

$$\Delta'=\beta(\Delta,u,t).$$

Next we introduce another operation.

Theorem 3. *Let $\Delta=(V,E,T)$ be a deltahedron with $\{a,b,c\},\{a,b,d\}\in T$. If $\{c,d\}\notin E$, we remove $\{a,b\}$ and add in $\{c,d\}$:*

$$E'=(E\setminus\{\{a,b\}\})\cup\{\{c,d\}\},$$

$$T'=(T\setminus\{\{a,b,c\},\{a,b,d\}\})\cup\{\{a,c,d\},\{b,c,d\}\},$$

then $\Delta'=(V,E',T')$ is a deltahedron.

The Δ' is said to be obtained by applying an α_1-operation to Δ, and we write

$$\Delta'=\alpha_1(\Delta,\{a,b\}).$$

Theorem 4. *Let $\Delta=(V,E,T)$ be a deltahedron with $\{a,b,c\},\{a,b,d\},\{c,d,e\}$ and*

$\{c, d, f\} \in T$ (the vertices appearing may be identical). Then $\{e, f\} \notin E$ unless Δ is a tetrahedron, and if we remove $\{a, b\}$ and add in $\{e, f\}$, that is, set

$$E' = (E \setminus \{\{a, b\}\}) \cup \{\{e, f\}\},$$

$$T' = (T \setminus \{\{a, b, c\}, \{a, b, d\}, \{c, d, e\}, \{c, d, f\}\})$$
$$\cup \{\{a, c, d\}, \{b, c, d\}, \{c, e, f\}, \{d, e, f\}\},$$

then $\Delta' = (V, E', T')$ is a deltahedron.

Δ' is said to be obtained by applying an α_2-operation to Δ, and we write

$$\Delta' = \alpha_2(\Delta, \{a, b\}).$$

Both α_1-operations and α_2-operations are called α-operations. The following theorem is the main result in [2].

Theorem 5. *Let $\Delta = (V, E, T)$ and $\Delta' = (V', E', T')$ be deltahedra with $V = V'$. Then there exists a finite sequence of α- and β-operations which transforms Δ into Δ'.*

[2] also conjectured that the theorem is true by using α-operations only. In the next section we give an affirmative answer to this conjecture.

Main result

Let $\Delta = (V, E, T)$ be a deltahedron. We construct another graph, $G(\Delta)$, as follows. The vertices of $G(\Delta)$ are the triangles in T. Two vertices are adjacent iff their corresponding triangles have an edge in common.

Lemma. *$G(\Delta)$ is a connected graph.*

Proof. Suppose the lemma is not true.

First we recall that (V, E) is a connected graph by Axiom $\Delta 1$. Let P be a connected component in $G(\Delta)$. P denotes a set of triangles in (V, E). Then any triangle which shares an edge with a triangle in P is itself in P, and if a vertex of V is incident with a triangle in P, all the triangles with which it is incident are in P.

Thus the vertices in V can be partitioned into two non-empty classes W, X. Every triangle incident with a vertex in W is in P, and no triangle incident with a vertex in X is in P.

But (V, E) is connected, so there will be an edge in E with one end in W and the other in X. This edge is incident with two triangles which must be both in P and not in P. Contradiction. Hence $G(\Delta)$ is connected. $\square$

Now we are ready to state our main result.

Theorem 6. *A β-operation can be replaced by a finite sequence of α_2-operations. Hence, any two deltahedra of the same order can be transformed into each other by a sequence of α-operations.*

Proof. Let $\Delta = (V, E, T)$ be a deltahedron, u be a vertex of valency 3 in V, t be a triangle not incident with u, and $\Delta' = \beta(\Delta, u, t)$.

We show that Δ' can be obtained by a sequence of α_2-operations. Let t' be the triangle obtained when u is removed from Δ. By the proceeding lemma, there exists a sequence of triangles, starting from t' and ending at t, such that every pair of successive triangles in the sequence have an edge in common. Hence we need only to consider the case when t' and t have a common edge. For otherwise, we can transform along the sequence of triangles.

Now let $t' = \{a, b, c\}$, and $t = \{b, c, f\}$ where $\{b, c\}$ be the common edge of t' and t. Since u has valency 3 in (V, E), we may assume the three triangles incident with u in Δ are $\{u, a, b\}$, $\{u, b, c\}$ and $\{u, c, a\}$. Consider the edge $\{a, u\} \in E$ and the two triangles $\{a, u, c\}$ and $\{a, u, b\}$ incident with $\{a, u\}$. By Theorem 4, $\{u, f\} \notin E$, and it is easy to check that

$$\beta(\Delta, u, t) = \alpha_2(\Delta, \{a, u\}).$$

(See Fig. 1 below.) This proves the theorem. $\square$

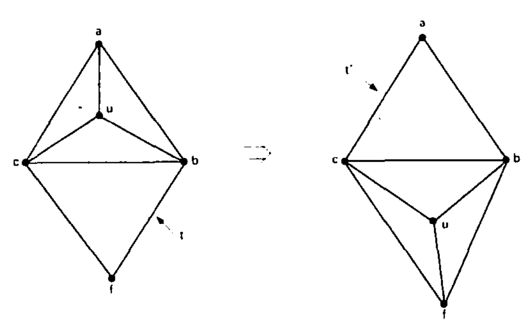

Fig. 1.

Acknowledgement

The author thanks very much the referees for their remarks on the original version of the paper which clarify the proof of the Lemma.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (MacMillan, London, 1976).

[2] L.R. Foulds and D.F. Robinson, Constructive properties of combinatorial deltahedra, Discrete Appl. Math. 1 (1979) 75–87.