Analytical solutions of cracks emanating from an elliptical hole under shear

Liu Shuhong a,*, Duan Shijie b

a Department of Engineering Mechanics, Shijiazhuang Tiedao University, Shijiazhuang 050043, China
b Department of Adult Education, Shijiazhuang Tiedao University, Shijiazhuang 050043, China

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Abstract Based on the complex variable method, the analytical solutions of stress functions and stress intensity factors (SIFs) are provided for the plane problem of two collinear edge cracks emanating from an elliptical hole in an infinite plate under shear. The stress distribution along the horizontal axis is given in graphical forms, which conforms to Saint-Venant’s principle. The influences of crack length and ellipse shape on the stress intensity factors are evaluated. Comparing the analytical solutions with finite element method (FEM) results shows good coincidence. These numerical examples show that the present solutions are accurate.

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1. Introduction

Cracks emanating from a hole are very common in engineering. Even if a crack is rather short, it can lead to a dangerous situation. Consequently, it is of great importance to deal with hole-edge crack problems. Bowie1 was the first to give solutions of a circular hole with a single edge crack and a pair of symmetrical edge cracks in a plate under uniform tension at infinity by using the complex mapping technique. Because the mapping functions adopted are complicated and inaccurate, a number of papers analyzing the stress intensity factor (SIF) for cracks originating from a hole have been published.2–11 The complex variable function method was used to calculate the SIF for a single edge crack or a pair of symmetrical edge cracks originating from an elliptical hole in an infinite plate under tension.12,13 Liu et al.14–16 studied the plane problem of an elliptic hole or a crack in transversely isotropic piezoelectric materials subjected to tension at infinity, internal pressure, and shear loads acting on the edge of the defect. By using the finite element method (FEM), Liu et al.17 obtained the stress distributions in the vicinity of the hole and the crack for the plane problem of a plate with a crack emanating from an elliptical hole.

From the literatures, it can be seen that an infinite plate containing hole-crack is mainly subjected to uniform remote tension loads. This paper concerns with two cracks of unequal lengths at the edge of an elliptic hole in an infinite plate under shear by means of the complex variable function method, and the analytical solutions of stress functions and SIFs are obtained. Numerical calculations are presented to graphically show the stress distribution along the horizontal axis. The
2. Basic equations

Muskhelishvili’s method is used for stress analysis, and the stress components \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \) in rectangular coordinates are given in terms of the complex potentials \( \phi(z) \) and \( \psi(z) \)

\[
\sigma_x + \sigma_y = 4 \text{Re}(\phi_1(z)) \\
\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\text{Re}\phi_1(z) + \psi_1(z)]
\]

(2)

where \( z = x + iy \) is the complex variable, the bar denotes the complex conjugate, and the prime notation denotes differentiation with respect to \( z \), and “Re(·)” represents the real part of a complex variable.

In order to make the boundary conditions more manageable, it is advantageous to replace the complex variable \( z \) for any point in the \( \sigma \)-plane by a new complex variable \( \zeta = re^{i\theta} \) in the \( \zeta \)-plane, by using a conformal transformation \( z = \omega(\zeta) \), and then the stress functions \( \phi(z) \) and \( \psi(z) \) will be considered as functions of parameter \( \zeta \). Thus, the new notation is introduced as

\[
\begin{align*}
\phi(\zeta) &= \phi_1(z) = \phi_1[\omega(\zeta)] \\
\psi(\zeta) &= \psi_1(z) = \psi_1[\omega(\zeta)] \\
\phi'(\zeta) &= \phi'_1(z) = \phi'_1[\omega(\zeta)] \\
\psi'(\zeta) &= \psi'_1(z) = \psi'_1[\omega(\zeta)]
\end{align*}
\]

(3)

Let \( \sigma_{\theta} \) and \( \sigma_{\phi} \) be the stress components in curvilinear coordinates, Eq. (1) can be rewritten as

\[
\sigma_{\theta} + \sigma_{\phi} = 4 \text{Re}(\Phi(\zeta))
\]

(4)

The functions \( \phi(\zeta) \) and \( \psi(\zeta) \) can be obtained by the following equations:

\[
\phi(\zeta) = \frac{1 + \mu}{8\pi}(X + iY) \ln \zeta + B_0(\zeta) + \phi_0(\zeta)
\]

(5)

\[
\psi(\zeta) = -\frac{3 - \mu}{8\pi}(X - iY) \ln \zeta + (B' + iC')\omega(\zeta) + \psi_0(\zeta)
\]

(6)

\[
\phi_0(\zeta) = \frac{1}{2\pi i} \int \frac{\omega(\sigma)}{\phi'(\sigma)} \frac{\phi_0'(\sigma)}{\sigma - \zeta} \ d\sigma = \frac{1}{2\pi i} \int \frac{f_0}{\sigma - \zeta} \ d\sigma
\]

(7)

\[
\psi_0(\zeta) = \frac{1}{2\pi i} \int \frac{\omega(\sigma)}{\psi'(\sigma)} \frac{\phi_0'(\sigma)}{\sigma - \zeta} \ d\sigma = \frac{1}{2\pi i} \int \frac{f_0}{\sigma - \zeta} \ d\sigma
\]

(8)

\[
f_0 = i \int (\mathfrak{X} + i\mathfrak{Y}) d\xi - \frac{X + iY}{2\pi} \ln \sigma - \frac{1 + \mu}{8\pi}(X - iY)
\]

\[
\times \frac{\omega(\sigma)}{\phi'(\sigma)} \frac{\phi_0'(\sigma)}{\sigma - B_0(\sigma)} - (B' - iC')\omega(\sigma)
\]

(9)

where \( \mathfrak{X} \) and \( \mathfrak{Y} \) represent the components of the surface forces per unit area at any point of the interior boundary, while \( X \) and \( Y \) are the algebraic sums of the surface force components on the interior boundaries in the \( x \) and \( y \) directions, respectively. The constants \( B \) and \( B' + iC' \) are related to the magnitudes of the principal stresses \( \sigma_1 \) and \( \sigma_2 \) at infinity.

3. Problem and exact solutions

Consider two asymmetrical collinear cracks emanating from an elliptical hole in an infinite solid, as shown in Fig. 1. The cracks and the hole are assumed to be traction-free, while the solid is subjected to shear (\( q \)) acting on the plate sides as shown in Fig. 1(a). Take the center of the elliptical hole as the origin and the line where the crack is located as the \( x \)-axis to build rectangular coordinates.

3.1. Stress functions

The conformal mapping function\(^18\) is

\[
z = \omega(\zeta) = \frac{a + b}{2} \mu(\zeta) + \frac{a - b}{2} \frac{1}{\mu(\zeta)}
\]

(11)

\[
\mu(\zeta) = \frac{\epsilon_1(1 + \zeta) \epsilon_1(1 + \zeta)}{4\zeta}
\]

\[
+ \frac{[(\epsilon_1(1 + \zeta) \epsilon_1(1 + \zeta) - 16\epsilon_1^2)^{1/2}]}{4\zeta}
\]

(12)

\[
e_i = \frac{(a + L_i)^2 + b^2 + ab + (a + L_i)\sqrt{L_i^2 + 2aL_i + b^2}}{(a + b)(a + L_i + \sqrt{L_i^2 + 2aL_i + b^2})}
\]

\[
\ (i = 1, 2)
\]

(13)

Eq. (11) provides a conformal mapping from the outside region of the elliptical hole and cracks into the interior of a unit circle in the \( \zeta \)-plane. By Eq. (11), the four points \( A(a + L_1, 0), I(0, -b), E_1(-a - L_2, 0), \) and \( G(0, b) \) in the \( \sigma \)-plane (see Fig. 1(a)) are mapped to \( A'(1, 0), I', E'(-1, 0), \) and \( G' \) (see Fig. 1(b)), respectively, and at the same time, the lower points \( H(a, 0) \) and \( D'(-a, 0) \) to points \( H' \) and \( D' \), the upper points \( H \) and \( D \) to points \( H'' \) and \( D'' \), respectively.

Under the loading condition in this paper, it can be seen that \( B = 0 \), \( B' + iC' = qi \), and \( \mathfrak{X} = \mathfrak{Y} = X = Y = 0. \) Thus Eq. (9) is simplified as

\[
f_0 = q\omega(\sigma)
\]

(14)

From Eqs. (11), (12), one can obtain the following equations for use:

\[
\omega'(\zeta) = -\frac{(1 - \zeta^2)(\epsilon_1 + \epsilon_2)}{4\zeta^2} \left( \frac{b(\epsilon_1(1 + \zeta) \epsilon_1(1 - \zeta)^2)}{\sqrt{(\epsilon_1(1 + \zeta) \epsilon_1(1 - \zeta)^2)^2 - 16\epsilon_1^2}} + a \right)
\]

(15)

\[
\omega'(\frac{1}{\zeta}) = \frac{(1 - \zeta^2)(\epsilon_1 + \epsilon_2)}{4} \left( \frac{b(\epsilon_1(1 + \zeta)^2 + \epsilon_1(1 - \zeta)^2)}{\sqrt{(\epsilon_1(1 + \zeta)^2 + \epsilon_1(1 - \zeta)^2)^2 - 16\epsilon_1^2}} + a \right)
\]

(16)
Analytical solutions of cracks emanating from an elliptical hole under shear

On the unit circle, since $\sigma = 1/\sigma$ and $\omega(\sigma) = \sigma^2(\sigma)$, it can be seen that $\frac{\omega(\zeta)}{\omega(\bar{\zeta})} \varphi_0(\zeta)$ is analytical outside the unit circle as well as continuous outside and on the circle, and then, using Cauchy integral, one can obtain

$$\frac{1}{2\pi i} \int \frac{\omega(\sigma)}{\omega(\bar{\sigma})} \varphi_0(\sigma) \frac{d\sigma}{\sigma - \zeta} = 0 \quad (17)$$

Substituting Eqs. (14) and (17) into Eq. (7) yields

$$\varphi_0(\zeta) = \frac{1}{2\pi i} \int \frac{\varphi_0(\sigma)}{\sigma - \zeta} d\sigma \quad (18)$$

Because $\zeta = \infty$ is the 1st order pole of $\omega(\sigma)$, $\text{Res}(\omega(\zeta), \infty) = \frac{(a+b)(e_1+e_2)}{4}$, where “Res” denotes the residue. According to the residue theorem, one can obtain

$$\varphi_0(\zeta) = \frac{qi(a+b)(e_1+e_2)}{4} \zeta \quad (19)$$

From Eq. (13), when $L_1 = L_2 = 0$, $e_1 = e_2 = 1$ can be obtained, and the elliptic hole with two unequal-length cracks degenerates to the elliptical hole. From Eqs. (11), (12), (13), and (17), one can obtain

$$\varphi_0(\zeta) = \frac{qi(a+b)}{2} \zeta \quad (20)$$

This is just the well-known result for the elliptical hole. For the line crack, $b = 0$, and Eq. (20) becomes $\varphi_0(\zeta) = \frac{qi a}{2} \zeta$.

Differentiating Eq. (20) with respect to $\zeta$, one can obtain

$$\varphi_0'(\zeta) = \frac{qi(a+b)(e_1+e_2)}{4} \zeta \quad (21)$$

We know that $\frac{\omega(\sigma)}{\omega(\bar{\sigma})} \varphi_0(\sigma)$ is analytical inside the circular hole, and also continuous inside and on the circle. Substituting Eqs. (14) and (19) into Eq. (8), and using Cauchy integral again, one can obtain

$$\psi_0(\zeta) = -\frac{\varphi_0(\zeta)}{\omega(\bar{\zeta})} - qi(\omega(\zeta) - (a+b)(e_1+e_2)) \frac{4}{4e} \quad (22)$$

Substituting Eqs. (10), (11), (19), and (22) into Eqs. (5), (6) yields

$$\varphi(\zeta) = \varphi_0(\zeta) - \frac{qi(a+b)(e_1+e_2)}{4} \zeta \quad (23)$$

$$\psi(\zeta) = -\frac{\varphi_0'(\zeta)}{\omega(\bar{\zeta})} + qi(a+b)(e_1+e_2) \frac{4}{4e} \quad (24)$$

Substituting Eqs. (3), (23) and (24) into Eqs. (1), (2), it is a straightforward matter to find the stress components $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ in terms of the complex variable $\zeta$ as follows:

$$\left\{ \begin{array}{l}
\sigma_x = 2\text{Re} \varphi'(\zeta) + \text{Re}[\varphi_0'(\zeta) + \psi'(\zeta)] \\
\sigma_y = 2\text{Re} \varphi'(\zeta) - \text{Re}[\varphi_0'(\zeta) + \psi'(\zeta)] \\
\tau_{xy} = \text{Im} [\varphi_0'(\zeta) + \psi'(\zeta)]
\end{array} \right. \quad (25)$$

where “Im” represents the imaginary part of a complex variable. At the edge of the elliptical hole, $\sigma_y = 0$, and then, from Eq. (4), one can obtain $\sigma_y = 4\text{Re}(\Phi(\zeta))$.

3.2. SIFs

For a mixed-mode problem, the SIFs $K_{II A}$, $K_{II A L}$, $K_{IE}$, and $K_{IE L}$ at the crack tips $A$ and $E$, as shown in Fig. 1(a), can be expressed in the $\zeta$-plane as follows:

$$\left\{ \begin{array}{l}
K_{II A} + iK_{II A L} = 2\sqrt{2\pi} \lim_{\zeta \to 1} \sqrt{\omega(\zeta) - \omega(1)} \frac{\varphi'(\zeta)}{\omega(\bar{\zeta})} \\
K_{IE} + iK_{IE L} = 2\sqrt{2\pi} \lim_{\zeta \to -1} \sqrt{-\omega(\zeta) - \omega(-1)} \frac{\varphi'(\zeta)}{\omega(\bar{\zeta})}
\end{array} \right. \quad (26)$$

Inserting Eqs. (11) and (23) into Eq. (26), one can obtain

$$\left\{ \begin{array}{l}
K_{II A} = 0 \\
K_{II L} = \frac{2\sqrt{2\pi}q(a+b)\sqrt{e_1+e_2}\sqrt{e_1^2-1}}{2\sqrt{b_0+ae_1-1}} \quad (27)
\end{array} \right.$$
Eqs. (27)–(28) show that the SIFs of two unequal-length cracks emanating from the edge of an elliptical hole in an infinite solid are related to the applied mechanical loading and the geometries of the cracks and the hole. Under limited conditions, many new configurations can be simulated from the present results. If the length of the short semi-axis $b$ tends to be zero and the left crack length $L_2$ is $L_1$, using Eq. (13), Eqs. (27)–(28) reduce to

$$K_{II}^A = K_{II}^E = \sqrt{\pi q(a + L)},$$

which is just the well-known result of the Griffith crack in an infinite solid.

4. Numerical results and discussions

Good agreement has been found between the analytical solutions and the FEM results, which is shown in Figs. 2–3. The plane stress finite element model of ANSYS is used. Fig. 2 shows the normalized stress distribution of $\tau_{xy}/q$ along the negative $x$-axis. Dimensions of the hole and the cracks are as follows: $a = 2$ mm, $b = 1$ mm, $L_1 = 2$ mm, and $L_2 = 1$ mm. The first point is 0.01 mm distant from the left crack tip, and the values of $x$ vary from $-3.01$ mm to $-12$ mm. It can be seen that there is obvious stress concentration at the crack tip. It is probably because the finite element meshing is not small enough that results in the difference between the FEM and present results, but the tendency is consistent. However, at distances which are longer than the size of the defect, the stress distribution is practically uniform, and tends to applied loads, which conforms to the conclusion usually made on the basis of...
Saint-Venant’s principle. Because of symmetry, $\sigma_x/q$ and $\sigma_y/q$ along the $x$-axis are always zero. The normalized SIFs $K_{II,a}/K_0$ and $K_{III,a}/K_0$ for two cracks of unequal lengths at the edge of an elliptical hole with $L/a$ are illustrated in Figs. 3–5, which are normalized by $K_0 = q\sqrt{\pi a}$, where $a = 2$ mm, $b = 1$ mm, and $L_2 = 3$ mm in Fig. 3, $L_2 = 3$ mm and $a = 2$ mm in Fig. 4, and $a = 2$ mm and $b = 1$ mm in Fig. 5. It is found that the values of $K_{II,a}$ and $K_{III,a}$ are related to the ratio of $a/b$ and increase with increasing crack length.

Table 1 $F_{II,a}$ of a crack emanating from an elliptical hole.

<table>
<thead>
<tr>
<th>$L/a$</th>
<th>$F_{II,a}$</th>
<th>$b/a = 1/2$</th>
<th>$b/a = 1$</th>
<th>$b/a = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.224(2.9143)</td>
<td>0.053(1.9852)</td>
<td>0.015(1.4972)</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>0.811(2.6380)</td>
<td>0.244(1.9290)</td>
<td>0.075(1.4860)</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>1.175(2.3902)</td>
<td>0.436(1.8652)</td>
<td>0.145(1.4723)</td>
<td></td>
</tr>
<tr>
<td>0.20</td>
<td>1.443(2.0644)</td>
<td>0.712(1.7553)</td>
<td>0.272(1.4456)</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>1.471(1.5935)</td>
<td>1.085(1.5215)</td>
<td>0.568(1.3716)</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>1.300(1.2831)</td>
<td>1.200(1.2990)</td>
<td>0.854(1.2689)</td>
<td></td>
</tr>
</tbody>
</table>

If $L_2 = 0$ mm, Eq. (27) is the result of a crack emanating from the edge of an elliptical hole under shear. Comparison with the results of Isida et al.1 is given in the following section. Table 1 is the dimensionless expression $F_{II,a} = K_{II,a}/\tau \sqrt{\pi L_1}$ of a crack emanating from an elliptical hole under shear (Fig. 6(a)), which corresponds to Table 5 given in Ref. 4. The results in the bracket are calculated based on the present paper. From Table 1, it can be seen that there is great difference between Isida’s results and the results of the present paper.

Table 2 $F_{II,a}$ of a crack emanating from an elliptical hole.

<table>
<thead>
<tr>
<th>$L/a$</th>
<th>$F_{II,a}$</th>
<th>$b/a = 1/2$</th>
<th>$b/a = 1$</th>
<th>$b/a = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>5.181(2.9143)</td>
<td>3.293(1.9852)</td>
<td>2.230(1.4972)</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>4.043(2.6380)</td>
<td>3.037(1.9290)</td>
<td>2.180(1.4860)</td>
<td></td>
</tr>
<tr>
<td>0.10</td>
<td>3.256(2.3902)</td>
<td>2.772(1.8652)</td>
<td>2.120(1.4723)</td>
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</tr>
<tr>
<td>0.20</td>
<td>2.460(2.0644)</td>
<td>2.374(1.7553)</td>
<td>2.007(1.4456)</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>1.640(1.5935)</td>
<td>1.728(1.5215)</td>
<td>1.126(1.3716)</td>
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</tr>
<tr>
<td>1.00</td>
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<td>1.306(1.2990)</td>
<td>1.410(1.2689)</td>
<td></td>
</tr>
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</table>

5. Conclusions

(1) For the plane problem of two cracks of unequal lengths at the edge of an elliptical hole in an infinite solid which is subject to shear, the closed-form solutions of stress functions and stress intensity factor are obtained. Based on the present results, the solutions for some particular hole-edge crack configurations can be obtained by adding specific conditions, e.g., two cracks of equal length or a single crack emanating from the edge of an elliptical or a circular hole, T-shaped cracks, and cross-shaped cracks.

(2) Numerical calculations show that the stress distribution along the horizontal axis conforms to Saint-Venant’s principle, and the values of stress intensity factors increase with increasing crack length and are related to the major-to-minor axial ratio of the elliptical hole. Good agreement is found between the FEM and the analytical solution derived, so the analytical results are reliable.

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References


Liu Shuhong received her ph.D. degree in Bridge and Tunnel engineering from Beijing Jiaotong University in 2004, and is currently a professor at Shijiazhuang Tiedao University. Her main research interests are solid mechanics, structural safety evaluation and mechanical bearing capacity of intelligent materials.

Duan Shijie received his bachelor’s degree from Shijiazhuang Tiedao University in 1992, and is currently an associate professor at Shijiazhuang Tiedao University. His research interest is computer application in engineering.