JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 73, 338–350 (1980)

On the Euler Equations for Nonhomogeneous Fluids (II)

HUGO BEIRÃO DA VEIGA AND ALBERTO VALLI

Dipartimento di Matematica e Fisica, Libera Università di Trento, 38050 Povo (Trento), Italy

Submitted by J. L. Lions

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the motion of a nonhomogeneous ideal incompressible fluid in a bounded connected open subset Ω of $\mathbb{R}^3$. We assume that the boundary Γ is a compact manifold of dimension 2, without boundary, and that Ω is locally situated on one side of Γ. Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ such that Γ_j $(j = 1, \ldots, m)$ are inside of Γ_0 and outside of one another. In Sections 2 and 3 we assume that Ω is simply connected; in Section 4 we drop this condition. We denote by $v(t, x)$ the velocity field, by $\rho(t, x)$ the mass density, and by $\pi(t, x)$ the pressure. The Euler equations of the motion are (see for instance Sédov [14, Chap. IV, Sect. 1, p. 164])

$$
\begin{aligned}
&\rho\left[\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v - b\right] = -\nabla\pi && \text{in} \quad Q_{T_0} = [0, T_0] \times \bar{\Omega},\\
&\operatorname{div} v = 0 && \text{in} \quad Q_{T_0},\\
&v \cdot n = 0 && \text{on} \quad [0, T_0] \times \Gamma,\\
&\frac{\partial \rho}{\partial t} + v \cdot \nabla\rho = 0 && \text{in} \quad Q_{T_0},\\
&\rho\,|_{t=0} = \rho_0 && \text{in} \quad \bar{\Omega},\\
&v\,|_{t=0} = a && \text{in} \quad \bar{\Omega},
\end{aligned}
\tag{E}
$$

where $n = n(x)$ is the unit outward normal to the boundary Γ, $b = b(t, x)$ is the external force field, and $a = a(x)$, $\rho_0 = \rho_0(x)$ are the initial velocity field and the initial mass density, respectively.

Nonhomogeneous ideal incompressible fluids have been studied by several authors; see, for instance, Sédov [14], Zeytounian [18], Yih [17]. In some problems concerning oceanography (see, for instance, LeBlond and Mysak [10])

0022-247X/80/020315-23$02.00/0

or, more generally, rotating systems (see also Kazhikhov [9]), Eq. $(E)_1$ is replaced by

$$\rho\left[\frac{\partial v}{\partial t}+(v\cdot\nabla)\,v+2\omega\wedge v-b\right]=-\nabla\pi,$$

where ω is the angular velocity. The perturbation term $2\,\omega\wedge v$ does not give rise to any difficulty and our results and proofs hold again if one assumes that $\omega\in C^{0,1+\lambda}(Q_{T_0})$.

For the case in which the fluid is homogeneous, i.e., the density ρ_0 (and consequently ρ) is constant, Eqs. (E) have been studied by several authors.

For the three-dimensional case see, for instance, Lichtenstein [11], Ebin and Marsden [6], Swann [15], Kato [8], Bourguignon and Brezis [5], Temam [16], Bardos and Frisch [2].

For nonhomogeneous fluids, Marsden [13] has proved (in the n-dimensional case) the existence of a local solution to problem (E), under the assumption that the external force field $b(t, x)$ is divergence free and tangential to the boundary, i.e., $\operatorname{div} b=0$ in Q_{T_0} and $b\cdot n=0$ on $[0, T_0]\times\Gamma$. The proof relies on techniques of Riemannian geometry on infinite-dimensional manifolds.[1] In a previous paper [4], we have proved, in the two-dimensional case, the existence of a local solution to problem (E) without any restriction on the external field $b(t, x)$. In this paper we prove the corresponding result for the three-dimensional case, i.e.,

THEOREM A. *Let Γ be of class $C^{3+\lambda}$, $0<\lambda<1$, and let $a\in C^{1+\lambda}(\bar\Omega)$ with $\operatorname{div} a=0$ in $\bar\Omega$ and $a\cdot n=0$ on Γ, $\rho_0\in C^{1+\lambda}(\bar\Omega)$ with $\rho_0(x)>0$ for each $x\in\bar\Omega$, and $b\in C^{0,1+\lambda}(Q_{T_0})$. Then there exists $T_1\in[0, T_0]$, $v\in C^{1,1+\lambda}(Q_{T_1})$, $\rho\in C^{1+\lambda,1+\lambda}(Q_{T_1})$ $\pi\in C^{0,2+\lambda}(Q_{T_1})$ such that (v, ρ, π) is a solution of* (E) *in Q_{T_1}.*

A uniqueness theorem for problem (E) is proved by Graffi [20]. See also [3].

For the study of nonhomogeneous *viscous* incompressible fluids see Kazhikhov [9], [21] Antoncev and Kazhikhov [1], Ladyzhenskaya and Solonnikov [22], Lions [12], and Simon [23].

2. Preliminaries and Existence of a Local Solution of the Auxiliary System (A)[2]

In this section and in Section 3 we assume that Ω is simply connected. We use the notations introduced in [4]. We need only to define for a vector function φ the operator

$$\operatorname{curl}\varphi=\left(\frac{\partial\varphi_3}{\partial x_2}-\frac{\partial\varphi_2}{\partial x_3},\frac{\partial\varphi_1}{\partial x_3}-\frac{\partial\varphi_3}{\partial x_1},\frac{\partial\varphi_2}{\partial x_1}-\frac{\partial\varphi_1}{\partial x_2}\right).$$

[1] For the analytic case on compact manifolds without boundary see [19].

[2] See the end of this section.

In the following $\varphi(t, x) \in C^{0,\lambda}(Q_T)$, $T \in]0, T_0]$, will be a generic element of the sphere

$$\| \varphi \|_{0,\lambda} \leqslant A \tag{2.1}$$

(where the radius A is a positive constant which we will specify below) such that for each $t \in [0, T]$

$$\operatorname{div} \varphi = 0 \quad \text{in} \quad \mathscr{D}'(\Omega), \qquad \int_{\Gamma_i} \varphi \cdot n \, d\Gamma = 0 \qquad \forall i = 1,..., m. \tag{2.2}$$

This condition is equivalent to the existence of a vector function v_0 such that $\varphi = \operatorname{curl} v_0$; see for instance Foias and Temam [7, Proposition 1.3]. We denote by $c, c_1, c_2,...,$ positive constants depending at most on λ and Ω.

Under our assumptions on Ω, the conditions on φ assure the existence of a unique solution $v \in C^{0,1+\lambda}(Q_T)$ of the elliptic system

$$\begin{aligned} \operatorname{curl} v &= \varphi && \text{in } Q_T, \\ \operatorname{div} v &= 0 && \text{in } Q_T, \\ v \cdot n &= 0 && \text{in } [0, T] \times \Gamma. \end{aligned} \tag{2.3}$$

Moreover,

$$\| v \|_{0,1+\lambda} \leqslant c \| \varphi \|_{0,\lambda} \leqslant cA, \tag{2.4}$$

which corresponds to inequality (3.4) in [4].

As in [4] we construct the functions $U(s, t, x)$, $\rho(t, x)$, and $w(t, x)$ and we prove the corresponding Lemmas 3.2, 3.3, 4.1, 4.2.

We want now to study the equation

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + (v \cdot \nabla)\, \zeta = \beta + w \wedge \frac{\nabla \rho}{\rho^2} + (\zeta \cdot \nabla)\, v \qquad &\text{in} \quad Q_T, \\ \zeta \,|_{t=0} = \alpha \qquad &\text{in} \quad \bar{\Omega}, \end{aligned} \tag{2.5}$$

where $\alpha \equiv \operatorname{curl} a$ and $\beta \equiv \operatorname{curl} b$.

To solve (2.5) we use the well known method of characteristics. Consider in Q_T the C^1-change of variable $(t, x) \to (t, x')$ defined by

$$x' = U(0, t, x), \qquad \text{i.e.} \quad x = U(t, 0, x'), \quad \forall t \in [0, T]. \tag{2.6}$$

Set $\gamma \equiv \beta + w \wedge (\nabla \rho / \rho^2)$; system (2.5) becomes then

$$\begin{aligned} \frac{d\tilde{\zeta}}{dt}(t, x') &= \gamma(t, U(t, 0, x')) + Dv(t, U(t, 0, x')) \cdot \tilde{\zeta}(t, x'). \\ \tilde{\zeta}(0, x') &= \alpha(x'), \end{aligned} \tag{2.7}$$

where

$$\tilde{\zeta}(t, x') = \zeta(t, U(t, 0, x')), \tag{2.8}$$

i.e.,

$$\zeta(t, x) = \tilde{\zeta}(t, U(0, t, x)). \tag{2.9}$$

Dv is the matrix with $\partial v_i/\partial x_j$ in the ith row, jth column, and $Dv \cdot \tilde{\zeta}$ is the matrix product.

The linear ordinary system (2.7) has a unique solution for each $x' \in \bar{\Omega}$. Since $\alpha(x')$, $\gamma(t, U(t, 0, x'))$ and $Dv(t, U(t, 0, x'))$ are not differentiable with respect to x', $\tilde{\zeta}(t, x')$ is generally not differentiable with respect to this last variable; hence $\zeta(t, x)$, being not differentiable in x, is not a classical solution of (2.5). For this reason we must define $\zeta(t, x)$ by (2.9). We denote by $\bar{c}$, $\bar{c}_1$, $\bar{c}_2$,..., positive constants depending at most on λ, Ω, ρ_0, and b.

LEMMA 2.1. *The solution $\tilde{\zeta}$ of system* (2.7) *satisfies*

$$\begin{aligned}
\| \tilde{\zeta} \|_\infty &\leqslant (\| \alpha \|_\infty + T \| \gamma \|_\infty)\, e^{T\|Dv\|_\infty} \leqslant \| \alpha \|_\infty\, e^{cTA} + T\bar{c}(A, T), \\
[\tilde{\zeta}]_{0,\lambda} &\leqslant ([\alpha]_\lambda + Te^{\lambda T[v]_{0,\mathrm{lip}}}[\gamma]_{0,\lambda})\, e^{T\|Dv\|_\infty} \\
&\quad + (\| \alpha \|_\infty + T \| \gamma \|_\infty)\, T[Dv]_{0,\lambda}\, e^{T(2\|Dv\|_\infty + \lambda[v]_{0,\mathrm{lip}})} \\
&\leqslant [\alpha]_\lambda\, e^{cTA} + T\bar{c}(A, T)\,(1 + \| \alpha \|_\infty), \\
[\tilde{\zeta}]_{\lambda,0} &\leqslant [\| \gamma \|_\infty + (\| \alpha \|_\infty + T \| \gamma \|_\infty)\, \| Dv \|_\infty\, e^{T\|Dv\|_\infty}]\, T^{1-\lambda} \\
&\leqslant T^{1-\lambda}\bar{c}(A, T)\,(1 + \| \alpha \|_\infty),
\end{aligned} \tag{2.10}$$

where $\bar{c}(A, T)$ is nondecreasing in the variables A and T

Proof From (2.7) we have

$$\frac{d\,|\tilde{\zeta}(t, x')|}{dt} \leqslant \left| \frac{d\tilde{\zeta}(t, x')}{dt} \right| \leqslant \| \gamma \|_\infty + \| Dv \|_\infty\, |\tilde{\zeta}(t, x')|,$$

$$|\tilde{\zeta}(0, x')| = |\alpha(x')|.$$

By comparison theorems and (2.4) one obtains $(2.10)_1$. Moreover we have

$$\begin{aligned}
\frac{d}{dt} |\tilde{\zeta}(t, x') - \tilde{\zeta}(t, x'')| &\leqslant \left| \frac{d}{dt} [\tilde{\zeta}(t, x') - \tilde{\zeta}(t, x'')] \right| \\
&\leqslant ([\gamma]_{0,\lambda}\, [U]^{\lambda}_{0,\mathrm{lip}} + \| \tilde{\zeta} \|_\infty\, [Dv]_{0,\lambda}\, [U]^{\lambda}_{0,\mathrm{lip}})\, |x' - x''|^{\lambda} \\
&\quad + \| Dv \|_\infty\, |\tilde{\zeta}(t, x') - \tilde{\zeta}(t, x'')|, \\
|\tilde{\zeta}(0, x') - \tilde{\zeta}(0, x'')| &\leqslant [\alpha]_\lambda\, |x' - x''|^{\lambda}.
\end{aligned}$$

From $(2.10)_1$ and estimate $(3.7)_1$ of [4] we obtain

$$\frac{d}{dt} \mid \tilde{\zeta}(t, x') - \tilde{\zeta}(t, x'')| \leqslant [[\gamma]_{0,\lambda} + (\| \alpha \|_\infty + T \| \gamma \|_\infty) [Dv]_{0,\lambda} e^{T\|Dv\|_\infty}] e^{\lambda T[v]_{0,1ip}} \times | x' - x'' |^\lambda + \| Dv \|_\infty | \tilde{\zeta}(t, x') - \tilde{\zeta}(t, x'')| .$$

By comparison theorems we have $(2.10)_2$.

Finally, from (2.7), $(2.10)_1$, and

$$| \tilde{\zeta}(t, x') - \tilde{\zeta}(s, x')| = \left| \int_s^t \frac{d}{d\tau} \tilde{\zeta}(\tau, x') \, d\tau \right|$$

one easily gets $(2.10)_3$. ∎

From this lemma, (2.9), and estimate $(3.7)_1$, $(3.7)_2$ of [4], one easily obtains

LEMMA 2.2. *The function $\zeta(t, x)$ defined in* (2.8) *is such that $\zeta \in C^{\lambda,\lambda}(Q_T)$ and*

$$\begin{aligned} &\| \tilde{\zeta} \|_\infty = \| \zeta \|_\infty \leqslant \| \alpha \|_\infty e^{cTA} + T\bar{c}(A, T), \\ &[\zeta]_{0,\lambda} \leqslant [\tilde{\zeta}]_{0,\lambda} [U]^\lambda_{0,1ip} \leqslant [\alpha]_\lambda e^{cTA} + T\bar{c}(A, T)(1 + \| \alpha \|_\infty), \\ &[\zeta]_{\lambda,0} \leqslant [\tilde{\zeta}]_{\lambda,0} + [\tilde{\zeta}]_{0,\lambda} [U]^\lambda_{1ip,0} \leqslant c_1 A^\lambda [\alpha]_\lambda e^{cTA} + T^{1-\lambda}\bar{c}(A, T)(1 + \| \alpha \|_\infty). \end{aligned} \tag{2.11}$$

Now we want to prove that for each $t \in [0, T]$ div $\zeta = 0$ in $\mathscr{D}'(\Omega)$ and $\int_{\Gamma_i} \zeta \cdot n d\Gamma = 0$ for each $i = 1,..., m$. First of all we observe that

$$\gamma = \text{curl } g, \qquad g \in C^{0,1+\lambda}(Q_T)$$

since $w \wedge \nabla\rho/\rho^2 = \text{curl } w/\rho$, as one easily sees.

LEMMA 2.3. *Let $\zeta(t, x)$ be defined by* (2.9). *Then*

$$\text{div } \zeta = 0 \quad \textit{in} \quad \mathscr{D}'(\Omega)$$

and (2.12)

$$\int_{\Gamma_i} \zeta \cdot n \, d\Gamma = 0 \qquad \forall i = 1,..., m, \quad \forall t \in [0, T].$$

Proof. Suppose that $a \in C^2(\bar{\Omega})$, $g \in C^{0,2}(Q_T)$, $v \in C^{0,2}(Q_T)$,

$$\text{div } v = 0 \text{ in } Q_T, \qquad \text{and} \qquad v \cdot n = 0 \text{ on } [0, T] \times \Gamma.$$

Then the solution $\tilde{\zeta}$ of (2.7) belongs to $C^1(Q_T)$, and consequently $\zeta \in C^1(Q_T)$ is a classical solution of (2.5).

Since

$$(v \cdot \nabla)\zeta - (\zeta \cdot \nabla)v = v \text{ div } \zeta - \zeta \text{ div } v - \text{curl}(v \wedge \zeta), \tag{2.13}$$

we obtain that ζ is the solution of

$$\frac{\partial \zeta}{\partial t} + v \operatorname{div} \zeta = \operatorname{curl}(v \wedge \zeta) + \gamma \qquad \text{in} \qquad Q_T.$$
$$\zeta|_{t=0} = \alpha \qquad \text{in} \qquad \bar{\Omega}.$$

Let $\theta \in C^{1,2}(Q_T)$, $\theta = 0$ on $[0, T] \times \Gamma$, $\theta(T, x) = 0$ for each $x \in \bar{\Omega}$. We obtain

$$\int_{Q_T} \frac{\partial \zeta}{\partial t} \cdot \nabla\theta \, dx\, dt + \int_{Q_T} (\operatorname{div} \zeta) v \cdot \nabla\theta \, dx\, dt = 0,$$

since curl grad $= 0$ and $\nabla\theta \wedge n = 0$ on $[0, T] \times \Gamma$. By integrating by parts

$$-\int_{Q_T} \zeta \cdot \nabla \frac{\partial\theta}{\partial t} \, dx\, dt + \int_{Q_T} (\operatorname{div} \zeta) v \cdot \nabla\theta \, dx\, dt = 0,$$

since $\theta|_{t=T} = 0$, $\operatorname{div} \zeta|_{t=0} = \operatorname{div} \alpha = 0$ and $\theta = 0$ on $[0, T] \times \Gamma$.

Moreover

$$-\int_{Q_T} \zeta \cdot \nabla \frac{\partial\theta}{\partial t} \, dx\, dt = \int_{Q_T} \operatorname{div} \zeta \frac{\partial\theta}{\partial t} \, dx\, dt$$

since $\theta|_{[0,T]\times\Gamma} = 0$.

Hence we have

$$\int_{Q_T} \operatorname{div} \zeta \left(\frac{\partial\theta}{\partial t} + v \cdot \nabla\theta\right) dx\, dt = 0,$$

and consequently

$$\int_{Q_T} (\operatorname{div} \zeta)\, \psi \, dx\, dt = 0 \qquad \forall \psi \in \mathscr{D}(Q_T),$$

since the solution $\theta(t, x)$ of

$$\frac{\partial\theta}{\partial t} + v \cdot \nabla\theta = \psi \qquad \text{in} \qquad Q_T,$$
$$\theta|_{t=T} = 0 \qquad \text{in} \qquad \bar{\Omega}$$

is in $C^{1,2}(Q_T)$ and satisfies $\theta|_{[0,T]\times\Gamma} = 0$.

In conclusion we have $\operatorname{div} \zeta = 0$ in Q_T. Moreover

$$\frac{d}{dt}\int_{\Gamma_1} \zeta \cdot n \, d\Gamma = \int_{\Gamma_1} \frac{\partial\zeta}{\partial t} \cdot n \, d\Gamma = \int_{\Gamma_1} \gamma \cdot n \, d\Gamma + \int_{\Gamma_1} [(\zeta \cdot \nabla) v - (v \cdot \nabla) \zeta] \cdot n \, d\Gamma$$
$$= 0$$

by using (2.13). Hence for each $i = 1, \dots, m$

$$\int_{\Gamma_i} \zeta \cdot n \, d\Gamma = \int_{\Gamma_i} \alpha \cdot n \, d\Gamma = 0 \qquad \forall t \in [0, T].$$

If a, g, and v are not regular, we can approximate them in the following way. By using the Friedrichs mollifiers we can find

$$a^m \in C^{2+\lambda}(\bar{\Omega}),\ a^m \to a \text{ in } C^{1+\lambda/2}(\bar{\Omega}); \qquad g^m \in C^{0,2+\lambda}(Q_T),$$
$$g^m \to g \text{ in } C^{0,1+\lambda/2}(Q_T);\ \tilde{v}^m \in C^{0,2+\lambda}(Q_T),\ \tilde{v}^m \to v \text{ in } C^{0,1+\lambda/2}(Q_T).$$

Hence we have that

$$\begin{aligned} \alpha^m &\equiv \operatorname{curl} a^m \to \alpha && \text{in } \ C^{\lambda/2}(\bar{\Omega}), \\ \gamma^m &\equiv \operatorname{curl} g^m \to \gamma && \text{in } \ C^{0,\lambda/2}(Q_T), \\ \varphi^m &\equiv \operatorname{curl} \tilde{v}^m \to \varphi && \text{in } \ C^{0,\lambda/2}(Q_T). \end{aligned}$$

From this last result we see that the solutions v^m of

$$\begin{aligned} \operatorname{curl} v^m &= \varphi^m && \text{in } \ Q_T, \\ \operatorname{div} v^m &= 0 && \text{in } \ Q_T, \\ v^m \cdot n &= 0 && \text{on } \ [0, T] \times \Gamma \end{aligned}$$

are such that $v^m \in C^{0,2+\lambda}(Q_T)$, $v^m \to v$ in $C^{0,1+\lambda/2}(Q_T)$. Define now the vector function ζ^m by using α^m, γ^m, and v^m; by the first part of the proof it follows that $\operatorname{div} \zeta^m = 0$ in Q_T and $\int_{\Gamma_i} \zeta^m \cdot n \, d\Gamma = 0$ for each $t \in [0, T]$. Moreover, by using (2.7), we easily see that $\zeta^m \to \zeta$ in $C^0(Q_T)$; hence the lemma is proved. ∎

The function ζ defined in (2.9) trivially satisfies $(2.5)_2$; moreover ζ is a solution of $(2.5)_1$ in the following weak sense:

LEMMA 2.4. *For each $\Phi \in C^1(\bar{\Omega})$ one has*

$$\frac{d}{dt}(\zeta, \Phi) = (\gamma, \Phi) + ((\zeta \cdot \nabla) v, \Phi) + ((v \cdot \nabla) \Phi, \zeta), \tag{2.14}$$

where $(\ ,\ )$ is the scalar product in $L^2(\Omega)$.

Proof. We have

$$\int_{\Omega} \zeta(t, x) \cdot \Phi(x)\, dx = \int_{\Omega} \tilde{\zeta}(t, x') \cdot \Phi(U(t, 0, x'))\, dx';$$

hence by $(2.7)_1$

$$\frac{d}{dt} \int_{\Omega} \zeta(t, x) \cdot \Phi(x)\, dx$$

$$= \sum_{i=1}^{3} \int_{\Omega} \left[\frac{d\tilde{\zeta}_i}{dt}(t, x')\, \Phi_i(U(t, 0, x')) \right.$$

$$\left. + \sum_{j=1}^{3} \tilde{\zeta}_i(t, x') \frac{\partial \Phi_i}{\partial x_j}(U(t, 0, x')) \cdot v_j(t, U(t, 0, x')) \right] dx'$$

$$= \sum_{i=1}^{3} \int_{\Omega} \left\{ \left[\gamma_i(t, x) + \sum_{j=1}^{3} \frac{\partial v_i}{\partial x_j}(t, x)\, \zeta_j(t, x) \right] \Phi_i(x) + \sum_{j=1}^{3} \zeta_i(t, x) \frac{\partial \Phi_i}{\partial x_j}(x)\, v_j(t, x) \right\} dx.$$

∎

Now we define the map F as follows. The domain of F consists of the functions φ of the sphere defined by (2.1) with A satisfying

$$A > \| \alpha \|_{\lambda}, \tag{2.15}$$

and such that (2.2) holds.

Finally we put $\zeta = F[\varphi]$.

It follows from estimates (2.11) and from Lemma 2.3 that there exists $T_1 \in]0, T_0]$ such that the set

$$S \equiv \{\varphi \in C^{\lambda,\lambda}(Q_{T_1}) \mid \| \varphi \|_{0,\lambda} \leqslant A,\ [\varphi]_{\lambda,0} \leqslant c_1 A^{1+\lambda},\ \varphi \text{ satisfies } (2.2)\}$$

satisfies $F[S] \subset S$, where F, the norms, and the seminorms correspond to the interval $[0, T_1]$.

S is a convex set and by the Ascoli–Arzelà theorem it follows that S is compact in $C^0(Q_{T_1})$.

Moreover, as in [4], we obtain

LEMMA 2.5. *The map F: $S \to S$ has a fixed point.*

Hence we have construct a solution ζ, v, ρ, w of the auxiliary system

$$
\begin{aligned}
&\frac{\partial \zeta}{\partial t} + (v \cdot \nabla)\, \zeta = \beta + w \wedge \frac{\nabla \rho}{\rho^2} + (\zeta \cdot \nabla)\, v && \text{in} \quad Q_{T_1},\\
&\operatorname{curl} v = \zeta && \text{in} \quad Q_{T_1},\\
&\operatorname{div} v = 0 && \text{in} \quad Q_{T_1},\\
&v \cdot n = 0 && \text{on} \quad [0, T_1] \times \Gamma,\\
&\frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0 && \text{in} \quad Q_{T_1},\\
&\rho\,|_{t=0} = \rho_0 && \text{in} \quad \bar{\Omega},\\
&\operatorname{curl} w = 0 && \text{in} \quad Q_{T_1},\\
&\operatorname{div} w = \frac{\nabla \rho}{\rho} \cdot w + \rho \sum_{i,j} (D_i v_j)(D_j v_i) - \rho \operatorname{div} b && \text{in} \quad Q_{T_1},\\
&w \cdot n = -\rho \sum_{i,j} (D_i n_j)\, v_i v_j - \rho b \cdot n && \text{on} \quad [0, T_1] \times \Gamma,\\
&\zeta\,|_{t=0} = \alpha && \text{in} \quad \bar{\Omega},
\end{aligned}
\tag{A}
$$

where equation $(\mathrm{A})_1$ is satisfied in the sense described in Lemma 2.4.

3. Existence of a Solution of System (E)

First we prove that $D_t v$ exists in the classical sense and belongs to $C^{0,\lambda}(Q_{T_1})$. We need two lemmas:

Lemma 3.1. *If* $v \in C^1(\bar{\Omega})$, $\operatorname{div} v = 0$ *in* Ω, *and* $v \cdot n = 0$ *on* Γ, *then*

$$
\begin{aligned}
\operatorname{div}[(v \cdot \nabla)\, v] &= \sum_{i,j} (D_i v_j)(D_j v_i) && \text{in} \quad \Omega,\\
[(v \cdot \nabla)\, v] \cdot n &= -\sum_{i,j} (D_i n_j)\, v_i v_j && \text{on} \quad \Gamma,
\end{aligned}
\tag{3.1}
$$

where the operator div *is in the sense of distributions in* Ω.

See Bourguignon and Brezis [5, Sect. 3] or Temam [16, Lemma 1.1].

Lemma 3.2. *If* $v \in C^1(\bar{\Omega})$, $\zeta = \operatorname{curl} v$, *we have*

$$((v \cdot \nabla)v, \operatorname{curl} \Phi) = -((v \cdot \nabla)\Phi, \zeta) - ((\zeta \cdot \nabla)v, \Phi) \qquad \forall \Phi \in C_0^\infty(\Omega). \tag{3.2}$$

Proof. If $v \in C^2(\bar{\Omega})$, by a direct computation we have

$$\operatorname{curl}[(v \cdot \nabla)v] = (v \cdot \nabla)\zeta - (\zeta \cdot \nabla)v + (\operatorname{div} v)\zeta,$$

and this leads easily to (3.2).

If $v \in C^1(\bar{\Omega})$, we approximate it with $v_n \in C^2(\bar{\Omega})$. ▮

Now we can prove the existence of $D_t v$.

LEMMA 3.3. *We have*

$$\frac{\partial v}{\partial t} = b + \frac{w}{\rho} - (v \cdot \nabla)\, v \qquad \text{in} \qquad Q_{T_1}; \tag{3.3}$$

hence $\partial v/\partial t \in C^{0,\lambda}(Q_{T_1})$.

Proof. Let $\Phi \in C_0^\infty(\Omega)$. We have

$$D_t(v, \operatorname{curl} \Phi) = D_t(\zeta, \Phi)$$

since $\operatorname{curl} v = \varphi = \zeta$. Moreover from (2.14), (3.2), and the equation $\gamma = \operatorname{curl}(b + w/\rho)$ we obtain

$$\begin{aligned} D_t(v, \operatorname{curl} \Phi) &= (\gamma, \Phi) + ((\zeta \cdot \nabla)\, v, \Phi) + ((v \cdot \nabla)\, \Phi, \zeta) \\ &= (\gamma, \Phi) - ((v \cdot \nabla)\, v, \operatorname{curl} \Phi) = \left(b + \frac{w}{\rho} - (v \cdot \nabla)\, v, \operatorname{curl} \Phi\right). \end{aligned}$$

Hence

$$\begin{aligned} (v, \operatorname{curl} \Phi) &= \left(v(0, \cdot), \operatorname{curl} \Phi\right) + \int_0^t \left(b + \frac{w}{\rho} - (v \cdot \nabla)\, v, \operatorname{curl} \Phi\right) d\tau \\ &= \left(v(0, \cdot) + \int_0^t \left[b + \frac{w}{\rho} - (v \cdot \nabla)\, v\right] (\tau, \cdot)\, d\tau, \operatorname{curl} \Phi\right), \end{aligned}$$

and consequently for each $t \in [0, T_1]$

$$v(t, x) - v(0, x) - \int_0^t \left[b + \frac{w}{\rho} - (v \cdot \nabla)\, v\right] (\tau, x)\, d\tau = \nabla \Xi(t, x),$$

where $\Xi \in C^{1+\lambda}(\bar{\Omega})$, $\forall t \in [0, T_1]$. From $(2.3)_2$, $(2.3)_3$, $(\mathrm{A})_8$, $(\mathrm{A})_9$, and (3.1) we conclude that $\operatorname{div} \nabla \Xi = 0$ in the distributions sense, and $\nabla \Xi \cdot n = 0$ on $[0, T_1] \times \Gamma$, hence (3.3). ▮

From (3.3) and $(\mathrm{A})_7$ it follows that

$$\rho \left[\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v - b\right] = -\nabla \pi \qquad \text{in} \qquad Q_{T_1},$$

i.e., $(E)_1$ holds, with $\pi \in C^{0,2+\lambda}(Q_{T_1})$ (see Lemmas 4.1 and 4.2 in [4]). Furthermore

$$\begin{aligned} \operatorname{curl}(v|_{t=0} - a) = \zeta|_{t=0} - \alpha = 0 \quad & \text{in } \bar{\Omega}, \\ \operatorname{div}(v|_{t=0} - a) = 0 \quad & \text{in } \bar{\Omega}, \\ (v|_{t=0} - a) \cdot n = 0 \quad & \text{on } \Gamma, \end{aligned}$$

and consequently $(E)_6$ holds.

Finally, as in Remark 5.5 in [4], we prove that $\rho \in C^{1+\lambda,1+\lambda}(Q_{T_1})$, and the proof of Theorem A is complete.

4. The Case Ω not Simply Connected

By the hypotheses on the domain Ω (see Section 1), it is clear that if Ω is not simply connected, one can make it so by means of a finite number of regular cuts. The number N of these cuts is the dimension of the first cohomology space $H_c(\Omega)$ of Ω, i.e., the quotient of the space of closed differential forms by the space of exact differential forms.

Moreover one can construct N functions $q_1, q_2, \dots, q_N$ such that $v^{(k)} = \operatorname{grad} q_k$ are linearly independent and satisfy $v^{(k)} \in C^{1+\lambda}(\bar{\Omega})$, $\operatorname{div} v^{(k)} = 0$, $\operatorname{curl} v^{(k)} = 0$, $v^{(k)} \cdot n = 0$ on Γ. These $v^{(k)}$ are a basis of the space $H_c(\Omega)$.

Finally, one sees that a function w is a gradient if and only if $\operatorname{curl} w = 0$ and $(w, v^{(k)}) = 0$ for each $k = 1, \dots, N$ (for these results see Foias and Temam [7, Remark 1.2, Lemma 1.3, and Proposition 1.1]).

We can orthonomalize the $v^{(k)}$; if we denote the orthonomal system thus obtained by $u^{(k)}$, we have constructed a system of vectors which has the properties of that introduced in [8, Sect. 1].

The difference between two solutions v_1 and v_2 of (2.3) is given by

$$v_1(t, x) - v_2(t, x) = \sum_k \theta_k(t)\, u^{(k)}(x),$$

where the $\theta_k(t) \in C^0([0, T])$ are arbitrary.

We denote by $v(t, x)$ the solution of (2.3) such that $(v, u^{(k)}) = 0$ for each $k = 1, \dots, N$. Such a solution is obviously unique, and we have

$$\| v \|_{0,1+\lambda} \leqslant c \| \varphi \|_{0,\lambda} \,.$$

Moreover each solution $\bar{v}$ of (2.3) can be written in the form

$$\bar{v}(t, x) = v(t, x) + \sum_k \theta_k(t)\, u^{(k)}(x).$$

Hence, arguing as is [4], we obtain a solution $\bar{v}$, $\bar{\rho}$, $\bar{w}$ of system (6.1)–(6.5) and we prove Lemmas 7.2 and 7.3 and Remark 7.4 of [4]. Hence, by proceeding as before, we construct a function $\bar{\zeta}$ which satisfies the usual properties and we find a fixed point $\varphi = \bar{\zeta}$ (see Section 2 of this paper). The regularity of $D_t\bar{v}$ is proved as in Lemma 3.3 of this paper, by also using the fact that

$$D_t(\bar{v}, u^{(k)}) = \left(\frac{\bar{w}}{\bar{\rho}} - (\bar{v} \cdot \nabla)\,\bar{v} + b, u^{(k)}\right), \qquad \forall t \in [0, T_1], \quad \forall k = 1,\dots, N;$$

finally one has

$$\begin{aligned} \operatorname{curl}(\bar{v}\,|_{t=0} - a) = \bar{\zeta}\,|_{t=0} - \alpha &= 0 && \text{in } \bar{\Omega}, \\ (\bar{v}\,|_{t=0} - a, u^{(k)}) &= 0, && \forall k = 1,\dots, N, \\ \operatorname{div}(\bar{v}\,|_{t=0} - a) &= 0 && \text{in } \bar{\Omega}, \\ (\bar{v}\,|_{t=0} - a) \cdot n &= 0 && \text{on } \Gamma, \end{aligned}$$

that is, $\bar{v}\,|_{t=0} = a$ in $\bar{\Omega}$.

References

1. S. N. Antoncev and A. V. Kazhikhov, Mathematical study of flows of non homogeneous fluids, Novosibirsk, Lectures at the University, 1973 (Russian).
2. C. Bardos and U. Frisch, Finite time regularity for bounded and unbounded ideal incompressible fluids using Hölder estimates, *in* "Turbulence and Navier–Stokes Equation" (R. Temam, Ed.), Lecture Notes in Mathematics No 565, Springer-Verlag, Berlin/Heidelberg, 1976.
3. H. Beirão da Veiga and A. Valli, On the motion of a non-homogeneous ideal incompressible fluid in an external force field, *Rend. Sem. Mat. Univ. Padova* **59** (1978), 117–145.
4. H. Beirão da Veiga and A. Valli, On the Euler equations for non-homogeneous fluids, I, *Rend. Sem. Mat. Univ. Padova*, in press.
5. J. P. Bourguignon and H. Brezis, Remarks on the Euler equation, *J. Functional Analysis* **15** (1974), 341–363.
6. D. G. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92** (1970), 102–163.
7. C. Foias and R. Temam, Remarques sur les équations de Navier–Stokes stationnaires et les phénomènes succesifs de bifurcation, *Ann. Scuola Norm. Sup. Pisa, Serie IV* **5** (1978), 29–63.
8. T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, *J. Functional Analysis* **9** (1972), 296–305.
9. A. V. Kazhikhov, Solvability of the initial and boundary-value problem for the equations of motion of an inhomogeneous viscous incompressible fluid, *Soviet Physics Dokl.* **19** (1974), 331–332 (previously in *Dokl. Akad. Nauk SSSR* **216** (1974), 1008–1010 (Russian)).
10. P. H. LeBlond and L. A. Mysak, "Waves in the Ocean," Elsevier, Amsterdam, 1977.

11. L. Lichtenstein, Über einige Existenzprobleme der Hydrodynamik..., *Math. Z.* **23** (1925), 89–154, 310–316; **26** (1927), 196–323; **28** (1928), 387–415, 752; **32** (1930), 608–640.
12. J. L. Lions, On some questions in boundary value problems of Mathematical Physics, Rio de Janeiro, Lectures at the Univ. Federal, Instituto de Mat., 1977.
13. J. E. Marsden, Well-posedness of the equations of a non-homogeneous perfect fluid, *Comm. Partial Differential Equations* **1** (1976), 215–230.
14. L. Sédov, "Mécanique des milieux continus," Vol. I, Editions MIR, Moscou, 1975 (trans. from Russian).
15. H. S. G. Swann, The convergence with vanishing viscosity of nonstationary Navier–Stokes flow to ideal flow in $\mathbb{R}^3$, *Trans. Amer. Math. Soc.* **157** (1971), 373–397.
16. R. Temam, On the Euler equations of incompressible perfect fluids, *J. Functional Analysis* **20** (1975), 32–43.
17. C.-S. Yih, "Dynamics of Non-Homogeneous Fluids," The Macmillan Co., New York, 1965.
18. R. K. Zeytounian, "Notes sur les écoulements rotationnels de fluides parfaits," Lecture Notes in Physics No. 27, Springer-Verlag, Berlin/Heidelberg, 1974.
19. M. S. Baouendi and C. Goulaouic, Solutions analytiques de l'équation d'Euler d'un fluide incompressible, Seminaire Goulaouic–Schwartz, 1976–77 (Paris).
20. D. Graffi, Il teorema di unicità per i fluidi incompressibili, perfetti, eterogenei, *Rev. Un. Mat. Argentina* **17** (1955), 73–77.
21. A. V. Kazhikhov, Theory of boundary value problems for the equations of a viscous, non-homogeneous fluid, *in* "Seminar on Numerical Methods in Mechanics (Seminar N. N. Yanenko), Novosibirsk," Part II (1975), pp. 65–76 (Russian).
22. O. A. Ladyzhenskaya and V. A. Solonnikov, Unique solvability of an initial and boundary value problem for viscous incompressible nonhomogeneous fluids, *J. Soviet Math.* **9** (1978) (previously in *Zap. Naučn. Sem. Leningrad.* **52** (1975), 52–109, 218–219 (Russian)).
23. J. Simon, Écoulement d'un fluide non homogène avec une densité initiale s'annullant, *C. R. Acad. Sci. Paris* **287** (1978), 1009–1012.