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Equivariant rational homotopy theory as a closed model category

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Abstract

In this note we present a variant of the algebraization of equivariant rational homotopy theory. For a finite group G, let $\mathcal{O}(G)$ be the category of its canonical orbits. We prove that the category $\mathcal{O}(G)$-$DGA_{\mathbb{Q}}$ of $\mathcal{O}(G)$-differential graded algebras over the rationals is a closed model category. Then, by means of the equivariant KS-minimal models constructed in this paper, we show that the homotopy category of $\mathcal{O}(G)$-$DGA_{\mathbb{Q}}$ is equivalent to the rational homotopy category of $\mathcal{O}(G)$-simplicial sets provided G is a Hamiltonian group. © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

Let k be a field of characteristic 0 and DGA_k (resp. $\mathbb{SS}$) the category of homologically connected (i.e., $H^0(A)=k$ for A in DGA_k) commutative differential graded k-algebras (resp. the category of connected simplicial sets). It has been proved [1, 10] that these categories form closed model categories in the sense of Quillen [10]: *weak equivalences* are homology isomorphisms (resp. weak homotopy equivalences); *fibrations* are surjections (resp. Kan fibrations) and *cofibrations* are maps having the left-lifting property with respect to all maps which are both fibrations and weak equivalences. An algebra A (resp. a simplicial set X) is *cofibrant* (resp. *fibrant*) if the canonical map $k \to A$ (resp. $X \to *$) is a cofibration (resp. a fibration).

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The most important cofibrant algebras are the *minimal* ones introduced by Sullivan [11]. They are sufficient for most homotopy theoretic purposes because every connected algebra A can be "approximated up to weak equivalence" by a unique-up-to-isomorphism minimal algebra M_A, i.e., there is a weak equivalence $\rho_A : M_A \to A$. Moreover, in [1] a pair of adjoint functors are constructed

$$DGA_{\mathbb{Q}} \underset{A^*}{\overset{F_*}{\rightleftarrows}} \mathbb{SS}$$

which determine the basic Sullivan–de Rham equivalence, where $\mathbb{Q}$ is the field of rationals. An equivariant version of the Sullivan minimal model theory was given in [12, 13] for nilpotent G-spaces of finite type with a basepoint, where G is a finite group. Later Fine was able to remove the basepoint hypothesis in his Chicago Ph.D. thesis in 1992. Our aim is to present a variant of the equivariant Sullivan–de Rham equivalence based on the Bousfield and Gugenheim categorical approach [1].

We now give an outline of the paper. In Section 1 (Theorem 1.3) we show how, by means of [3], a closed model structure on a category $\mathbb{C}$ can be extended to the functor category $\mathbb{I}$-$\mathbb{C}$ (called the category of $\mathbb{I}$-objects), where $\mathbb{I}$ is an EI-category i.e., each of its endomorphism is an isomorphism. In particular, on the category $\mathbb{I}$-$DGA_{\mathbb{Q}}$ a closed model structure is induced from such a structure on $DGA_{\mathbb{Q}}$ (see [1]) and on the category $\mathbb{I}$-$\mathbb{SS}$ from such a one (see [10]) on the category dual to $\mathbb{SS}$. Then, we consider a pair of functors

$$\text{ho}\,\mathbb{I}\text{-}DGA_{\mathbb{Q}} \underset{\mathscr{C}}{\overset{\mathscr{F}_*}{\rightleftarrows}} \text{ho}\,\mathbb{I}\text{-}\mathbb{SS}$$

between the associated homotopy categories. We point out that for any small category $\mathbb{I}$ the category $\mathbb{I}$-$\mathbb{SS}$ has also been endowed in [2] with a closed model category structure but inherited from such a one on $\mathbb{SS}$ and not on the dual.

In Section 2, assuming some properties of $\mathbb{I}$, we prove that $\mathscr{C}(\mathscr{X})$, for $\mathscr{X}$ in $\mathbb{I}$-$\mathbb{SS}$ could be chosen as an appropriate KS-minimal model. An idea for its construction for a special case has been given in [8] and is based on the notion of a Koszul–Sullivan extension presented in [7]. Then some geometric applications are presented. In particular, let G be a finite group and $\mathscr{O}(G)$ the associated EI-category of canonical orbits. Its objects are orbits G/H for all subgroups $H \subseteq G$ and morphisms are G-maps between them. Then, as a result of Theorem 1.3, we may state

Theorem. *The categories $\mathscr{O}(G)$-$\mathbb{SS}$ and $\mathscr{O}(G)$-$DGA_{\mathbb{Q}}$ are closed model categories.*

For any G-connected simplicial set X (i.e., such that all fixed point simplicial subsets X^H are connected for subgroups $H \subseteq G$), we can consider differential graded $\mathbb{Q}$-algebras of polynomial forms $A^*(X^H)$ for all subgroups $H \subseteq G$. Therefore, we obtain a functor

$$\mathscr{A}^* : G\text{-}\mathbb{SS} \to \mathscr{C}(\mathbb{G})\text{-}\mathbb{DGA}_{\mathbb{Q}},$$

where G-$\mathbb{SS}$ is the category of G-connected simplicial sets. On the other hand, from [4] one could deduce the existence of an equivalence of homotopy categories

$$\text{ho}\, G\text{-}\mathbb{SS} \underset{\approx}{\rightleftarrows} \text{ho}\, \mathscr{C}(G)\text{-}\mathbb{SS}.$$

Now let $DGA^0_{\mathbb{Q}}$ (resp. $\mathbb{SS}^0$) be the category of homologically connected augmented differential graded $\mathbb{Q}$-algebras (resp. the category of pointed simplicial sets) and let G be a finite Hamiltonian group (i.e., each subgroup of G is normal). We show that for a nilpotent X in G-$\mathbb{SS}^0$, the equivariant KS-minimal model of $\mathscr{A}^*(X)$ has the strong homotopy type of its injective model considered in [12, 13] and we prove the following equivariant version of the Sullivan–de Rham equivalence.

Theorem 2.7. *If G is a Hamiltonian group, then there exists a pair of adjoint functors*

$$\text{ho}\, G\text{-}\mathbb{SS}^0 \rightleftarrows \text{ho}\, \mathscr{C}(G)\text{-}DGA^0_{\mathbb{Q}}$$

which restrict to inverse equivalences

$$f\mathbb{Q}N\text{-ho}\, G\text{-}\mathbb{SS}^0 \underset{\approx}{\rightleftarrows} f\mathbb{Q}\text{-ho}\, \mathscr{C}(G)\text{-}DGA^0_{\mathbb{Q}},$$

where $f\mathbb{Q}N$-ho G-$\mathbb{SS}^0$ is the full subcategory of ho G-$\mathbb{SS}^0$ induced by those G-connected pointed simplicial sets which are nilpotent and of finite type and $f\mathbb{Q}$-ho $\mathscr{C}(G)$-$DGA^0_{\mathbb{Q}}$ is the full subcategory of ho $\mathscr{C}(G)$-$DGA^0_{\mathbb{Q}}$ induced by those augmented $\mathscr{C}(G)$-algebras which are equivalent to equivariant KS-minimal $\mathscr{C}(G)$-algebras and with finitely many multiplicative generators.

In a forthcoming paper, we plan to extend this result to G-disconnected unpointed simplicial sets.

1. Systems of algebras

Various categories considered in algebraic topology have the property that endomorphisms are isomorphisms. Therefore, let $\mathbb{I}$ be a small *EI-category* which by definition, is a small category in which each endomorphism is an isomorphism and denote by $\text{Ob}(\mathbb{I})$ the set of its objects. Following [9] we define a partial order, crucial for the sequel, on the set $\text{Is}(\mathbb{I})$ of isomorphism classes $\bar{i}$ of objects $i \in \text{Ob}(\mathbb{I})$ by

$$\bar{i} \le \bar{j} \quad \text{if } \mathbb{I}(i,j) \ne \emptyset.$$

This induces a partial ordering on the set $\mathrm{Is}(\mathbb{I})$, since the EI-property ensures that $\bar{i} \le \bar{j}$ and $\bar{j} \le \bar{i}$ implies $\bar{i} = \bar{j}$. We write that $\bar{i} < \bar{j}$ if $\bar{i} \le \bar{j}$ and $\bar{i} \ne \bar{j}$.

Throughout, $\mathbb{I}$ is a *cofinite* EI-category i.e., each isomorphism class $\bar{i}$ has only finitely many predecessors. For any $i \in \mathrm{Ob}(\mathbb{I})$ we define its *height* as the number of its predecessors. Observe that any group G can be treated as an EI-category with a single object.

Fix a complete and cocomplete category $\mathbb{C}$ with a closed model structure. Our aim is to define, by means of [3], such a structure on the category $\mathbb{I}$-$\mathbb{C}$ of all covariant functors from $\mathbb{I}$ to $\mathbb{C}$, called $\mathbb{I}$-*objects* of $\mathbb{C}$ or *systems of objects* indexed by $\mathbb{I}$. For this purpose, we distinguish in this category the following three classes of maps. A map $f : \mathscr{A} \to \mathscr{B}$ of $\mathbb{I}$-objects is called a *weak equivalence* (resp. *fibration*) if for all $i \in \mathrm{Ob}(\mathbb{I})$ the maps $f(i) : \mathscr{A}(i) \to \mathscr{B}(i)$ are weak equivalences (resp. fibrations) in the category $\mathbb{C}$. A map $f : \mathscr{A} \to \mathscr{B}$ is a *cofibration* if it has the left-lifting property with respect to all maps which are both fibrations and weak equivalences i.e., trivial fibrations. In particular, for a group G the category G-$\mathbb{C}$ of G-*objects* inherits a closed model structure from $\mathbb{C}$.

Let $\mathrm{Aut}(i)$ be the automorphism group of $i \in \mathrm{Ob}(\mathbb{I})$ and $\mathscr{A}$ an $\mathbb{I}$-object. Then on $\mathscr{A}(i)$ there is the natural $\mathrm{Aut}(i)$-action and, for a map $f : \mathscr{A} \to \mathscr{B}$ of $\mathbb{I}$-objects, the maps $f(i) : \mathscr{A}(i) \to \mathscr{B}(i)$ preserve the $\mathrm{Aut}(i)$-action. Therefore, for a fixed $i \in \mathrm{Ob}(\mathbb{I})$, we have the *restriction* functor

$$\mathrm{Res}_i : \mathbb{I}\text{-}\mathbb{C} \to \mathrm{Aut}(i)\text{-}\mathbb{C}$$

such that $\mathrm{Res}_i(\mathscr{A}) = \mathscr{A}(i)$ for an $\mathbb{I}$-object $\mathscr{A}$ and its right adjoint $F_i : \mathrm{Aut}(i)\text{-}\mathbb{C} \to \mathbb{I}\text{-}\mathbb{C}$ is called the *coextension* functor which is defined as follows. For $i' \in \mathrm{Ob}(\mathbb{I})$, let $\mathbb{I}_i^{i'}$ be the category with objects being maps $\phi : i' \to i$ and maps from $\phi : i' \to i$ to $\psi : i' \to i$ are determined by maps $\rho : i \to i$ such that $\rho\phi = \psi$. Then any $\mathrm{Aut}(i)$-object C determines an $\mathbb{I}_i^{i'}$-object $\mathscr{F}_i^{i'}(C)$ such that $\mathscr{F}_i^{i'}(C)(\phi : i' \to i) = C$ and we put $F_i(C)(i') = \varprojlim_{\mathbb{I}_i^{i'}} \mathscr{F}_i^{i'}(C)$. Of course, any map $\phi : i'' \to i'$ in the category $\mathbb{I}$ determines a map $F_i(C)(\phi) : F_i(C)(i'') \to F_i(C)(i')$ and this construction is functorial with respect to $\mathrm{Aut}(i)$-objects C as well. Note that an isomorphism $i' \overset{\approx}{\to} i$ determines an isomorphism $C \overset{\approx}{\to} F_i(C)(i')$.

For a fixed $i \in \mathrm{Ob}(\mathbb{I})$, let $\mathbb{I}_i$ be the category which objects are pairs (i', ϕ), where $\phi : i' \to i$ is a non-isomorphism and maps from (i'_1, ϕ_1) to (i'_2, ϕ_2) are determined by maps $\psi : i'_1 \to i'_2$ such that $\phi_2\psi = \phi_1$. Then any $\mathbb{I}$-object $\mathscr{A}$ determines an $\mathbb{I}_i$-object $\mathscr{A}_i$ such that $\mathscr{A}_i(i', \phi) = \mathscr{A}(i')$ and a map $\varinjlim_{\mathbb{I}_i} \mathscr{A}_i \to \mathscr{A}(i)$ in the category $\mathbb{C}$. Note that $\varinjlim_{\mathbb{I}_i} \mathscr{A}_i$ is isomorphic to the initial object in the category $\mathbb{C}$ for i of height 0. We now state the following description of cofibrations in the category $\mathbb{I}$-$\mathbb{C}$.

Proposition 1.1. *Let $\mathbb{I}$ be a cofinite small EI-category. A map $f : \mathscr{A} \to \mathscr{B}$ in $\mathbb{I}$-$\mathbb{C}$ is a (trivial) cofibration if and only if for each $i \in \mathrm{Ob}(\mathbb{I})$ the induced* $\mathrm{Aut}(i)$*-map* $h(i)$

in the pushout diagram

$$\begin{array}{ccccc}
\varinjlim_{\mathbb{I}_i} \mathscr{A}_i & \longrightarrow & \mathscr{A}(i) & & \\
\downarrow & & \vdots & \searrow & \\
\varinjlim_{\mathbb{I}_i} \mathscr{B}_i & \overset{\gamma}{\dashrightarrow} & \mathscr{C}(i) & & \\
& \searrow & & \overset{h(i)}{\searrow} & \\
& & & & \mathscr{B}(i)
\end{array}$$

is a (trivial) cofibration in the category Aut(i)-$\mathbb{C}$.

Proof. First, let f be a (trivial) cofibration in $\mathbb{I}$-$\mathbb{C}$ and for a fixed $i \in \mathrm{Ob}(\mathbb{I})$ consider a commutative diagram

$$(\star) \qquad \begin{array}{ccc}
\mathscr{C}(i) & \xrightarrow{\delta} & D \\
{\scriptstyle h(i)}\downarrow & & \downarrow{\scriptstyle p} \\
\mathscr{B}(i) & \longrightarrow & E
\end{array}$$

in the category Aut(i)-$\mathbb{C}$, where p is a (trivial) fibration. Define the objects $\mathscr{D}, \mathscr{E}$ in $\mathbb{I}$-$\mathbb{C}$ as follows:

$$\mathscr{D}(i') = \begin{cases} F_i(D)(i') & \text{for } \overline{i'} \le \bar{i}, \\ * & \text{otherwise,} \end{cases} \qquad \mathscr{E}(i') = \begin{cases} F_i(D)(i') & \text{for } \overline{i'} < \bar{i}, \\ F_i(E)(i') & \text{for } \overline{i'} = \bar{i}, \\ * & \text{otherwise,} \end{cases}$$

where $*$ is the terminal object in $\mathbb{C}$ and maps are induced either by projections or p or being trivial. Then we obtain the commutative diagram in the category $\mathbb{I}$-$\mathbb{C}$

$$(\star\star) \qquad \begin{array}{ccc}
\mathscr{A} & \xrightarrow{\alpha} & \mathscr{D} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\
\mathscr{B} & \xrightarrow[\beta]{} & \mathscr{E},
\end{array}$$

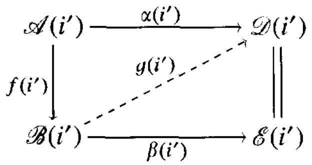
where $\bar{p}(i')$ is either the identity or induced by p. Thus, the map $\bar{p}$ is a (trivial) fibration. The maps $\alpha(i')$ are induced from the composite maps

$$\mathscr{A}(i') \to F_i(\mathscr{A}(i))(i') \to F_i(\mathscr{C}(i))(i') \xrightarrow{F_i(\delta)(i')} \mathscr{D}(i')$$

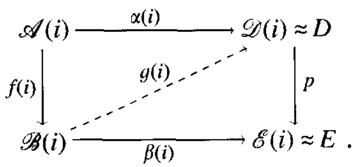
for $\bar{i'} \le \bar{i}$, the maps $\beta(i')$ from

$$\mathscr{B}(i') \to F_i(\varinjlim_{\mathbb{I}_i} \mathscr{B}_i)(i') \xrightarrow{F_i(\gamma)(i')} F_i(\mathscr{C}(i))(i') \xrightarrow{F_i(\delta)(i')} \mathscr{E}(i')$$

for $\bar{i'} < \bar{i}$ and $\mathscr{B}(i) \to E \overset{\approx}{\to} \mathscr{E}(i')$ for $\bar{i'} = \bar{i}$. So, there exists a filler g in $(\star\star)$ and we have the commutative diagrams

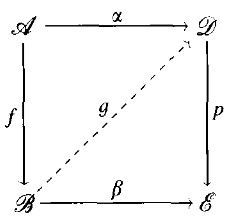


for $\bar{i'} < \bar{i}$ and

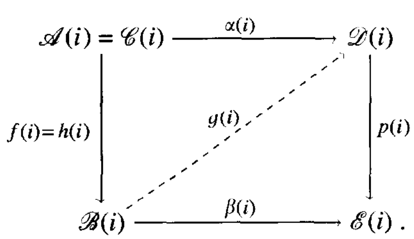


To show that $g(i)$ is a filler in $(\star)$, it is sufficient to prove that $g(i)h(i) = \delta$.

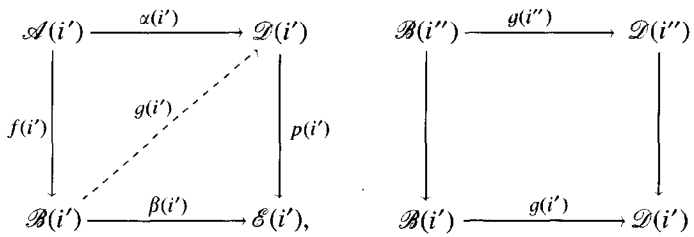
Now let all $h(i)$ be (trivial) cofibrations and consider a solid-arrow commutative diagram in the category $\mathbb{I}$-$\mathbb{C}$



in which the map p is a (trivial) fibration. We construct components $g(i)$ of a filler g inductively with respect to the height of i. If $i \in \mathrm{Ob}(\mathbb{I})$ has height 0, then $\mathscr{C}(i) = \mathscr{A}(i)$, $f(i) = h(i)$ and there exists a filler $g(i)$:

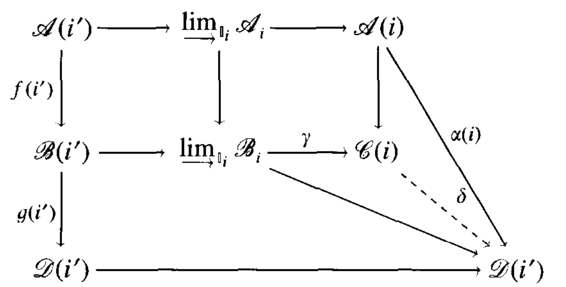


Suppose that for all $i' \in \mathrm{Ob}(\mathbb{I})$ of height smaller than that of i there exist $g(i'): \mathscr{B}(i') \to \mathscr{D}(i')$ such that diagrams

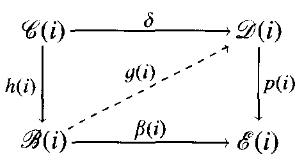


commute for $\overline{i''} \leq \overline{i'}$.

At first we define a map $\delta : \mathscr{C}(i) \to \mathscr{D}(i)$ assuming commutativity of the diagrams



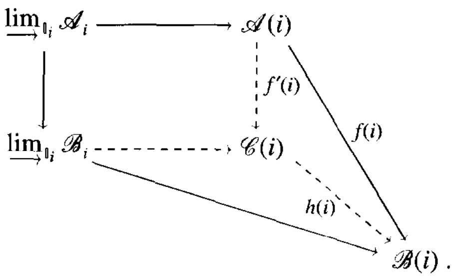
and we get the solid-arrow diagram



which is commutative because $p(i) \circ \delta \circ (\mathscr{A}(i) \to \mathscr{C}(i)) = p(i) \circ \alpha(i) = \beta(i) \circ f(i) = \beta(i) \circ h(i) \circ (\mathscr{A}(i) \to \mathscr{C}(i))$ and $p(i) \circ \delta \circ \gamma \circ (\mathscr{B}(i') \to \underrightarrow{\lim}_{\mathbb{I}_i} \mathscr{B}_i) = p(i) \circ \mathscr{D}(i' \to i) \circ g(i') = \mathscr{E}(i' \to i) \circ p(i') \circ g(i') = \mathscr{E}(i' \to i) \circ \beta(i') = \beta(i) \circ \mathscr{B}(i' \to i) = \beta(i) \circ h(i) \circ \gamma \circ (\mathscr{B}(i') \to \underrightarrow{\lim}_{\mathbb{I}_i} \mathscr{B}_i)$. Thus, there exists a filler $g(i)$ and we have $g(i) \circ f(i) = g(i) \circ h(i) \circ (\mathscr{A}(i) \to \mathscr{C}(i)) = \delta \circ (\mathscr{A}(i) \to \mathscr{C}(i)) = \alpha(i)$, $p(i) \circ g(i) = \beta(i)$ and $g(i) \circ \mathscr{B}(i' \to i) = g(i) \circ h(i) \circ \gamma \circ (\mathscr{B}(i') \to \underrightarrow{\lim}_{\mathbb{I}_i} \mathscr{B}_i) = \mathscr{D}(i' \to i) \circ g(i')$ for $\overline{i'} \leq \overline{i}$. So, the inductive step is done. □

Corollary 1.2. *Let $\mathbb{I}$ be a cofinite small EI-category and $f : \mathscr{A} \to \mathscr{B}$ a cofibration in $\mathbb{I}$-$\mathbb{C}$. Then for each $i \in \mathrm{Ob}(\mathbb{I})$ the map $f(i) : \mathscr{A}(i) \to \mathscr{B}(i)$ is a cofibration in the category* $\mathrm{Aut}(i)$-$\mathbb{C}$.

Proof. For $i \in \mathrm{Ob}(\mathbb{I})$, consider the commutative diagram



Then we see that the map $\varinjlim_{\mathbb{I}_i} f_i$ is a cofibration in $\mathbb{C}$ and $f'(i)$ is also. By Proposition 1.1 the map $h(i)$ is a cofibration and, consequently, the composite map $h(i)f'(i) = f(i)$ is a cofibration. □

The above results and a dualization of the procedure presented in [3, Section 3] yield

Theorem 1.3. *If* $\mathbb{I}$ *is a cofinite small EI-category, then the category* $\mathbb{I}$-$\mathbb{C}$, *together with the above structure, is a closed model category.*

Now let k be a field and DGA_k (resp. $\mathbb{SS}$) the category of homologically connected commutative differential graded k-algebras (resp. the category of connected simplicial sets). On the category $\mathbb{I}$-DGA_k of all $\mathbb{I}$-algebras, a closed model structure is determined from such a structure on DGA_k (considered in [1]) and, on the category $\mathbb{I}$-$\mathbb{SS}$, from such a structure (considered in [10]) on the dual category to $\mathbb{SS}$. For $k = \mathbb{Q}$, the pair of adjoint functors

$$DGA_{\mathbb{Q}} \underset{A^*}{\overset{F_*}{\rightleftarrows}} \mathbb{SS}$$

considered in [1] induces such a pair between functor categories

$$\mathbb{I}\text{-}DGA_{\mathbb{Q}} \underset{\mathscr{A}^*}{\overset{\mathscr{F}_*}{\rightleftarrows}} \mathbb{I}\text{-}\mathbb{SS}.$$

For $\mathscr{A}$ in $\mathbb{I}$-DGA_k and A in DGA_k define an $\mathbb{I}$-algebra $A \otimes \mathscr{A} \in \mathbb{I}$-$DGA_k$ by $(A \otimes \mathscr{A})(i) = A \otimes \mathscr{A}(i)$ for $i \in \mathrm{Ob}(\mathbb{I})$. Then we get a functor

$$F : \mathbb{I}\text{-}DGA_k \times \mathbb{I}\text{-}DGA_k \to \mathbb{SS}$$

such that $F(\mathscr{A}, \mathscr{B})_n = \mathbb{I}\text{-}DGA_k(\mathscr{A}, A^*(\Delta[n]) \otimes \mathscr{B})$ for $n \geq 0$, where $A^*(\Delta[n])$ is the de Rham k-algebra on the n-simplex $\Delta[n]$ ([1]).

Proposition 1.4. *Let $\mathbb{I}$ be a cofinite small EI-category.*

(1) If $p:\mathscr{E}\to\mathscr{B}$ is a (trivial) fibration and $\mathscr{C}$ cofibrant in $\mathbb{I}$-DGA_k then the induced map $p_*:F(\mathscr{C},\mathscr{E})\to F(\mathscr{C},\mathscr{B})$ is a (trivial) fibration in the category $\mathbb{SS}$.

(2) If $i:\mathscr{C}\to\mathscr{D}$ is a (trivial) cofibration in the category $\mathbb{I}$-DGA_k then the induced map $\mathscr{F}_*(i):\mathscr{F}_*(\mathscr{D})\to\mathscr{F}_*(\mathscr{C})$ is a (trivial) fibration in the category $\mathbb{I}$-$\mathbb{SS}$.

Proof. (1) We must show that the map $p_*:F(\mathscr{C},\mathscr{E})\to F(\mathscr{C},\mathscr{B})$ has the right lifting property with respect to the canonical maps $u:\Lambda^m[n]\to\Delta[n]$ (resp. $u:\dot{\Delta}[n]\to\Delta[n]$) for $n\geq 0$ and $0\leq m\leq n$, where $\Lambda^m[n]$ (resp. $\dot{\Delta}[n]$) is the mth "boundary cone" (resp. "boundary") of the n-simplex $\Delta[n]$. But this means that the cofibration $\underline{k}\to\mathscr{C}$ should have the left lifting property for the map

$$(A^*u\otimes\mathrm{id},\mathrm{id}\otimes p):A^*(\dot{\Delta}[n])\otimes\mathscr{E}\to A^*(\Delta[n])\otimes\mathscr{E}\times_{A^*(\dot{\Delta}[n])\otimes\mathscr{B}}A^*(\Delta[n])\otimes\mathscr{B}$$

in the category $\mathbb{I}$-DGA_k. By [1] this map is a (trivial) fibration and this completes the proof of (1).

(2) follows from Proposition 1.1 and its dual in the category $\mathbb{I}$-$\mathbb{SS}$. □

Observe that $A^*(\Delta[1])$ is the free DGA_k on two generators t and dt of degree 0 and 1, respectively, with $d(t)=dt$. We say that two maps $f,g:\mathscr{A}\to\mathscr{B}$ are *homotopic* (denoted by $f\simeq g$) if there is a map $H:\mathscr{A}\to\mathscr{B}\otimes A^*(\Delta[1])$ such that $p_0\circ H=f$ and $p_1\circ H=g$, where p_0 is the projection $\mathscr{B}\otimes A^*(\Delta[1])\to\mathscr{B}$ with $t=0$, $dt=0$ and p_1 the projection with $t=1$, $dt=0$. We define the notion of homotopy between two maps $f,g:\mathscr{X}\to\mathscr{Y}$ of $\mathbb{I}$-simplicial sets similarly.

For any closed model category $\mathbb{C}$, a homotopy category ho$\mathbb{C}$ is constructed in [10] by adjoining formal inverses of weak equivalences in $\mathbb{C}$. This category is equivalent to the more simple homotopy category, ho$\mathbb{C}=\pi\mathbb{C}_{cf}$, whose objects are the "fibrant–cofibrant" objects of $\mathbb{C}$ and maps are "homotopy classes" of maps in $\mathbb{C}$. We will use the homotopy category ho$\mathbb{I}$-$\mathbb{SS}$ (resp. ho$\mathbb{I}$-DGA_k), whose objects are fibrant $\mathbb{I}$-simplicial sets (resp. cofibrant $\mathbb{I}$-algebras) and maps are given by ho$\mathbb{I}$-$\mathbb{SS}(\mathscr{X},\mathscr{Y})=[\mathscr{X},\mathscr{Y}]$ (resp. ho$\mathbb{I}$-$DGA_k(\mathscr{A},\mathscr{B})=[\mathscr{A},\mathscr{B}]$), where $[\mathscr{X},\mathscr{Y}]$ (resp. $[\mathscr{A},\mathscr{B}]$) denotes the set of homotopy classes of maps from $\mathscr{X}$ to $\mathscr{Y}$ (resp. from $\mathscr{A}$ to $\mathscr{B}$). Then we may state the following:

Corollary 1.5. *Let $\mathbb{I}$ be a cofinite small EI-category. If $f:\mathscr{A}\to\mathscr{B}$ is a weak equivalence and $\mathscr{C}$ is cofibrant in $\mathbb{I}$-DGA_k then the induced map $f_*:F(\mathscr{C},\mathscr{A})\to F(\mathscr{C},\mathscr{B})$ is a weak equivalence in the category $\mathbb{SS}$. In particular, the induced map of homotopy classes $[\mathscr{C},\mathscr{A}]\to[\mathscr{C},\mathscr{B}]$ is a bijection.*

Proof. By Theorem 1.3, we can factor f as $\mathscr{A}\xrightarrow{q}\mathscr{A}'\xrightarrow{p}\mathscr{B}$ with q a cofibration and p a fibration and both are weak equivalences. But any object in $\mathbb{I}$-DGA_k is fibrant, hence there is a map $q':\mathscr{A}'\to\mathscr{A}$ such that $q'\circ q=\mathrm{id}_{\mathscr{A}}$. Thus, q' is a trivial fibration and by Proposition 1.4 the induced maps $q'_*:F(\mathscr{C},\mathscr{A}')\to F(\mathscr{C},\mathscr{A})$ and $p_*:F(\mathscr{C},\mathscr{A}')\to F(\mathscr{C},\mathscr{B})$ are trivial fibrations. Therefore, the induced maps q_* and $f_*=p_*\circ q_*$ are weak equivalences. □

It follows from Proposition 1.4 that the functor $\mathscr{F}_* : \mathbb{I}\text{-}DGA_{\mathbb{Q}} \to \mathbb{I}\text{-}\mathbb{SS}$ carries cofibrant objects to fibrant and one may easily show that this functor preserves the homotopy relation, hence we get the induced functor $\mathscr{F}_* : \mathrm{ho}\mathbb{I}\text{-}DGA_{\mathbb{Q}} \to \mathrm{ho}\mathbb{I}\text{-}\mathbb{SS}$ in an obvious way. Although the functor $\mathscr{A}^* : \mathbb{I}\text{-}\mathbb{SS} \to \mathbb{I}\text{-}DGA_{\mathbb{Q}}$ may not carry fibrant to cofibrant objects, the induced adjoint functor $\mathscr{C} : \mathrm{ho}\mathbb{I}\text{-}\mathbb{SS} \to \mathrm{ho}\mathbb{I}\text{-}DGA_{\mathbb{Q}}$ may be constructed as well. For each $\mathbb{I}$-simplicial set $\mathscr{X}$, choose a weak equivalence $\mathscr{C}_{\mathscr{X}} \to \mathscr{A}^*(\mathscr{X})$ with $\mathscr{C}_{\mathscr{X}}$ cofibrant, and for each $f : \mathscr{X} \to \mathscr{Y}$ choose (by Corollary 1.5) a map $\mathscr{C}_f : \mathscr{C}_{\mathscr{Y}} \to \mathscr{C}_{\mathscr{X}}$ such that the diagram

$$\begin{array}{ccc} \mathscr{C}_{\mathscr{Y}} & \xrightarrow{\mathscr{C}_f} & \mathscr{C}_{\mathscr{X}} \\ \downarrow & & \downarrow \\ \mathscr{A}^*(\mathscr{Y}) & \xrightarrow{\mathscr{A}(f)} & \mathscr{A}^*(\mathscr{X}) \end{array}$$

commutes up to homotopy. We define the functor $\mathscr{C}$ by $\mathscr{C}(\mathscr{X}) = \mathscr{C}_{\mathscr{X}}$ and $\mathscr{C}(f) = [\mathscr{C}_f]$.

Remark 1.6. If $DGA^0_{\mathbb{Q}}$ (resp. $\mathbb{SS}^0$) is the category of homologically connected augmented commutative differential graded $\mathbb{Q}$-algebras (resp. the category of connected pointed simplicial sets), then by [1] there exists also a pair of adjoint functors

$$DGA^0_{\mathbb{Q}} \underset{A^*_0}{\overset{F^0_*}{\rightleftarrows}} \mathbb{SS}^0$$

which induces a pair between functor categories

$$\mathbb{I}\text{-}DGA^0_{\mathbb{Q}} \underset{\mathscr{A}^*_0}{\overset{\mathscr{F}^0_*}{\rightleftarrows}} \mathbb{I}\text{-}\mathbb{SS}^0$$

with the above properties.

2. Applications to rational homotopy theory

For a map $\gamma : B \to E$ in DGA_k, where B is augmented, Halperin [7] considers its "minimal factorization". Namely, he defines a *minimal KS-extension* as a special sequence of augmented DGA_k's

$$\mathbb{E} : B \xrightarrow{i} C \xrightarrow{\pi} A.$$

In [7] the following result is proved.

Theorem 2.1. *For any map $\gamma : B \to E$ of connected DGA_k's, where B is augmented, there is a unique (up to isomorphism) minimal KS-extension*

$$\mathbb{E} : B \xrightarrow{i} C \xrightarrow{\pi} A$$

and a homology isomorphism $\rho : C \to E$ such that $\rho \circ i = \gamma$.

The extension $\mathbb{E}$ together with the map $\rho : C \to E$ is called a *KS-minimal model* for γ. In particular, a minimal algebra M_A together with a homology isomorphism $\rho_A : M_A \to A$ is isomorphic to the *minimal model* for A.

Now let G be a finite group and G-DGA_k the category of differential graded algebras with an action of G. Then a notion of a minimal KS-extension may be considered in G-DGA_k as well and in [6] it has been shown that an equivariant version of Theorem 2.1 yields a G-KS-minimal model of a map $\gamma : B \to E$ in G-DGA_k.

For further convenience, we will suppose that a cofinite small EI-category $\mathbb{I}$ has the additional property:

(∗) for any its map $\phi : i' \to i$, there is an epimorphism $\hat{\phi} : \mathrm{Aut}(i') \to \mathrm{Aut}(i)$ with $\phi \circ \gamma = \hat{\phi}(\gamma) \circ \phi$ for all $\gamma \in \mathrm{Aut}(i')$.

Then for a given $\mathscr{A}$ in $\mathbb{I}$-DGA_k and a map $\phi : i' \to i$ there is an action of $\mathrm{Aut}(i')$ on $\mathscr{A}(i)$ and $\mathscr{A}(\phi) : \mathscr{A}(i') \to \mathscr{A}(i)$ is an $\mathrm{Aut}(i')$-map. Denote by $I_\phi(i')(\mathscr{A})$ the ideal in $\mathscr{A}(i')$ generated by elements $a - ga$ for $a \in \mathscr{A}(i')$ and $g \in \ker \hat{\phi}$. Then $\mathscr{A}_\phi(i') = \mathscr{A}(i')/I_\phi(i')(\mathscr{A})$ is an $\mathrm{Aut}(i)$-DGA_k and the induced map $\mathscr{A}_\phi(i') \to \mathscr{A}(i)$ preserves the $\mathrm{Aut}(i)$-action. Moreover, we get a functor $\overline{\mathscr{A}_i} : \mathbb{I}_i \to DGA_k$ such that $\overline{\mathscr{A}_i}(i', \phi) = \mathscr{A}_\phi(i')$. Hence $\overline{\mathscr{A}}(i) = \lim_{\overrightarrow{\mathbb{I}_i}} \overline{\mathscr{A}_i}$ is an $\mathrm{Aut}(i)$-DGA_k and there is the induced $\mathrm{Aut}(i)$-map $\overline{\rho}(i) : \overline{\mathscr{A}}(i) \to \mathscr{A}(i)$. The algebra $\overline{\mathscr{A}}(i)$ is augmented, hence we may take the $\mathrm{Aut}(i)$-KS-minimal model

$$\begin{array}{ccc} \overline{\mathscr{A}}(i) & \xrightarrow{\overline{\rho(i)}} & \mathscr{A}(i) \\ \big\downarrow & \nearrow{\scriptstyle \rho(i)} & \\ \tilde{\mathscr{A}}(i) & & \end{array}$$

of the map $\overline{\rho}(i)$.

We say that an object $\mathscr{M}$ in $\mathbb{I}$-DGA_k is *KS-minimal* if $\mathscr{M}(i) = \widetilde{\overline{\mathscr{M}}}(i)$ for any object $i \in \mathrm{Ob}(\mathbb{I})$.

Proposition 2.2. *If a cofinite small EI-category $\mathbb{I}$ satisfies the condition (∗), then any minimal object $\mathscr{M}$ in $\mathbb{I}$-DGA_k is cofibrant.*

Proof. Consider a commutative diagram

$$\begin{array}{ccc} \underline{k} & \longrightarrow & \mathscr{D} \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathscr{M} & \xrightarrow{\alpha} & \mathscr{E} \end{array}$$

in $\mathbb{I}$-DGA_k, where $\underline{k}$ is the constant $\mathbb{I}$-algebra determined by the field k and p is a trivial fibration. For any object $i \in \mathrm{Ob}(\mathbb{I})$ of height 0, there is a map $\beta(i) : \mathscr{M}(i) \to \mathscr{D}(i)$ such that $p(i)\circ\beta(i)=\alpha(i)$. Now suppose that for all $i'\in \mathrm{Ob}(\mathbb{I})$ of height smaller than height of i there are maps $\beta(i'):\mathscr{M}(i')\to\mathscr{D}(i')$ such that $p(i')\circ\beta(i')=\alpha(i')$. Hence, we get a map $\bar\beta(i):\bar{\mathscr{M}}(i)=\varinjlim_{\mathbb{I}_i}\mathscr{M}_i\to\mathscr{D}(i)$. Then in the commutative diagram

$$\begin{array}{ccc} \bar{\mathscr{M}}(i) & \xrightarrow{\bar\beta(i)} & \mathscr{D}(i) \\ \big\downarrow & \nearrow^{\beta(i)} & \big\downarrow{\scriptstyle p(i)} \\ \mathscr{M}(i) & \xrightarrow{\alpha(i)} & \mathscr{E}(i) \end{array}$$

there is a filler $\beta(i)$ since the map $\bar{\mathscr{M}}(i)\to\mathscr{M}(i)$ is a cofibration in the category $\mathrm{Aut}(i)$-DGA_k. □

Let $\mathscr{A}$ be in $\mathbb{I}$-DGA_k and let $\rho:\mathscr{M}\to\mathscr{A}$ be a weak equivalence, where $\mathscr{M}$ is KS-minimal. Then $\mathscr{M}$ is called the *KS-minimal model* of $\mathscr{A}$. Proposition 2.4 (cf. [1,8]) implies that this definition is meaningful.

Lemma 2.3. *If a cofinite small EI-category $\mathbb{I}$ satisfies the condition $(*)$, then for a commutative up to homotopy diagram in $\mathbb{I}$-DGA_k*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\alpha} & \mathscr{C} \\ {\scriptstyle q}\big\downarrow & \overset{\gamma}{\nearrow} & \big\downarrow{\scriptstyle f} \\ \mathscr{B} & \xrightarrow{\beta} & \mathscr{D}, \end{array}$$

where q is a cofibration and f a weak equivalence, there exists an arrow γ making this diagram commutative up to homotopy.

Proof. Using Theorem 1.3, we may factor f as $\mathscr{C}\xrightarrow{q'}\mathscr{C}'\xrightarrow{p}\mathscr{D}$ with q' a trivial cofibration and p a trivial fibration. Every object in $\mathbb{I}$-DGA_k is fibrant, hence by [10] the map $q':\mathscr{C}\to\mathscr{C}'$ has a homotopy inverse $q'':\mathscr{C}'\to\mathscr{C}$. But the map $q:\mathscr{A}\to\mathscr{B}$ is a cofibration, so there is a map $\beta':\mathscr{B}\to\mathscr{D}$ such that $\beta\simeq\beta'$ and the diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{q'\circ\alpha} & \mathscr{C}' \\ {\scriptstyle q}\big\downarrow & \overset{\gamma'}{\nearrow} & \big\downarrow{\scriptstyle p} \\ \mathscr{B} & \xrightarrow{\beta'} & \mathscr{D} \end{array}$$

strictly commutes, where the map $\gamma' : \mathcal{B} \to \mathcal{C}'$ is determined by Theorem 1.3. Then $\gamma = q'' \circ \gamma'$ is the required map. □

Proposition 2.4. *Let $\mathbb{I}$ be a cofinite small EI-category satisfying the condition* $(*)$, *let $\mathcal{M}$ and $\mathcal{M}'$ be KS-minimal $\mathbb{I}$-algebras and $\rho : \mathcal{M} \to \mathcal{A}$, $\rho' : \mathcal{M}' \to \mathcal{A}$ weak equivalences. Then*:

(1) *there is an isomorphism $\theta : \mathcal{M} \to \mathcal{M}'$ in $\mathbb{I}$-DGA_k such that $\rho'(i) \circ \theta(i) \simeq \rho(i)$ in the category* Aut(i)-DGA_k *for all $i \in$* Ob$(\mathbb{I})$;

(2) *if $\hat{\theta} : \mathcal{M} \to \mathcal{M}'$ is a map in $\mathbb{I}$-DGA_k such that $\rho'(i) \circ \hat{\theta}(i) \simeq \rho(i)$ in the category* Aut(i)-DGA_k *then $\hat{\theta}$ is an isomorphism and $\hat{\theta}(i) \simeq \theta(i)$ in the category* Aut(i)-DGA_k, *for all $i \in$* Ob$(\mathbb{I})$.

Proof. (1) We proceed inductively with respect to the height of $i \in \mathrm{Ob}(\mathbb{I})$. If $i \in \mathrm{Ob}(\mathbb{I})$ has height 0, then $\mathcal{M}(i)$ and $\mathcal{M}'(i)$ are Aut(i)-minimal and by [7, Proposition 4.3] there is an Aut(i)-isomorphism $\theta(i) : \mathcal{M}(i) \to \mathcal{M}'(i)$ such that $\rho'(i) \circ \theta(i) \simeq \rho(i)$ in the category Aut(i)-DGA_k.

Suppose that for all $i' \in \mathrm{Ob}(\mathbb{I})$ of height smaller than that of i, there exists $\theta(i')$: $\mathcal{M}(i') \to \mathcal{M}'(i')$ such that $\rho'(i') \circ \theta(i') \simeq \rho(i')$ in the category Aut(i')-DGA_k and the diagrams commute

$$\begin{array}{ccc} \mathcal{M}(i'') & \xrightarrow{\theta(i'')} & \mathcal{M}'(i'') \\ \downarrow & & \downarrow \\ \mathcal{M}(i') & \xrightarrow{\theta(i')} & \mathcal{M}'(i') \end{array}$$

for $\overline{i''} < \overline{i'}$. Then we get the induced isomorphism $\bar{\theta}(i) : \bar{\mathcal{M}}(i) \to \bar{\mathcal{M}}'(i)$. But the map $\alpha(i)$: $\bar{\mathcal{M}}(i) \to \mathcal{M}(i)$ is a cofibration in the category Aut(i)-DGA_k and $\rho(i)$: $\mathcal{M}(i) \to \mathcal{A}(i)$ is a weak equivalence, hence by Lemma 2.3 there is an Aut(i)-map $\theta'(i) : \mathcal{M}(i) \to \mathcal{M}'(i)$ such that the diagram

$$\begin{array}{ccccc} \bar{\mathcal{M}}(i) & \xrightarrow{\alpha(i)} & \mathcal{M}(i) & & \\ & & & \searrow{\scriptstyle \rho(i)} & \\ \downarrow{\scriptstyle \bar{\theta}(i)} & & \downarrow{\scriptstyle \theta'(i)} & & \mathcal{A}(i) \\ & & & \nearrow{\scriptstyle \rho(i)} & \\ \bar{\mathcal{M}}'(i) & \xrightarrow{\alpha'(i)} & \mathcal{M}'(i) & & \end{array}$$

commutes up to homotopy. In particular, $\theta'(i)\circ\alpha(i)\simeq\alpha'(i)\circ\bar{\theta}(i)$. But the map $\alpha(i)$ is a cofibration, hence there is a map $\theta(i)$ such that $\theta'(i)\simeq\theta(i)$ and $\theta(i)\circ\alpha(i)=\alpha'(i)\circ\bar{\theta}(i)$. The maps $\alpha(i)$ and $\alpha'(i)$ are Aut(i)-KS-minimal extensions and $\bar{\theta}(i):\bar{\mathscr{M}}(i)\to\bar{\mathscr{M}}'(i)$ is an isomorphism, hence by [7, Proposition 4.6] the map $\theta(i)$ is an isomorphism.

(2) If $i\in\mathrm{Ob}(\mathbb{I})$ has height 0, then $\mathscr{M}(i)$ and $\mathscr{M}'(i)$ are Aut(i)-minimal and by [7, Proposition 4.3] the map $\hat{\theta}(i)$ is an Aut(i)-isomorphism and $\theta(i)\simeq\hat{\theta}(i)$ in the category Aut(i)-DGA_k.

Suppose that, for all $i'\in\mathrm{Ob}(\mathbb{I})$ of height smaller than that of i, the maps $\hat{\theta}(i')$ are Aut(i')-isomorphisms and there exists an Aut(i')-homotopy $\theta(i')\simeq\hat{\theta}(i')$. Then the diagram

$$\begin{array}{ccc}
\bar{\mathscr{M}}(i) & \xrightarrow{\alpha(i)} & \mathscr{M}(i) \\
{\scriptstyle \bar{\theta}(i)}\downarrow & & \downarrow{\scriptstyle \theta'(i)} \\
\bar{\mathscr{M}}'(i) & \xrightarrow{\alpha'(i)} & \mathscr{M}'(i)
\end{array}$$

satisfies the hypothesis of Theorem 10.4 in [7], hence $\theta(i)\simeq\hat{\theta}(i)$ in the category Aut(i)-DGA_k and the map $\hat{\theta}(i)$ is an isomorphism. □

We now show the existence of a KS-minimal model.

Proposition 2.5. *Let $\mathbb{I}$ be a cofinite small EI-category satisfying* $(*)$. *Then for any $\mathscr{A}$ in $\mathbb{I}$-DGA_k there exist a KS-minimal model $\mathscr{M}_{\mathscr{A}}$ and a weak equivalence $\rho:\mathscr{M}_{\mathscr{A}}\to\mathscr{A}$.*

Proof. For any $\mathscr{A}$ in $\mathbb{I}$-DGA_k, we construct its KS-minimal model $\mathscr{M}_{\mathscr{A}}$ as follows:

(1) if $i\in\mathrm{Ob}(\mathbb{I})$ has height 0, then for $\mathscr{M}_{\mathscr{A}}(i)$ take the Aut(i)-minimal model of $\mathscr{A}(i)$. Let $\rho(i):\mathscr{M}_{\mathscr{A}}(i)\to\mathscr{A}(i)$ be a fixed Aut(i)-weak equivalence;

(2) suppose that for all $i'\in\mathrm{Ob}(\mathbb{I})$ of height smaller than height of i there are Aut(i')-weak equivalences $\rho(i'):\mathscr{M}_{\mathscr{A}}(i')\to\mathscr{A}(i')$ such that for $\overline{i'_1},\overline{i'_2}<\overline{i'}$ with $\overline{i'_1}<\overline{i'_2}$ all diagrams

$$\begin{array}{ccc}
\mathscr{M}_{\mathscr{A}}(i'_1) & \xrightarrow{\rho(i'_1)} & \mathscr{A}(i'_1) \\
\downarrow & & \downarrow \\
\mathscr{M}_{\mathscr{A}}(i'_2) & \xrightarrow{\rho(i'_2)} & \mathscr{A}(i'_2)
\end{array}$$

commute. To get $\mathcal{M}_{\mathcal{A}}(i)$ and an Aut(i)-weak equivalence $\rho(i):\mathcal{M}_{\mathcal{A}}(i)\to\mathcal{A}(i)$, consider the induced Aut(i)-map $\overline{\rho}(i):\overline{\mathcal{M}_{\mathcal{A}}}(i)\to\mathcal{A}(i)$ and its Aut(i)-KS-minimal model

$$\begin{array}{ccc} \overline{\mathcal{M}_{\mathcal{A}}}(i) & \xrightarrow{\overline{\rho(i)}} & \mathcal{A}(i) \\ \downarrow & \nearrow_{\rho(i)} & \\ \mathcal{M}_{\mathcal{A}}(i). & & \end{array} \qquad \square$$

Now let G be a finite Hamiltonian group (i.e., each subgroup of G is normal). Then the category $\mathcal{O}(G)$ of canonical orbits is a cofinite EI-category satisfying the condition $(*)$. We say that a G-simplicial set X is *G-connected* if all fixed point simplicial subsets X^H are connected, for subgroups $H\subseteq G$. Write G-$\mathbb{SS}$ (resp. G-$\mathbb{SS}^0$) for the category of all G-connected (resp. pointed) simplicial sets. Then from [4] it follows that there is an equivalence of homotopy categories

$$\mathrm{ho}\,G\text{-}\mathbb{SS} \overset{\approx}{\rightleftarrows} \mathrm{ho}\,\mathcal{O}(G)\text{-}\mathbb{SS}.$$

On the other hand, the de Rham functor A^* (resp. A_0^*) of polynomial forms determines a functor

$$\mathcal{A}^*: G\text{-}\mathbb{SS}\longrightarrow\mathcal{O}(G)\text{-}DGA_{\mathbb{Q}} \quad (\text{resp. } \mathcal{A}_0^*: G\text{-}\mathbb{SS}^0\longrightarrow\mathcal{O}(G)\text{-}DGA^0_{\mathbb{Q}}),$$

such that $\mathcal{A}^*(X)(G/H)=A^*(X^H)$ (resp. $\mathcal{A}_0^*(X)(G/H)=A_0^*(X^H)$) for X in G-$\mathbb{SS}$ (resp. in G-$\mathbb{SS}^0$) and $H\subseteq G$, where $\mathbb{Q}$ is the field of rationals. Choosing a weak equivalence $\mathcal{M}_X\to\mathcal{A}^*(X)$ in the category $\mathcal{O}(G)$-DGA_k with $\mathcal{M}_X$ a KS-minimal model of $\mathcal{A}^*(X)$, we consider a pair of adjoint functors

$$\mathrm{ho}\,G\text{-}\mathbb{SS} \rightleftarrows \mathrm{ho}\,\mathcal{O}(G)\text{-}DGA_{\mathbb{Q}} \quad (\text{resp. } \mathrm{ho}\,G\text{-}\mathbb{SS}^0 \rightleftarrows \mathrm{ho}\,\mathcal{O}(G)\text{-}DGA^0_{\mathbb{Q}})$$

constructed in Section 1.

By [12, 13], for any X in G-$\mathbb{SS}^0$ there is a minimal model $\mathcal{M}^i_X$, injective as an $\mathcal{O}(G)$-module, and a weak equivalence $\mathcal{M}^i_X\to\mathcal{A}^*(X)$ such that, for nilpotent G-connected pointed simplicial sets X, Y of finite type, there is a bijection

$$[X,Y]_G\approx[\mathcal{M}^i_Y,\mathcal{M}^i_X],$$

where $[X,Y]_G$ is the set of pointed G-homotopy classes of G-maps from X to Y. From Proposition 2.4 one gets that the KS-minimal models of $\mathcal{M}^i_X$ and $\mathcal{A}^*(X)$ are isomorphic. Hence, there is a weak equivalence $\rho:\mathcal{M}_X\to\mathcal{M}^i_X$. By [12, Proposition 5.5] there is a map $\rho':\mathcal{M}^i_X\to\mathcal{M}_X$ such that $\rho\circ\rho'\simeq\mathrm{id}_{\mathcal{M}^i_X}$. Thus, the map ρ' is a weak equivalence and by Corollary 1.5 and Proposition 2.2 there is a map $\rho'' : \mathcal{M}_X \to \mathcal{M}_X^i$ such that $\rho' \circ \rho'' \simeq \mathrm{id}_{\mathcal{M}_X}$. Therefore, we have

Proposition 2.6. *Let G be a finite Hamiltonian group. If X and Y are nilpotent G-connected pointed simplicial sets of finite type, then there is a bijection*

$$[X, Y]_G \approx [\mathcal{M}_Y, \mathcal{M}_X],$$

provided Y is rational.

Finally, we can extend the Sullivan–de Rham equivalence to the equivariant case.

Theorem 2.7. *If G is a Hamiltonian group, then there exists a pair of adjoint functors*

$$\mathrm{ho}\, G\text{-}\mathbb{SS}^0 \rightleftarrows \mathrm{ho}\, \mathscr{O}(G)\text{-}DGA^0_{\mathbb{Q}}$$

which restrict to inverse equivalences

$$f\mathbb{Q}N\text{-ho}\, G\text{-}\mathbb{SS}^0 \overset{\approx}{\rightleftarrows} f\mathbb{Q}\text{-ho}\, \mathscr{O}(G)\text{-}DGA^0_{\mathbb{Q}},$$

where $f\mathbb{Q}N$-ho G-$\mathbb{SS}^0$ is the full subcategory of ho G-$\mathbb{SS}^0$ induced by those G-connected pointed simplicial sets which are nilpotent and of finite type and $f\mathbb{Q}$-ho$\mathscr{O}(G)$-$DGA^0_{\mathbb{Q}}$ is the full subcategory of ho $\mathscr{O}(G)$-$DGA^0_{\mathbb{Q}}$ induced by those $\mathscr{O}(G)$-augmented algebras which are equivalent to equivariant KS-minimal $\mathscr{O}(G)$-algebras and with finitely many multiplicative generators.

Remark 2.8. (1) In [5] it was shown that any system of $\mathscr{O}(G)$-differential graded algebras can be mapped into an injective $\mathscr{O}(G)$-system of such algebras via a homology isomorphism.

(2) The above result also holds for nilpotent G-connected unpointed simplicial sets X (of finite type) with $X^G \neq \emptyset$.

(3) A construction of the equivariant KS-minimal model of any nilpotent G-disconnected simplicial set and a formulation of an appropriate version of the equivariant Sullivan–de Rham equivalence require more subtle methods and will be published elsewhere.

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