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The Bishop–Phelps–Bollobás property and lush spaces ☆

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ABSTRACT

We prove that for every lush space X, the couple (ℓ_1, X) has the Bishop–Phelps–Bollobás property for operators, that is, every lush space has the *AHSP* (standing for the approximate hyperplane series property). While every lush space has the alternative Daugavet property, there exists a space with the alternative Daugavet property that does not have the *AHSP*. We also show that there is a Banach space with both the *AHSP* and the alternative Daugavet property which is not lush.

1. Introduction

In this paper we show an interesting result that the geometric property "lushness" of a Banach space is closely related with the Bishop–Phelps–Bollobás property (*BPBP* for short) for operators, though they appeared to be apparently unrelated each other.

The concept of lushness was introduced to characterize an infinite dimensional Banach space with numerical index 1 [7]. The fact that a Banach space X has numerical index 1 means that the norm of any bounded operator on X is the same as its numerical radius. The lushness has been known to be the weakest among quite a few isometric properties in the literature which are sufficient conditions for a Banach space to have numerical index 1.

On the other hand, some attention has been recently paid to the question if a given couple of Banach spaces satisfies the *BPBP* for operators, a strong form of denseness of the set of norm-attaining operators [1–3,9].

The Bishop–Phelps theorem [4] shows that the set of norm-attaining functionals on a Banach space X is dense in its dual space X^*. This theorem has been extended to bounded linear operators between Banach spaces, and also to non-linear mappings like multilinear mappings, polynomials and holomorphic mappings.

Afterwards Bollobás [5] sharpened the Bishop–Phelps theorem. More precisely, he obtained that for an arbitrary $\epsilon > 0$, if $x \in B_X$ and $x^* \in S_{X^*}$ satisfy $|1 - x^*(x)| < \frac{\epsilon^2}{4}$, then there are $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \epsilon$ and $\|y^* - x^*\| < \epsilon$, which is now called the Bishop–Phelps–Bollobás theorem. Very recently Acosta et al. [1] extended this theorem to bounded linear operators between Banach spaces.

The *BPBP* for operators is a much stronger property than the denseness of norm-attaining operators. It has been known that the set of norm-attaining operators from ℓ_1 to any Banach space X is dense, but the pair (ℓ_1, X) has the *BPBP* for operators only when X satisfies the so-called "approximate hyperplane series property" (*AHSP* for short).

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Our main result is to show that every lush space has the *AHSP*. Very recently it was shown that every almost-CL-space has the *AHSP* [8], which is also a lush space. In general, every lush space has numerical index 1, and every Banach space with numerical index 1 has the so-called "alternative Daugavet property" (*ADP* for short) [16]. There exists a Banach space with the *ADP*, but not the *AHSP*. However, we don't know if every Banach space with numerical index 1 has the *AHSP*. We also show that there is a Banach space with both the *AHSP* and *ADP*, which is not lush.

2. Results

We begin by recalling some relevant definitions and reviewing several recent results. Given Banach spaces X, Y over $\mathbb{K}$ $(=\mathbb{R}$ or $\mathbb{C})$, by B_X we denote the closed unit ball, by S_X the unit sphere of X and by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X into Y.

Definition 1. (See [1, Definition 1.1].) We say that the couple (X,Y) satisfies the *BPBP*, if given $\epsilon>0$ there exist $\beta(\epsilon)>0$ and $\eta(\epsilon)>0$ with $\lim_{\epsilon\to 0^+}\beta(\epsilon)=0$ such that for $T\in S_{\mathcal{L}(X,Y)}$, if $x_0\in S_X$ is such that $\|Tx_0\|>1-\eta(\epsilon)$, then there exist a point $u_0\in S_X$ and an operator $S\in S_{\mathcal{L}(X,Y)}$ that satisfy the following conditions:

$$\|Su_0\|=1,\qquad \|x_0-u_0\|<\beta(\epsilon)\quad\text{and}\quad \|T-S\|<\epsilon.$$

Acosta et al. introduced the *AHSP*, with which they characterized the Banach space Y such that the couple (ℓ_1, Y) has the *BPBP*.

Definition 2. (See [1, Remark 3.2].) A Banach space X is said to have the *AHSP* if for every $\epsilon>0$ there exist $\gamma(\epsilon)>0$ and $\rho(\epsilon)>0$ with $\lim_{\epsilon\to 0^+}\gamma(\epsilon)=0$ such that for every sequence $(x_k)_{k=1}^\infty\subset B_X$ and for every convex series $\sum_{k=1}^\infty\alpha_k$ satisfying

$$\left\|\sum_{k=1}^\infty \alpha_k x_k\right\|>1-\rho(\epsilon)$$

there exist a subset $A\subset\mathbb{N}$, a subset $\{z_k\colon k\in A\}\subset S_X$ and $x^*\in S_{X^*}$ such that

(i) $\sum_{k\in A}\alpha_k>1-\gamma(\epsilon)$,
(ii) $\|z_k-x_k\|<\epsilon$ for all $k\in A$, and
(iii) $x^*(z_k)=1$ for all $k\in A$.

The following Banach spaces were shown to have the *AHSP*: (a) a finite dimensional normed space, (b) a real or complex space $L_1(\mu)$ for a σ-finite measure μ, (c) a real or complex space $C(K)$ for a compact Hausdorff space K, and (d) a uniformly convex space.

On the other hand, the concept of lushness was introduced in [7] as a geometric property of a Banach space which insures that the space has numerical index 1. Before the lush spaces were studied, the basic examples of Banach spaces with numerical index 1 had been known to be almost-CL-spaces [13,14]. Clearly every almost-CL-space is lush.

Definition 3. A Banach space X is said to be *lush* if for every $x, y\in S_X$ and for every $\epsilon>0$ there is a slice

$$S=S\big(B_X,x^*,\epsilon\big)=\big\{x\in B_X\colon \operatorname{Re}x^*(x)>1-\epsilon\big\},\quad x^*\in S_{X^*}$$

such that $x\in S$ and $\operatorname{dist}(y,\operatorname{aconv}(S))<\epsilon$.

The following Banach spaces were shown to be lush in [6]: (a) the preduals of any $L_1(\mu)$, (b) any Banach space which nicely embeds into $C_b(\Omega)$, where Ω is a completely regular Hausdorff topological space, hence the disc algebra and $H^\infty(D)$ (see [19]), (c) C-rich subspaces of $C(K)$. It was also shown in [6] that every separable Banach space containing an isomorphic copy of c_0 can be equivalently renormed to be lush.

We are now ready to prove our main result. We first state the following propositions and lemma.

Proposition 4. *(See [11, Corollary 4.5], [12, Proposition 2.1].) For a separable lush space X, there exists a norming set $C\subset S_{X^*}$ such that $B_X=\overline{\operatorname{conv}}(\mathbb{T}F(x^*))$ for all $x^*\in C$, where $F(x^*)=\{x\in B_X\colon x^*(x)=1\}$ is the face generated by x^* and $\mathbb{T}$ is the set of modulus-one scalars.*

Proposition 5. *(See [6, Theorem 4.2].) A Banach space X is lush if and only if for every separable subspace Y of X there exists a separable lush subspace Z of X containing Y.*

Lemma 6. *(See [1, Lemma 3.3].) Let $\{c_n\}$ be a sequence of complex numbers with $|c_n| \leqslant 1$ for every n, and let $\eta > 0$ be such that for a convex series $\sum_{n=1}^{\infty} \alpha_n$, $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$. Then for every $0 < r < 1$, the set $A = \{i \in \mathbf{N}\colon \operatorname{Re} c_i > r\}$ satisfies the estimate*

$$\sum_{i \in A} \alpha_i \geqslant 1 - \frac{\eta}{1-r}.$$

Theorem 7. *Every lush space X has the* AHSP. *In particular, (ℓ_1, X) has the* BPBP *for every lush space X.*

Proof. Let $\epsilon > 0$ be given. Choose $0 < \delta < \epsilon$ so that $\sqrt{2\delta} + 2\delta + \frac{\delta^2}{2} < \epsilon$.

Given a sequence $(x_i)_{i=1}^{\infty} \subset B_X$ and a convex series $\sum_{i=1}^{\infty} \alpha_i$, assume that

$$\left\| \sum_{i=1}^{\infty} \alpha_i x_i \right\| > 1 - \frac{\delta^3}{2}.$$

By Proposition 5, the sequence (x_i) lies in a separable lush subspace of X, call it Z. It follows from Proposition 4 that there exists a norming set $C \subset S_{Z^*}$ such that $B_Z = \overline{\operatorname{conv}}(\mathbb{T}F(z^*))$ for all $z^* \in C$.

Choose $z^* \in C$ such that

$$\operatorname{Re} z^* \left(\sum_{i=1}^{\infty} \alpha_i x_i \right) > 1 - \frac{\delta^3}{2}.$$

Let

$$A = \left\{ i \in \mathbf{N}\colon \operatorname{Re} z^*(x_i) > 1 - \frac{\delta^2}{2} \right\}.$$

It follows from Lemma 6 that

$$\sum_{i \in A} \alpha_i > 1 - \delta.$$

Since $B_Z = \overline{\operatorname{conv}}(\mathbb{T}F(z^*))$ for every $i \in A$, we can find $y_i = \sum_{k=1}^{m_i} \lambda_k^i \theta_k^i u_k^i$ such that $\|x_i - y_i\| < \frac{\delta^2}{2}$, $\sum_{k=1}^{m_i} \lambda_k^i = 1$, $u_k^i \in F(z^*)$, $0 \leqslant \lambda_k^i \leqslant 1$ and $\theta_k^i \in \mathbb{T}$ for every $k = 1, \ldots, m_i$.

We can get

$$\sum_{k=1}^{m_i} \lambda_k^i \operatorname{Re} \theta_k^i > 1 - \delta^2$$

for every $i \in A$, because $\operatorname{Re} z^*(y_i) > 1 - \delta^2$ for every $i \in A$.

For every $i \in A$ let

$$B_i = \left\{ k \in \{1, 2, \ldots, m_i\}\colon \operatorname{Re} \theta_k^i > 1 - \delta \right\},$$

$\mu_{B_i} = \sum_{k \in B_i} \lambda_k^i$, and $B_i^c = \{1, \ldots, m_i\} \setminus B_i$. Apply Lemma 6 again and we get $\mu_{B_i} > 1 - \delta$, and $|\theta_k^i - 1| < \sqrt{2\delta}$ for all $k \in B_i$.

Define $z_i = \sum_{k \in B_i} \frac{\lambda_k^i}{\mu_{B_i}} u_k^i$ for every $i \in A$. Let $x^* \in S_{X^*}$ be any Hahn–Banach extension of z^*. For every $i \in A$ we have $x^*(z_i) = 1$ and

$$\begin{aligned}
\|x_i - z_i\| &< \|y_i - z_i\| + \frac{\delta^2}{2} = \left\| \sum_{k=1}^{m_i} \lambda_k^i \theta_k^i u_k^i - \sum_{k \in B_i} \frac{\lambda_k^i}{\mu_{B_i}} u_k^i \right\| + \frac{\delta^2}{2} \\
&\leqslant \left\| \sum_{k \in B_i} \lambda_k^i (\theta_k^i - 1) u_k^i \right\| + \left\| \sum_{k \in B_i} \lambda_k^i \left(\frac{1}{\mu_{B_i}} - 1 \right) u_k^i \right\| + \left\| \sum_{k \in B_i^c} \lambda_k^i \theta_k^i u_k^i \right\| + \frac{\delta^2}{2} \\
&< \sum_{k \in B_i} \lambda_k^i \sqrt{2\delta} + \sum_{k \in B_i} \lambda_k^i \left(\frac{1}{\mu_{B_i}} - 1 \right) + (1 - \mu_{B_i}) + \frac{\delta^2}{2} \\
&< \mu_{B_i} \sqrt{2\delta} + 2(1 - \mu_{B_i}) + \frac{\delta^2}{2} < \sqrt{2\delta} + 2\delta + \frac{\delta^2}{2} < \epsilon. \qquad \square
\end{aligned}$$

It follows from the proof of Theorem 7 that given $\epsilon > 0$ we can find the same $\eta(\epsilon)$ and $\gamma(\epsilon)$ in the definition of the *AHSP* for all lush spaces. It was shown in [1, Theorem 4.1] that X has the *AHSP* if and only if the couple (ℓ_1, X) has the

BPBP. It follows from its proof that given $\epsilon > 0$ we can find the same $\eta(\epsilon)$ and $\beta(\epsilon)$ in the definition of the *BPBP* of the couple (ℓ_1, X) for all lush spaces X.

The converse of Theorem 7 is not true. Indeed, every finite dimensional Banach space has the *AHSP* [1, Proposition 3.5], but there is no Hilbert lush space with dimension $n > 1$.

Let us now replace ℓ_1 with a more general space L_1-space.

Theorem 8. *(See [9, Theorem 2.2].) Suppose that X has the Radon–Nikodým property and (Ω, Σ, μ) is a σ-finite measure space, where Σ is an infinite σ-algebra. Then the couple $(L_1(\mu), X)$ has the* BPBP *if and only if X has the* AHSP.

Corollary 9. *Suppose that X is a lush space having the Radon–Nikodým property, and that (Ω, Σ, μ) is a σ-finite measure space, where Σ is an infinite σ-algebra. Then the couple $(L_1(\mu), X)$ has the* BPBP.

Schachermayer [18] showed that the norm-attaining operators in $\mathcal{L}(L_1[0,1], C[0,1])$ are not dense. From this we can see that the lushness is not a sufficient condition on X for the couple $(L_1(\mu), X)$ to have the *BPBP*.

The notion of the *numerical index* of a Banach space was first introduced by G. Lumer in 1968 (see [10]), and it is the greatest constant of equivalence between the numerical radius and the usual norm in the Banach algebra $\mathcal{L}(X)$ of all bounded linear operators on X.

We denote the set

$$\Pi(X) = \big\{(x, x^*)\colon x \in S_X,\ x^* \in S_{X^*},\ \text{and } x^*(x) = 1\big\}.$$

For $T \in \mathcal{L}(X)$, the *numerical range of T* is the set of scalars

$$V(T) = \big\{x^*\big(T(x)\big)\colon \big(x, x^*\big) \in \Pi(X)\big\},$$

and the *numerical radius of T* is $v(f) = \sup\{|x^*(f(x))|\colon (x, x^*) \in \Pi(X)\}$. We define

$$n(X) = \inf\big\{v(T)\colon T \in \mathcal{L}(X; X),\ \|T\| = 1\big\}$$

and call it the *numerical index of X*.

A Banach space X is said to have the *ADP* if the norm identity

$$\max_{|\omega|=1} \|Id + \omega T\| = 1 + \|T\|$$

holds for every rank-one operator $T \in L(X)$. Since every Banach space with numerical index 1 has the *ADP*, every lush space has the *ADP*. Moreover, for the Banach space with the Radon–Nikodým property the lushness and the *ADP* are equivalent (see [6, Remark 2.2] and [15, Remark 6]). We are now interested in the question that every Banach space with the *ADP* has the *AHSP*. Indeed, there is a Banach space with the *ADP*, but not the *AHSP*. However, we don't know if every Banach space with numerical index 1 has the *AHSP*.

Proposition 10. *(See [16, Theorem 3.4].) Let X be a Banach space, K a compact Hausdorff space and μ a positive measure. Then*

(a) *$C(K, X)$ has the* ADP *if and only if K is perfect or X has the* ADP.
(b) *$L_1(\mu, X)$ has the* ADP *if and only if μ is atomless or X has the* ADP.

Theorem 11. *If $C(K, X)$ has the* AHSP, *then X has the* AHSP.

Proof. Given $\epsilon > 0$, let $\rho(\epsilon)$ and $\gamma(\epsilon)$ be the positive numbers which appear in the definition of *AHSP* of $C(K, X)$.

Given a sequence $(x_k)_{k=1}^{\infty} \subset B_X$ and a convex series $\sum_{k=1}^{\infty} \alpha_k$, assume that

$$\left\|\sum_{k=1}^{\infty} \alpha_k x_k\right\| > 1 - \rho(\epsilon).$$

For every $k \in \mathbf{N}$ define $f_k \in B_{C(K,X)}$ by $f_k(t) = x_k$ for all $t \in K$.

We can get

$$\left\|\sum_{k=1}^{\infty} \alpha_k f_k\right\| > 1 - \rho(\epsilon).$$

By the assumption there exist a subset $A \subset \mathbb{N}$ and a subset $\{g_k\colon k \in A\} \subset S_{C(K,X)}$ and $\phi \in S_{C(K,X)^*}$ such that (i) $\sum_{k \in A} \alpha_k > 1 - \gamma(\epsilon)$, (ii) $\|g_k - f_k\| < \epsilon$ for all $k \in A$, and (iii) $\phi(g_k) = 1$ for all $k \in A$.

From these it follows that $\|\sum_{k\in A}\alpha_k g_k\| = \sum_{k\in A}\alpha_k$. Choose $t_0 \in K$ so that

$$\left\|\sum_{k\in A}\alpha_k g_k(t_0)\right\| = \sum_{k\in A}\alpha_k.$$

Choose also $x^* \in S_X$ so that $x^*(\sum_{k\in A}\alpha_k g_k(t_0)) = \sum_{k\in A}\alpha_k$. Put $z_k = g_k(t_0)$ for every $k \in A$. Clearly $\|x_k - z_k\| < \epsilon$ and $x^*(z_k) = 1$ for every $k \in A$, hence X has the *AHSP*. □

Example 12. Let X be a strictly convex Banach space isomorphic to ℓ_1. Then X cannot have the *AHSP* by [1, Proposition 3.9], because it is not uniformly convex. It follows from Proposition 10 that $C([0,1],X)$ has the *ADP*, but it cannot have the *AHSP* by Theorem 11.

We finally wonder if every Banach space with both the *ADP* and the *AHSP* is lush. In fact, it is not true. We don't know if every Banach space with both numerical index 1 and the *AHSP* is lush.

Lemma 13. *Let X be an uniformly convex Banach space. Given $0 < \epsilon < 1$, there exists $0 < \eta(\epsilon)$ with $\lim_{\epsilon\to 0^+}\eta(\epsilon) = 0$ such that if $\operatorname{Re} x^*(x) > 1 - \eta(\epsilon)$ for $x^* \in S_{X^*}$ and $x \in B_X$, then there exists $y \in S_X$ satisfying $x^*(y) = 1$ and $\|y - x\| < \epsilon$.*

Proof. Given $0 < \epsilon < 1$, let $\delta(\epsilon)$ be the modulus of convexity of X and put $\eta(\epsilon) = \min\{2\delta(\frac{\epsilon}{2}), \frac{\epsilon}{2}\}$. Suppose that $\operatorname{Re} x^*(x) > 1 - \eta(\epsilon)$ for $x^* \in S_{X^*}$ and $x \in B_X$. Clearly $\|x\| \geqslant 1 - \eta(\epsilon) \geqslant 1 - \frac{\epsilon}{2}$, hence $\|x - \frac{x}{\|x\|}\| \leqslant \frac{\epsilon}{2}$. Choose $x_0 \in S_X$ so that $x^*(x_0) = 1$. If $\|x_0 - \frac{x}{\|x\|}\| \geqslant \frac{\epsilon}{2}$, then

$$\delta\left(\frac{\epsilon}{2}\right) \leqslant 1 - \left\|\frac{x_0 + \frac{x}{\|x\|}}{2}\right\|,$$

hence

$$\operatorname{Re} x^*\left(x_0 + \frac{x}{\|x\|}\right) \leqslant \left\|x_0 + \frac{x}{\|x\|}\right\| \leqslant 2 - 2\delta\left(\frac{\epsilon}{2}\right).$$

An easy computation shows that

$$\operatorname{Re} x^*(x) \leqslant \operatorname{Re} x^*\left(\frac{x}{\|x\|}\right) \leqslant 1 - 2\delta\left(\frac{\epsilon}{2}\right) \leqslant 1 - \eta(\epsilon),$$

which is a contradiction. Therefore, $\|x_0 - \frac{x}{\|x\|}\| \leqslant \frac{\epsilon}{2}$, and we obtain $\|x_0 - x\| \leqslant \epsilon$. □

Theorem 14. *Let (Ω, Σ, μ) be a σ-finite measure space. For a uniformly convex Banach space X, $L_1(\mu, X)$ has the* AHSP.

Proof. Given $\epsilon > 0$, let $\eta(\epsilon)$ the same positive number as in Lemma 13. We set

$$s(\epsilon) = \max\left(1 - \eta(\epsilon), \frac{2}{\sqrt{4+\epsilon^2}}\right), \qquad r(\epsilon) = \frac{4 + \epsilon(s(\epsilon) - 1)}{4}, \quad \text{and} \quad \rho(\epsilon) = \epsilon(1 - r(\epsilon)).$$

Note that $0 < s(\epsilon) < r(\epsilon) < 1$, and $\rho(\epsilon) > 0$.

It is enough only to check the conditions of *AHSP* for a finite convex combination (instead of an infinite convex series) in order to prove that $L_1(\mu, X)$ has the *AHSP*. Given a finite sequence $(f_k)_{k=1}^n \subset S_{L_1(\mu,X)}$ and a finite convex series $\sum_{k=1}^n \alpha_k$, assume that

$$\left\|\sum_{k=1}^n \alpha_k f_k\right\| > 1 - \rho(\epsilon).$$

For each $1 \leqslant k \leqslant n$ there is a simple function $g_k \in S_{L_1(\mu,X)}$ such that $\|f_k - g_k\| < \epsilon$ and

$$\left\|\sum_{k=1}^n \alpha_k g_k\right\| > 1 - \rho(\epsilon).$$

We can find $\phi \in S_{L_1(\mu,X)^*}$ such that

$$\operatorname{Re}\phi\left(\sum_{k=1}^n \alpha_k g_k\right) > 1 - \rho(\epsilon).$$

Note that $L_1(\mu, X)^* = L_\infty(\mu, X^*)$. We may assume that

$$g_k = \sum_{i=1}^{N} x_i^k \chi_{A_i}$$

and

$$\phi = \sum_{i=1}^{N} x_i^* \chi_{A_i},$$

where $N \in \mathbf{N}$, $x_i^k \in X$, $x_i^* \in B_{X^*}$, $A_i \in \Sigma$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. Then $\sum_{i=1}^{N} \|x_i^k\| \mu(A_i) = 1$ for every $1 \leqslant k \leqslant n$. We may also assume all $x_i^* \in S_{X^*}$. Indeed, if $0 < \|x_i^*\| < 1$ for some $1 \leqslant i \leqslant N$, then we verify the value $\operatorname{Re} x_i^*(\sum_{k=1}^{n} \alpha_k x_i^k)$. If it is nonnegative, then we replace x_i^* by $\frac{x_i^*}{\|x_i^*\|}$. For the other cases, we can similarly replace x_i^* by an element in S_{X^*}.

Define

$$A = \{k: 1 \leqslant k \leqslant n,\ \phi(g_k) > r(\epsilon)\}.$$

By Lemma 6

$$\sum_{k \in A} \alpha_k > 1 - \frac{\rho(\epsilon)}{1 - r(\epsilon)} = 1 - \epsilon.$$

For every $k \in A$ define

$$E_k = \{i: 1 \leqslant i \leqslant N,\ \operatorname{Re} x_i^*(x_i^k) > s(\epsilon)\|x_i^k\|\}.$$

For every $k \in A$ we have

$$\begin{aligned} r(\epsilon) < \operatorname{Re}\phi(g_k) &= \sum_{i \in E_k} \operatorname{Re} x_i^*(x_i^k)\mu(A_i) + \sum_{i \in E_k^c} \operatorname{Re} x_i^*(x_i^k)\mu(A_i) \\ &\leqslant \sum_{i \in E_k} \operatorname{Re} x_i^*(x_i^k)\mu(A_i) + \sum_{i \in E_k^c} s(\epsilon)\|x_i^k\|\mu(A_i) \\ &= \sum_{i \in E_k} \operatorname{Re} x_i^*(x_i^k)\mu(A_i) + s(\epsilon)\left(1 - \sum_{i \in E_k} \|(x_i^k)\|\mu(A_i)\right) \\ &\leqslant (1 - s(\epsilon)) \sum_{i \in E_k} \operatorname{Re} x_i^*(x_i^k)\mu(A_i) + s(\epsilon), \end{aligned}$$

which implies that

$$\sum_{i \in E_k} \operatorname{Re} x_i^*(x_i^k)\mu(A_i) > \frac{r(\epsilon) - s(\epsilon)}{1 - s(\epsilon)}.$$

Hence,

$$\sum_{i \in E_k^c} \|x_i^k\|\mu(A_i) = 1 - \sum_{i \in E_k} \|x_i^k\|\mu(A_i) \leqslant 1 - \sum_{i \in E_k} \operatorname{Re} x_i^*(x_i^k)\mu(A_i) < 1 - \frac{r(\epsilon) - s(\epsilon)}{1 - s(\epsilon)}.$$

Since X is uniformly convex, it follows from Lemma 13 that for every $i \in E_k$ we can find $y_i^k \in X$ such that $x_k^*(y_i^k) = \|y_i^k\| = \|x_i^k\|$ and $\|y_i^k - x_i^k\| \leqslant \epsilon\|x_i^k\|$. For every $k \in A$ let

$$\beta_k = \sum_{i \in E_k} \|y_i^k\|\mu(A_i) > \frac{r(\epsilon) - s(\epsilon)}{1 - s(\epsilon)},$$

and define $\tilde{g}_k \in S_{L(\mu, X)}$ by

$$\tilde{g}_k = \sum_{i \in E_k} \frac{y_i^k \chi_{A_i}}{\beta_k}.$$

For every $k \in A$ we have $\phi(\tilde{g}_k) = 1$ and

$$\begin{aligned}\|\tilde{g}_k - g_k\| &\leqslant \sum_{i\in E_k}\left\|\frac{y_i^k}{\beta_k} - x_i^k\right\|\mu(A_i) + \sum_{i\in E_k^c}\|x_i^k\|\mu(A_i)\\ &\leqslant \sum_{i\in E_k}\|y_i^k - x_i^k\|\mu(A_i) + \sum_{i\in E_k}\left\|\frac{y_i^k}{\beta_k} - y_i^k\right\|\mu(A_i) + \sum_{i\in E_k^c}\|x_i^k\|\mu(A_i)\\ &\leqslant \sum_{k\in E_n}\epsilon\|x_i^k\|\mu(A_i) + 2(1-\beta_k)\\ &\leqslant \epsilon + 2\left(1 - \frac{r(\epsilon)-s(\epsilon)}{1-s(\epsilon)}\right) = \frac{3}{2}\epsilon,\end{aligned}$$

hence $\|\tilde{g}_k - f_k\| < \frac{5}{2}\epsilon$. □

Example 15. It follows from Proposition 10 and Theorem 14 that $L_1([0,1],\ell_2)$ has both the *AHSP* and the *ADP*. But, [17, Theorem 8] shows that the numerical index of $L_1([0,1],\ell_2)$ is the same as that of ℓ_2, which is smaller than 1. Hence, $L_1([0,1],\ell_2)$ is not a lush space.

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