The Bargmann analytic representation in signal analysis

L.K. Stergioulas\textsuperscript{a,}\textsuperscript{*}, A. Vourdas\textsuperscript{b}

\textsuperscript{a}Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, UK
\textsuperscript{b}Department of Computing, University of Bradford, Bradford BD7 1DP, UK

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Abstract

The Bargmann analytic representation of quantum mechanics is used in the context of time–frequency signal analysis. Transformations between the various representations are derived. Examples that show how this method can be implemented numerically and demonstrate its robustness to noise are discussed.

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1. Introduction

Since the pioneering work by Gabor and Ville [8,20], several authors (e.g., [6,19,24]) have emphasized the analogy of signal and image processing methods with quantum optical techniques. The time–frequency methods (e.g., [3,7,9,12,13,15]) in signal processing are very similar to phase-space methods in quantum mechanics and quantum optics. Gaussian signals corresponding to coherent states in quantum mechanics, Wigner distribution functions, Zak transforms, wavelets [4,5,10,14], etc. are examples of techniques that can be applied to both signal processing and quantum mechanics.

The theory of analytic functions has played an important role in both areas. In quantum mechanics, there are several representations which are using analytic functions. The Bargmann representation [1,11,16] is the most well-known one, and uses analytic functions in the complex plane (Euclidean geometry). Representations that use analytic functions in the unit disc or equivalently half-plane (Lobachevsky geometry) have also been used and with them $SU(1,1)$ transformations are easily implemented with the use of conformal mappings in the unit disc. Representations that use analytic functions in the extended complex plane (which is stereographically equivalent to a sphere) have also

\textsuperscript{*} Corresponding author. Tel.: +44-1895-274000x3684; fax: +44-1895-251686.

\textit{E-mail address:} lampros.stergioulas@brunel.ac.uk (L.K. Stergioulas).
been used and with them SU(2) transformations are easily implemented with the use of conformal mappings in the extended complex plane. In digital signal processing, the \( z \)-transform which is defined in the exterior of the unit disc has been used extensively, and this is equivalent to the use of analytic functions in the unit disc in quantum mechanics. Control theory also extensively uses analytic representations.

The purpose of this paper is to use the Bargmann analytic representation in the context of signal and image processing. It should be clear from the above discussion that this is different from the \( z \)-transform. The Bargmann representation uses analytic functions in the complex Euclidean plane, while the \( z \)-transform uses analytic functions in the exterior of the unit disc (Lobachevsky geometry). The \( z \)-transform has been studied extensively in signal processing and in this paper we study the Bargmann representation.

In the Bargmann approach, signals (and images) are represented by functions which are analytic in the complex plane, and the powerful theory of analytic functions could be exploited for the accurate representation of a signal in terms of a small number of parameters. The Bargmann representation is known to be a powerful technique for quantum mechanics and in this paper we explore its potential for signal processing.

In Section 2, we consider one-dimensional problems and define the Bargmann representation. We also derive transforms between the Bargmann representation and other representations. Some other analytic representations are compared and contrasted to the Bargmann representation. In Section 3, we consider examples of optical signals and describe how we can implement the Bargmann representation in a numerically effective way. We conclude in Section 4 with a discussion of our results.

2. The Bargmann function, its growth and its relationship with other representations

We consider a one-dimensional signal represented by the function \( f_x(x) \). In this case, the variable \( x \) (and also \( p \) that appears later) represents time (and frequency). It can also be a one-dimensional image in which case \( f_x(x) \) gives the level of greyness at the point \( x \) \((-\infty < x < \infty)\). It is assumed that the function \( f_x(x) \) is normalised

\[
\int |f_x(x)|^2 \, dx = 1.
\]  

(1)

We call the representation of a signal by the function \( f_x(x) \), as “\( x \)-representation”, the index \( x \) indicating explicitly the representation. The image can also be represented by the Fourier transform of the function \( f(x) \):

\[
f_p(p) = \int_{-\infty}^{+\infty} f_x(x) \exp(i \, px) \, dx.
\]  

(2)

The representation of a signal by the function \( f_p(p) \) is called “\( p \)-representation”. Let \( H \) be the Hilbert space of all the functions \( f(x) \). In this Hilbert space, we consider the Hermitian orthonormal basis:

\[
U_N(x) = \pi^{-1/2} 2^{-N/2} (N!)^{-1/2} H_N(x) \exp(-\frac{1}{2} x^2),
\]  

(3)
\[
\int U_N(x)U_M(x) \, dx = 1, \tag{4}
\]
\[
\sum_{N=0}^{\infty} \int U_N(x)U_N(y) \, dx = \delta(x - y), \tag{5}
\]
where \(H_N(x)\) are the Hermitian polynomials. The function \(f_x(x)\) can be analysed in this basis as follows:
\[
f_x(x) = \sum_{N=0}^{\infty} f_N U_N(x),
\]
\[
f_N = \int f_x(x)U_N(x) \, dx,
\]
\[
\sum_{N=0}^{\infty} |f_N|^2 = 1. \tag{6}
\]

The Bargmann representation of the signal is defined to be
\[
f_B(z) = \sum_{N=0}^{\infty} (N!)^{-1/2} f_N z^N \tag{7}
\]
and is an analytic function in the complex plane. The index \(B\) indicates “Bargmann”. For a two-dimensional image, the Bargmann function is a function of two variables, one for each dimension.

The scalar product of two Bargmann functions \(f_B(z), g_B(z)\) is given by
\[
(f_B, g_B) = \int_C [f_B(z)]^* g_B(z) \exp(-|z|^2) \frac{d^2z}{\pi}. \tag{8}
\]

The growth (at infinity) of an analytic function \(f(z)\) is characterised by the order \(\rho\) and the type \(\sigma\). If \(M(R)\) is the maximum modulus of \(f(z)\) for \(|z| = R\), then
\[
\rho = \lim_{R \to \infty} \sup \frac{\ln \ln M(R)}{\ln R}, \quad \sigma = \lim_{R \to \infty} \sup \frac{\ln M(R)}{R^\rho}. \tag{9}
\]
From the requirement of convergence for the scalar product of Eq. (8), it is clear that the maximum growth of Bargmann functions is \(\rho = 2\) and \(\sigma = \frac{1}{2}\) \([2,17,18,22,21]\).

It is not difficult to prove that the Bargmann function can be calculated directly from the \(x\) and \(p\) representations \([23]\):
\[
f_B(z) = \pi^{-1/4} \exp\left(-\frac{z^2}{2}\right) \int_{-\infty}^{+\infty} \exp\left(\sqrt{2}zx - \frac{x^2}{2}\right) f_x(x) \, dx, \tag{10}
\]
\[
f_B(z) = \pi^{-1/4} \exp\left(-\frac{p^2}{2}\right) \int_{-\infty}^{+\infty} \exp\left(\sqrt{2}ipz - \frac{p^2}{2}\right) f_p(p) \, dp. \tag{11}
\]

The inverse relations can be easily found using the relations of Eqs. (10) and (11):
\[
f_x(x) = \pi^{-5/4} e^{-x^2/2} \int \int e^{-|z|^2 - (z^2/2) + \sqrt{2}xz^*} f_B(z) \, d^2z, \tag{12}
\]
\[
f_p(p) = \pi^{-5/4} e^{-p^2/2} \int \int e^{-|z|^2 + (p^2/2) - \sqrt{2}ipz^*} f_B(z) \, d^2z. \tag{13}
\]
If $z_R$, $z_I$ are the real and imaginary parts of $z$, we can derive the rather useful forms

\[ f_x(z_R) = \pi^{-3/4} e^{-z_R^2/2} \int_{-\infty}^{+\infty} e^{-z_I^2} f_B(\sqrt{2}z) \, dz_I, \]  
(14)

\[ f_p(z_I) = \pi^{-3/4} e^{-z_I^2/2} \int_{-\infty}^{+\infty} e^{-z_R^2} f_B(\sqrt{2}z^*) \, dz_R. \]  
(15)

Another analytic representation is the representation in the unit disk where the signal $f_x(x)$ of Eq. (6) is represented by the function

\[ f_D(z) = \sum_{N=0}^{\infty} f_N z^N, \quad |z| < 1. \]  
(16)

The space of all these functions is known as Hardy space $H_2(D)$. If we replace $z$ by $1/z$ in Eq. (16), we get the same representation defined in the exterior of the unit disk:

\[ f_Z(z) = \sum_{N=0}^{\infty} f_N z^{-N}, \quad |z| > 1. \]  
(17)

Eq. (17) is simply a $z$-transform. The inverse $z$-transform is given by

\[ f_N = \frac{1}{2\pi i} \oint_C f_Z(z) z^{N-1} \, dz. \]  
(18)

This should be compared and contrasted with the inverse Bargmann relation that provides $f_N$ from a known Bargmann function $f_B(z)$:

\[ f_N = \frac{(N!)^{1/2}}{2\pi i} \oint_C f_B(z) z^{N-1} \, dz. \]  
(19)

It is obvious from the above relations that the Bargmann analytic representation in the complex plane is very different from the $z$-transform in the unit disc (or in the exterior of the unit disc).

### 3. Numerical implementation

Numerical examples for two “difficult” one-dimensional pulse signals with sharp discontinuities, namely a triangular and a rectangular function, are considered. Starting from the function $f_x(x)$ (which is normalised as in Eq. (1)) and using the integral of Eq. (10), we get the Bargmann function in the whole complex plane. Numerically, we only need to calculate the Bargmann function on a ‘suitable finite grid’ in the complex plane. For the reconstruction of $f_x(x)$ we use Eq. (14), which in our case is a sum over the points of the grid that belong to an axis parallel to the imaginary axis. It is clear from Eq. (14) that it is sufficient for the grid to cover only regions where $e^{-z_I^2} f_B(\sqrt{2}z)$ takes significant values. We recall here that the maximum growth of Bargmann functions is $\rho = 2$, $\sigma = \frac{1}{2}$, and this shows that $e^{-z_I^2} f_B(\sqrt{2}z)$ will be effectively zero outside a limited region. In this region, the grid needs to be dense enough for a good accuracy.
As a measure of how good the reconstruction is, we evaluate the “reconstruction error” $D_B$ defined to be

$$D_B = \left[ \int [f'_x(x) - f_x(x)]^2 \, dx \right]^{1/2},$$

where the prime indicates the reconstructed function. The values of this quantity for the examples considered are given below. Another measure of the accuracy of the approximations involved in our calculation is as follows. The Bargmann method is in general valid for complex functions $f_x(x)$. In the examples considered here, the functions are real and we have verified that the reconstructed functions have imaginary parts equal to zero. In all the examples considered, the imaginary parts were found to be always less than $10^{-6}$.

From a practical point of view, it is important to study the robustness of the method when Gaussian noise is added to the Bargmann function. We consider

$$f'_B(z) = f_B(z) + r(z),$$

where $r(z) = r_R(z) + i r_I(z)$ with $r_R(z), r_I(z) \in [0, U]$ normally distributed random numbers. Using Eq. (17), the noise-distorted Bargmann function gives a noisy reconstruction $f'_x(x)$.

Numerical results have been obtained for the energy-normalised triangular pulse

$$f_x(x) = \begin{cases} 
0.185x + 0.6543 & \text{if } -\frac{5}{\sqrt{2}} \leq x \leq 0, \\
-0.185x + 0.6543 & \text{if } 0 < x \leq \frac{5}{\sqrt{2}}, \\
0 & \text{otherwise}
\end{cases}$$ (22)

and the energy-normalised rectangular pulse

$$f_x(x) = \begin{cases} 
0.4428 & \text{if } -\frac{5}{\sqrt{2}} \leq x \leq \frac{5}{\sqrt{2}}, \\
0 & \text{otherwise}
\end{cases}$$ (23)

Reconstruction results are shown in Figs. 1–6. For the reconstruction of the triangular signal, the following grids have been used:

- $61 \times 61$ $z$-plane grid with points
  $$z = A_{kl} = (-5 + 0.164k, -5 + 0.164l)$$ (24)
  with $k = 0, 1, \ldots, 60$ and $l = 0, 1, \ldots, 60$,

- reduced $7 \times 7$ $z$-plane grid with points
  $$z = B_{kl} = (-5 + 1.667k, -5 + 1.667l)$$ (25)
  with $k = 0, 1, \ldots, 6$ and $l = 0, 1, \ldots, 6$.

It can be seen that the truncated Bargmann function reconstructs accurately the triangular pulse using a grid of $61 \times 61$ points ($D_B = 0.0085$, Fig. 1) and the result is also good in the presence of noise with $r_R(z), r_I(z) \in [0, 0.1]$ ($D_B = 0.0246$, Fig. 2). Reasonably good accuracy is achieved with the reduced $7 \times 7$ Bargmann grid (Fig. 3).
Fig. 1. Triangular pulse signal $f_x(x)$ (61 points, dashed line) and its reconstruction $f'_x(x)$ (solid line) from a truncated Bargmann function (no noise, $61 \times 61$ Bargmann grid).

Fig. 2. Triangular pulse signal $f_x(x)$ (61 points, dashed line) and its reconstruction $f'_x(x)$ (solid line) from a truncated Bargmann function (with noise $r_R(z), r_I(z) \in [0, 0.1]$, $61 \times 61$ Bargmann grid).

For the reconstruction of the rectangular signal, the following grids have been used:

- $101 \times 101$ $z$-plane grid with points
  \[ z = C_{kl} = (-5 + 0.1k, -5 + 0.1l) \]  
with $k = 0, 1, \ldots, 100$ and $l = 0, 1, \ldots, 100$,

- reduced $10 \times 8$ $z$-plane grid with points
  \[ z = D_{kl} = (-5 + 1.111k, -5 + 1.429l) \]  
with $k = 0, 1, \ldots, 9$ and $l = 0, 1, \ldots, 7$. 

\[ z = C_{kl} = (-5 + 0.1k, -5 + 0.1l) \]  
\[ z = D_{kl} = (-5 + 1.111k, -5 + 1.429l) \]
It can be seen that the results for the case of a rectangular pulse are similar, but slightly worse as this is clearly a more difficult case. Reconstruction with 101 points (error $D_B = 0.1187$) is shown in Fig. 4, while noisy reconstruction with $r_R(z), r_I(z) \in [0, 0.1]$ is shown in Fig. 5 (error $D_B = 0.1206$). Again, the robustness of the method with respect to random noise is clearly demonstrated. Reconstruction with the reduced $10 \times 8$ Bargmann grid is shown in Fig. 6.

The examples exemplify the general comment made earlier about Eqs. (14) and (15), which are used for the reconstruction of the signal. Although the integration along an infinite line appears in these expressions, the maximum growth of Bargmann functions is $\rho = 2, \sigma = \frac{1}{2}$ and, consequently,
Fig. 5. Rectangular pulse signal \( f_x(x) \) (101 points, dashed line) and its reconstruction \( f'_x(x) \) (solid line) from a truncated Bargmann function (with noise \( r_R(z), r_I(z) \in [0, 0.1], 101 \times 101 \) Bargmann grid).

Fig. 6. Rectangular pulse signal \( f_x(x) \) (101 points, dashed line) and its reconstruction \( f'_x(x) \) (solid line) from a reduced \( 10 \times 8 \) Bargmann grid (no noise).

e^{-z^2} f_B(\sqrt{2}z) \) takes significant values only in a limited region. The use of a dense grid in this region gives very accurate results. The error due to the noise term \( r(z) \) in Eq. (21) is the integral \( e^{-z^2} r(\sqrt{2}z) \), which for Gaussian noise is small.

Most techniques that use the truncated expansion of a signal in terms of localised signals (Gabor analysis, wavelets, etc.) to reconstruct approximately the original signal are more accurate and more efficient than, for example, a truncated Fourier transform. In the general context, we have shown in this paper that the Bargmann representation (which is intimately related to Gaussian signals as it is seen in Eqs. (10) and (11)) is a very accurate and robust (insensitive to noise) technique.
4. Discussion

The $z$-transform, which is an analytic representation in the exterior of the unit disc (Lobachevsky geometry), has been used extensively in signal processing. Here we study another analytic representation, namely the Bargmann representation. It uses analytic functions in the complex plane (Euclidean geometry), it has been used extensively in quantum mechanics and here we have explored its potential for signal processing applications. It represents a signal with an analytic function in the complex plane defined in Eq. (9), where $f_N$ are coefficients given in Eq. (7). Eqs. (17) and (18) relate directly the Bargmann representation of a signal with its $x$ and $p$ representations. The Bargmann representation defined in the complex plane is very different from the formalistically similar and well-known $z$-transform, which is defined in the unit disc (or in the exterior of the unit disc).

In numerical calculations, it is usually a finite grid of a truncated Bargmann function which is used. We have seen that a relatively small number of terms is sufficient for the representation of fairly complicated functions. The Bargmann representation is amenable to inversion, and the inversion formulae proposed in Eqs. (17) and (18) can indeed be applied numerically with sufficient accuracy. The efficiency of the numerical approximations has been demonstrated with the use of an appropriate error measure. The robustness (insensitivity) of the method to noise has also been shown. Further development of this work should endeavour to exploit the analytic nature of the Bargmann representation for practical signal processing purposes and related areas.

References