Weak $C$-cleft extensions and weak Galois extensions

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Abstract
For a weak entwining structure $(A, C, \psi)$ we formulate the notion of weak $C$-Galois extension with normal basis and we show that these Galois extensions are equivalent to the weak $C$-cleft extensions introduced in [J.N. Alonso Álvarez, J.M. Fernández Vilaboab, R. González Rodríguez, A.B. Rodríguez Raposo, Weak $C$-cleft extensions, weak entwining structures and weak Hopf algebras, J. Algebra 284 (2005) 679–704].

Introduction
The notion of Hopf–Galois extension has its origin in the approach to Galois theory of groups acting on commutative rings developed by Chase, Harrison and Rosenberg. In 1969, Chase and Sweedler [18] extended this theory to coactions of Hopf algebra $H$ acting on a commutative $k$-algebra over a commutative ring $k$. In 1981, Kreimer and Takeuchi [29] give the following more general definition: let $H$ be a Hopf algebra and $A$ be a right
$H$-comodule algebra with coaction $\rho_A(a) = a_{(0)} \otimes a_{(1)}$, then the extension $B \subset A$, being $B = A^{coH} = \{a \in A; \rho_A(a) = a \otimes 1\}$ the subalgebra of coinvariant elements, is $H$-Galois if the canonical morphism $\gamma_A : A \otimes B \to A \otimes H$, defined by $\gamma_A(a \otimes b) = ab_{(0)} \otimes b_{(1)}$, is an isomorphism.

A well-known result in Galois theory says that if $B \subset A$ is a finite Galois extension of fields with Galois group $H$, then $A/B$ has a normal basis, i.e. there exists $a \in A$ such that the set $\{x.a; x \in H\}$ is a basis for $A$ over $B$. Kreimer and Takeuchi introduce in [29] the notion of normal basis for extensions, associated to Hopf algebras in categories of modules over a commutative ring, and in [22] Doi and Takeuchi characterized the $H$-Galois extensions with normal basis in terms of $H$-coleft extensions. This result can be extended to symmetric closed categories [25] and in the works of Brzeziński [10] and Abuhlail [1] we can find a more general formulation in the context of entwining structures (see also [15]).

In this paper, we formulate the definition of weak $C$-Galois extension with normal basis for a weak entwining structure $(A, C, \psi)$ living in a braided monoidal category $\mathcal{C}$ with equalizers and coequalizers and in Theorem 2.11 we characterize this extensions using the notion of cleftness introduced in [4]. Then, as a consequence, we obtain the following: first, the results related in the last paragraph are particular instances of Theorem 2.11; in second place, with this level of generality our approach can be applied to the study of Galois theory for weak Hopf algebras; finally, Theorem 2.11 proves that the notion of weak $C$-coleft extension formulated in [4] not only has applications in the theory of weak Hopf algebras with projection (for example, the proof of Radford’s theorem for weak Hopf algebras [2–4]) but it also can be applied to obtain the classical characterization of Galois extensions with normal basis for weak entwining structures.

1. Weak entwining structures and weak Galois extensions

We are working throughout in strict braided monoidal categories [27]. In what follows $(\mathcal{C}, \otimes, K, c)$ denotes a strict braided monoidal category with equalizers and coequalizers where $\otimes$ is the tensor product, $K$ the unit object and $c$ the braiding. It is an easy exercise to prove that if we have equalizers and coequalizers, then there exist split idempotents, i.e. for every morphism $q : Y \to Y$ such that $q = q \circ q$, there exist an object $Z$ and morphisms $i : Z \to Y$ and $p : Y \to Z$ satisfying $q = i \circ p$ and $p \circ i = id_Z$.

An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where $A$ is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A), \mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two algebras $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B), f : A \to B$ is an algebra morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A, f \circ \eta_A = \eta_B$. Also, if $A, B$ are algebras in $\mathcal{C}$, the object $A \otimes B$ is also an algebra in $\mathcal{C}$ where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where $D$ is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D, (\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are coalgebras, $f : D \to E$ is a coalgebra morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f, \varepsilon_E \circ f = \varepsilon_D$. When $D, E$ are coalgebras in $\mathcal{C}$, $D \otimes E$ is a coalgebra in $\mathcal{C}$ where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$. 

H-comodule algebra with coaction $\rho_A(a) = a_{(0)} \otimes a_{(1)}$, then the extension $B \subset A$, being $B = A^{coH} = \{a \in A; \rho_A(a) = a \otimes 1\}$ the subalgebra of coinvariant elements, is $H$-Galois if the canonical morphism $\gamma_A : A \otimes B \to A \otimes H$, defined by $\gamma_A(a \otimes b) = ab_{(0)} \otimes b_{(1)}$, is an isomorphism.
Weak entwining structures have been introduced by Caenepeel and de Groot [14] as a generalization of entwining structures defined by Brzeziński [10] and Brzeziński and Majid [11]. They introduce the so-called entwining structures, consisting of an algebra $A$, a coalgebra $C$, and an intertwining $\psi : C \otimes A \rightarrow A \otimes C$ satisfying four technical conditions which have been replaced by weaker axioms in the definition of Caenepeel and de Groot. The definition is the following.

**Definition 1.1.** A weak entwining structure on $C$ consists of a triple $(A, C, \psi)$, where $A$ is an algebra, $C$ a coalgebra, and $\psi : C \otimes A \rightarrow A \otimes C$ a morphism satisfying the relations:

1. $\psi \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A),$
2. $(A \otimes \delta_C) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes A),$
3. $\psi \circ (C \otimes \eta_A) = (e_{RR} \otimes C) \circ \delta_C,$
4. $(A \otimes \epsilon_C) \circ \psi = \mu_A \circ (e_{RR} \otimes A),$

where $e_{RR} : C \rightarrow A$ is the morphism defined by $e_{RR} = (A \otimes \epsilon_C) \circ \psi \circ (C \otimes \eta_A)$. The morphism $\psi$ is called intertwining.

In the definition of entwining structure the morphism $e_{RR} = \eta_A \otimes \epsilon_C$ and, obviously, any entwining structure is a weak entwining structure. Moreover, a weak entwining structure is an entwining structure if and only if $e_{RR} = \eta_A \otimes \epsilon_C$.

**Definition 1.2.** Let $(A, C, \psi)$ be a weak entwining structure in $C$. We denote by $\mathcal{M}_A^C(\psi)$ the category whose objects are triples $(M, \phi_M, \rho_M)$, where $(M, \phi_M)$ is a right $A$-module (i.e. $\phi_M \circ (\phi_M \otimes A) = \phi_M \circ (M \otimes \mu_A)$, $id_M = \phi_M \circ (M \otimes \eta_A)$), $(M, \rho_M)$ is a right $C$-comodule (i.e. $(\rho_M \otimes C) \circ \rho_M = (M \otimes \delta_C) \circ \rho_M$, $(M \otimes \epsilon_C) \circ \rho_M = id_M$), and

$$\rho_M \circ \phi_M = (\phi_M \otimes C) \circ (M \otimes \psi) \circ (\rho_M \otimes A).$$

The objects of $\mathcal{M}_A^C(\psi)$ will be called weak entwined modules and a morphism in $\mathcal{M}_A^C(\psi)$ is a morphism of $A$-modules and $C$-comodules. If $(A, C, \psi)$ is an entwining structure then we find the category of entwined modules introduced by Brzeziński in [10].

Using entwining structures it is possible to unify some categories of modules associated to a Hopf algebra as categories of entwined modules. For example, if $C = H$ is a Hopf algebra, $A$ is a right $H$-comodule algebra and $\psi = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A)$ an object $M$ in $\mathcal{M}_A^H(\psi)$ is a Hopf module [20]. If $C = A = H$ is a Hopf algebra and

$$\psi = (H \otimes \mu_H) \circ (H \otimes H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (c_{H,H} \otimes H \otimes H) \circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes \lambda_H \otimes \delta_H) \circ (H \otimes \delta_H),$$

then an object $M$ in $\mathcal{M}_A^C(\psi)$ is a Yetter–Drinfeld module [31,32]. Finally, let $H$ be a Hopf algebra. If $A$ is a right $H$-comodule algebra, $C$ is a right $H$-module coalgebra and $\psi = (A \otimes \phi_C) \circ (c_{C,A} \otimes H) \circ (C \otimes \rho_A)$ then an object in $\mathcal{M}_A^C(\psi)$ is a Doi–Koppinen module [23,28].
The category of weak Doi–Hopf modules, introduced in [7] can be identify as a category of weak entwined modules (see [14]).

1.3. We have the following (see [4]). Let \((A, C, \psi)\) be a weak entwining structure such that there exists a coaction \(\rho_A\) satisfying that \((A, \mu_A, \rho_A)\) belongs to \(\mathcal{M}_A^C(\psi)\). If for all \((M, \phi_M, \rho_M) \in \mathcal{M}_A^C(\psi)\), we denote by \(M_C\) the equalizer of \(\rho_M\) and \(\zeta_M = (\phi_M \otimes C) \circ (M \otimes (\rho_A \circ \eta_A))\) and by \(i^M_C\) the injection of \(M_C\) in \(M\), then:

(i) The triple \((AC, \eta_{AC}, \mu_{AC})\) is an algebra in \(C\), where \(\eta_{AC}: K \to AC\) and \(\mu_{AC}: AC \otimes AC \to AC\) are the factorizations of \(\eta_A\) and \(\mu_A \circ (i^A_C \otimes i^A_C)\) respectively, through the equalizer \(i^A_C\).

(ii) The pair \((MC, \phi_{MC})\) is a right \(AC\)-module, where \(\phi_{MC}: MC \otimes AC \to MC\) is the factorization of \(\phi_M \circ (i^M_C \otimes i^A_C)\) through the equalizer \(i^M_C\).

It is obvious that \((A, \phi_A^1 = \mu_A \circ (i^A_C \otimes A))\) is a left \(AC\)-module and \((A, \phi_A^2 = \mu_A \circ (A \otimes i^A_C))\) is a right \(AC\)-module. With \((q(A), A \otimes_{AC} A)\) we denote the coequalizer of \(A \otimes \phi_A^1\) and \(\phi_A^2 \otimes A\).

\[
\begin{array}{ccc}
A \otimes AC \otimes A & \xrightarrow{A \otimes \phi_A^1} & A \otimes A & \xrightarrow{q(A)} & A \otimes_{AC} A
\end{array}
\]

The following proposition was proved by Brzeziński in [12] and is the key to find a good definition of canonical morphism in this weak context.

**Proposition 1.4.** Let \((A, C, \psi)\) be a weak entwining structure. The morphism \(\Delta_{A \otimes C} : A \otimes C \to A \otimes C\) defined by

\[
\Delta_{A \otimes C} = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (A \otimes C \otimes \eta_A)
\]

is idempotent.

1.5. Let \((A, C, \psi)\) be a weak entwining structure such that there exists a coaction \(\rho_A\) satisfying that \((A, \mu_A, \rho_A)\) belongs to \(\mathcal{M}_A^C(\psi)\). As a consequence of 1.4 there exist an object \(A \varpi C\) and morphisms \(i_{A \otimes C} : A \varpi C \to A \otimes C\) and \(p_{A \otimes C} : A \otimes C \to A \varpi C\) satisfying \(\Delta_{A \otimes C} = i_{A \otimes C} \circ p_{A \otimes C}\) and \(p_{A \otimes C} \circ i_{A \otimes C} = id_{A \varpi C}\). Moreover, the following diagram

\[
\begin{array}{ccc}
A \varpi C & \xrightarrow{i_{A \otimes C}} & A \otimes C & \xrightarrow{\Delta_{A \otimes C}} & A \otimes C
\end{array}
\]

is an equalizer diagram.

Let \(i_A : A \otimes A \to A \otimes C\) be the morphism defined by \(i_A = (\mu_A \otimes C) \circ (A \otimes \rho_A)\). We have that
\[ \Delta_{A \otimes C} \circ t_A = (\mu_A \otimes C) \circ (A \otimes \mu_A \otimes C) \circ (A \otimes A \otimes \psi) \circ (A \otimes \rho_A \otimes \eta_A) \]
\[ = (\mu_A \otimes C) \circ (A \otimes \rho_A) \circ \left( A \otimes \left( \mu_A \circ (A \otimes \eta_A) \right) \right) \]
\[ = t_A. \]

Therefore, there exists an unique morphism \( r_{A \otimes C} : A \otimes A \rightarrow A \otimes C \) such that \( i_{A \otimes C} \circ r_{A \otimes C} = t_A \).

On the other hand, the morphism \( r_{A \otimes C} \) verifies:

\[ i_{A \otimes C} \circ r_{A \otimes C} \circ \left( A \otimes \phi_A^1 \right) = t_A \circ (A \otimes \rho_A) \circ \left( A \otimes \left( \Delta_A \otimes C \right) \otimes A \right) \]
\[ = \mu_A \otimes C \circ \left( A \otimes \left( \Delta_A \otimes C \right) \otimes A \right) \]
\[ = i_{A \otimes C} \circ r_{A \otimes C} \circ \left( \phi_A^2 \otimes A \right). \]

Then \( r_{A \otimes C} \circ (A \otimes \phi_A^1) = r_{A \otimes C} \circ (\phi_A^2 \otimes A) \) and, as a consequence, there exists an unique morphism (called the canonical morphism)

\[ \gamma_A : A \otimes_{AC} A \rightarrow A \otimes C \]

such that \( \gamma_A \circ q(A) = r_{A \otimes C} \).

Suppose that \( A \otimes - \) preserves coequalizers. Then \( \gamma_A \) is a morphism of left \( A \)-modules being \( \varphi_{A \otimes AC} : A \otimes (A \otimes_{AC} A) \rightarrow A \otimes_{AC} A \) the factorization of \( q(A) \circ (\mu_A \otimes A) \) through the coequalizer \( A \otimes q(A) \), i.e. \( \varphi_{A \otimes AC} \) is the unique morphism such that \( \varphi_{A \otimes AC} \circ (A \otimes q(A)) = q(A) \circ (\mu_A \otimes A) \), and \( \varphi_{A \otimes AC} : A \otimes A \otimes C \rightarrow A \otimes C \) is defined by \( \varphi_{A \otimes AC} = p_{A \otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \Delta_{A \otimes C}) \).

Indeed:

\[ \gamma_A \circ \varphi_{A \otimes AC} \circ \left( A \otimes q(A) \right) = r_{A \otimes C} \circ (\mu_A \otimes A) \]
\[ = p_{A \otimes C} \circ \left( \mu_A \otimes A \right) \]
\[ = p_{A \otimes C} \circ (A \otimes \mu_A \otimes C) \circ (A \otimes A \otimes q(A)) \]
\[ = \varphi_{A \otimes AC} \circ \left( A \otimes (p_{A \otimes C} \circ t_A) \right) \]
\[ = \varphi_{A \otimes AC} \circ (A \otimes \gamma_A) \circ (A \otimes q(A)). \]
Therefore, $\gamma_A \circ \varphi_{A \otimes C} = \varphi_{A \square C} \circ (A \otimes \gamma_A)$. Notice that in the last computations we use the equalities $p_{A \otimes C} \circ (\mu_A \otimes C) = p_{A \otimes C} \circ (\mu_A \otimes C) \circ (A \otimes \Delta_{A \otimes C}) = \varphi_{A \square C} \circ (A \otimes p_{A \otimes C})$, i.e. $p_{A \otimes C}$ is a morphism of left $A$-modules being $\varphi_{A \square C} = \mu_A \otimes C$.

Also, $\gamma_A$ is a morphism of right $C$-comodules where $\rho_{A \otimes A_C A} : A \otimes A_C A \to (A \otimes A_C A) \otimes C$ is the factorization of $(q(A) \otimes C) \circ (A \otimes \rho_A)$ through the coequalizer $q(A)$, i.e. $\rho_{A \otimes A_C A}$ is the unique morphism such that $\rho_{A \otimes A_C A} \circ q(A) = (q(A) \otimes C) \circ (A \otimes \rho_A)$, and $\rho_{A \square C} : A \square C \to A \square C \otimes C$ is defined by $\rho_{A \square C} = (p_{A \otimes C} \otimes C) \circ (A \otimes \delta_C) \circ i_{A \otimes C}$. Indeed, we have the equality $(\gamma_A \otimes C) \circ \rho_{A \otimes A_C A} = \rho_{A \square C} \circ \gamma_A$ because:

$$((\gamma_A \otimes C) \circ \rho_{A \otimes A_C A} \circ q(A)) = (r_{A \otimes C} \otimes C) \circ (A \otimes \rho_A)$$

$$= (p_{A \otimes C} \otimes C) \circ (A \otimes \delta_C) \circ t_A$$

$$= \rho_{A \square C} \circ r_{A \otimes C}$$

$$= \rho_{A \square C} \circ \gamma_A \circ q(A).$$

**Definition 1.6.** Let $(A, C, \psi)$ be a weak entwining structure such that $A \otimes -$ preserves coequalizers and there exists a coaction $\rho_A$ satisfying that $(A, \mu_A, \rho_A)$ belongs to $M_C^\psi$. We say that $A$ is a coalgebra weak Galois extension (or weak $C$-Galois extension) if the canonical morphism $\gamma_A$ defined in 1.5 is an isomorphism.

Notice that $\gamma_A$ is always a morphism of right $C$-comodules and is a morphism of left $A$-modules if $A \otimes -$ preserves coequalizers (see 1.5).

Recall that the notion of weak $C$-Galois extension have been introduced by Brzeziński in a category of modules for a commutative ring (see Example 2.4 of [12]).

**Examples 1.7.** (i) The last definition generalizes the notion of Hopf–Galois extension that arose from the works of Chase and Sweedler [18] and Kreimer and Takeuchi [29]. The latter is a $C$-Galois extension with $C = H$ being $H$ a Hopf algebra over a commutative ring $k$ and $A$ a right $H$-comodule algebra with coaction $\rho_A$ (see [17] for more details). For example, if $B = A^{coH} = \{a \in A; \rho_A(a) = a \otimes 1\} = AH$ is the subalgebra of coinvariant elements, a faithfully flat (as $B$-module) $H$-Galois extension $A$ is the noncommutative-geometric version of a principal fiber bundle or torsor in the terminology of Demazure and Gabriel [19].

(ii) In this example we work over a ground field $k$. Let $A$ be an algebra, let $C$ be a coalgebra and suppose that $(A, \rho_A)$ is a right $C$-comodule (using the Sweedler notation $\rho_A(a) = a_{(0)} \otimes a_{(1)}$). Put $B = A^{coC} = \{a \in A; \forall b \in A: \rho_A(ba) = ba_{(0)} \otimes a_{(1)}\}$. We say that $A$ is a coalgebra Galois extension (or $C$-Galois extension) if the canonical morphism $\gamma_A : A \otimes_B A \to A \otimes C$ defined by $\gamma_A(a \otimes b) = ab_{(0)} \otimes b_{(1)}$ is an isomorphism (see [9]).

The notion of entwining structure plays an important role in the study of coalgebra Galois extensions because an entwining between $C$ and $A$ arises from every $C$-Galois extension. It is shown in [9] that if there is an entwining $\psi : C \otimes A \to A \otimes C$ for which $A \in M_C^\psi$, then the morphism $\gamma_A$ is a morphism of entwined modules. Moreover, If $A$ is a $C$-Galois extension, then there is an unique entwining $\psi : C \otimes A \to A \otimes C$ satisfying this condition. It is called the canonical entwining associated to the $C$-Galois extension $A$, and is given by $\psi(c \otimes a) = \gamma_A(\gamma_A^{-1}(1 \otimes c)a)$. Therefore, the definition of $C$-Galois extension introduced
by Brzeziński and Hajac in [9] is an example of Definition 1.6 because in this case $\otimes = \circ$ and $A_C = A^{coC}$.

(iii) Weak Hopf algebras are generalizations of Hopf algebras and was introduced by Böhm, Nill and Szlachányi in [5,6]. The definition is the following:

A weak Hopf algebra $H$, in a symmetric monoidal category $\mathcal{C}$, is an algebra $(H, \eta_H, \mu_H)$ and coalgebra $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

(a1) $\delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_H \otimes H$.
(a2) $\varepsilon_H \circ \mu_H \circ (\mu_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes \delta_H \otimes H) = (\varepsilon_H \otimes \varepsilon_H) \circ (\mu_H \otimes \mu_H) \circ (H \otimes (\varepsilon_H \otimes H) \otimes H)$.
(a3) $\delta_H \circ \eta_H = (H \otimes \mu_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H) = (H \otimes (\mu_H \circ c_H,H) \otimes H) \circ (\delta_H \otimes \delta_H) \circ (\eta_H \otimes \eta_H)$.
(a4) There exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called antipode of $H$) verifying:
   (a4-1) $\mu_H \circ (H \otimes \lambda_H) \otimes H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_H,H) \circ ((\delta_H \circ \eta_H) \otimes H)$.
   (a4-2) $\mu_H \circ (\lambda_H \otimes H) \otimes \delta_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_H,H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H))$.
   (a4-3) $\mu_H \circ (\mu_H \otimes H) \circ (\lambda_H \otimes H \otimes \lambda_H) \circ (\delta_H \otimes H) \circ \delta_H = \lambda_H$.

As a consequence of this definition it is an easy exercise to prove that a weak Hopf algebra is a Hopf algebra if an only if the morphism $\delta_H$ (coproduct) is unit-preserving (i.e. $\eta_H \otimes \eta_H = \delta_H \circ \eta_H$) and if and only if the counit is a homomorphism of algebras (i.e. $\varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H$).

If $H$ is a weak Hopf algebra, the antipode $\lambda_H$ is unique, antimultiplicative, anti-comultiplicative and leaves the unit $\eta_H$ and the counit $\varepsilon_H$ invariant, i.e. $\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_H,H$, $\delta_H \circ \lambda_H = c_H,H \circ (\lambda_H \otimes \lambda_H) \circ \delta_H$, $\lambda_H \circ \eta_H = \eta_H$, $\varepsilon_H \circ \lambda_H = \varepsilon_H$.

If we define the morphisms $\Pi^L_H, \Pi^R_H, \overline{\Pi}^L_H$ and $\overline{\Pi}^R_H$ by

$$
\Pi^L_H = ((\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes c_H,H) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,
$$
$$
\Pi^R_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ (c_H,H \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H,
$$
$$
\overline{\Pi}^L_H = (H \otimes (\varepsilon_H \circ \mu_H)) \circ ((\delta_H \circ \eta_H) \otimes H) : H \to H,
$$
$$
\overline{\Pi}^R_H = (\varepsilon_H \circ \mu_H) \otimes H) \circ (H \otimes (\delta_H \circ \eta_H)) : H \to H,
$$

it is straightforward to show (see [5]) that they are idempotent. Moreover, we have that (see [14])

$$
\overline{\Pi}^R_H \circ \Pi^L_H = \Pi^L_H, \quad \Pi^L_H \circ \overline{\Pi}^R_H = \Pi^R_H, \quad \overline{\Pi}^L_H \circ \Pi^R_H = \Pi^R_H, \quad \Pi^R_H \circ \overline{\Pi}^L_H = \Pi^L_H.
$$

Also it is easy to show the formulas:

$$
\Pi^L_H \circ \lambda_H = \Pi^L_H \circ \lambda_H = \lambda_H \circ \Pi^L_H, \quad \Pi^R_H \circ \lambda_H = \Pi^R_H \circ \lambda_H = \lambda_H \circ \Pi^R_H,
$$

$$
\Pi^L_H \circ \lambda_H = \Pi^L_H \circ \lambda_H = \lambda_H \circ \Pi^L_H, \quad \Pi^R_H \circ \lambda_H = \Pi^R_H \circ \lambda_H = \lambda_H \circ \Pi^R_H.
$$
A morphism between weak Hopf algebras $H$ and $B$ is a morphism $f : H \to B$ which is both algebra and coalgebra morphism. If $f : H \to B$ is a weak Hopf algebra morphism, then $\lambda_B \circ f = f \circ \lambda_H$ (see [2, 1.4]).

Let be the triple $(H, H, \psi)$ where $\psi = (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes \delta_H)$. Then $(H, H, \psi)$ is a weak entwining structure with $e_{RR} = \Pi^R_H$.

Let $g : B \to H$ be a morphism of weak Hopf algebras and $f : H \to B$ be a morphism of coalgebras such that $g \circ f = id_H$ and $f \circ \eta_H = \eta_B$. If we define $\rho_B : B \to B \otimes H$ and the interwining $\psi : H \otimes B \to B \otimes H$ by

$$\rho_B = (B \otimes g) \circ \delta_B, \quad \psi = (B \otimes \mu_H) \circ (c_{H,B} \otimes H) \circ (H \otimes \rho_B),$$

we have that $(B, H, \psi)$ is a weak entwining structure where $e_{RR} = \Pi^R_H \circ f$.

These previous entwining structures are particular instances of the following general situation. Let $H$ be a weak Hopf algebra and let $(A, \rho_A)$ be an algebra, which is also a right $H$-comodule, such that $\mu_{A \otimes H} \circ (\rho_A \otimes \rho_A) = \rho_A \circ \mu_A$. We call $A$ a right $H$-comodule algebra if the following equivalent conditions hold:

1. $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes (\mu_H \circ c_{H,H}) \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$.
2. $(\rho_A \otimes H) \circ \rho_A \circ \eta_A = (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes \delta_H) \circ (\eta_A \otimes \eta_H)$.
3. $(A \otimes \Pi^R_H) \circ \rho_A = (\mu_A \otimes H) \circ (A \otimes \rho_A) \circ (\eta_A \otimes \eta_H)$.
4. $(A \otimes \Pi^R_H) \circ \rho_A = ((\mu_A \circ c_{A,A}) \otimes H) \circ (A \otimes \rho_A) \circ (A \otimes \eta_A)$.
5. $(A \otimes \Pi^R_H) \circ \rho_A \circ \eta_A = \rho_A \circ \rho_A \circ \eta_A$.
6. $(A \otimes \Pi^R_H) \circ \rho_A \circ \eta_A = \rho_A \circ \eta_A$.

Under these conditions, it is easy to show that $(A, H, \psi = (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes \rho_A))$ is a weak entwining structure and $(A, \mu_A, \rho_A) \in \mathcal{M}_A^C(\psi)$ (see [14, Theorem 4.14]). Also, we recover as examples the weak entwining structures $(H, H, \psi)$, $(B, H, \psi)$ defined in the previous paragraphs.

Then, using Definition 1.6, in the weak Hopf algebra context a weak $H$-Galois extension is a right $H$-comodule algebra $(A, \rho_A)$ such that $A \otimes -$ preserves coequalizers and the morphism $\gamma_A : A \otimes_{AH} A \to A \otimes H$ is an isomorphism. For example, if $H \otimes -$ preserves coequalizers, $(H, \delta_H)$ is weak $H$-Galois extension. Indeed:

Take the idempotent morphism $\Pi^L_H$. There exists an object $\Pi^L_H(H)$ and a pair of morphisms $i_{\Pi^L_H} : \Pi^L_H(H) \to H$, $p_{\Pi^L_H} : H \to \Pi^L_H(H)$, such that

$$\Pi^L_H = i_{\Pi^L_H} \circ p_{\Pi^L_H} \quad \text{and} \quad id_{\Pi^L_H(H)} = p_{\Pi^L_H} \circ i_{\Pi^L_H}.$$ 

Moreover, it is easy to prove that $\zeta_H = (H \otimes \Pi^R_H) \circ \delta_H$ and the following diagram

$$\begin{array}{ccc}
\Pi^L_H(H) & \xrightarrow{i_{\Pi^L_H}} & H \\
\Pi^R_H \downarrow & & \downarrow \delta_H \\
H \otimes H & \xrightarrow{\zeta_H} & H \otimes H
\end{array}$$

is an equalizer diagram. Therefore, we can put $H_H = \Pi^L_H(H)$. Let $\gamma_H$ the canonical morphism. We claim that $\gamma_H$ is an isomorphism with inverse.
\[
\gamma_H^{-1} = q(H) \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \circ i_{H \otimes H}.
\]

Indeed: First, notice that \( \gamma_H \circ \gamma_H^{-1} = \text{id}_{H \otimes H} \) iff \( i_{H \otimes H} \circ \gamma_H \circ \gamma_H^{-1} = i_{H \otimes H} \).

Finally, recall that the examples (i), (ii), (iii) can be described in terms of Galois corings (see [12, Example 5.5] or [13,16]) living in a category of modules over a commutative ring. Galois corings in monoidal categories with equalizers and coequalizers have been introduced by Böhm in [8].

**Definition 1.8.** Let \( A \) be a weak \( C \)-Galois extension. We will say that \( A \) satisfies the normal basis property (or \( A \) is a weak \( C \)-Galois extension with normal basis) if there exists an idempotent morphism of left \( AC \)-modules and right \( C \)-comodules \( \Omega_A: AC \otimes C \to AC \otimes C \) \((\varphi_{AC\otimes C} = \mu_{AC} \otimes C, \rho_{AC\otimes C} = AC \otimes \delta_C)\) and an isomorphism of left \( AC \)-modules and right \( C \)-comodules \( b_A: A \to AC \times C \) where \( AC \times C \) is the image of \( \Omega_A \) and

\[
\varphi_{AC \times C} = r_A \circ (\mu_{AC} \otimes C) \circ (AC \otimes s_A), \quad \rho_{AC \times C} = (r_A \otimes C) \circ (AC \otimes \delta_C) \circ s_A,
\]

being \( s_A: AC \times C \to AC \otimes C \) and \( r_A: AC \otimes C \to AC \times C \) the morphisms such that \( s_A \circ r_A = \Omega_A \) and \( r_A \circ s_A = \text{id}_{AC \times C} \).

Note that if \( C \) is a category of modules over a commutative ring, this notion of normal basis is equivalent to say that \( A \) is a direct summand of \( AC \otimes C \) in the category of left \( AC \)-modules and right \( C \)-comodules.
Observe that $\varphi_{AC \times C}$ and $\rho_{AC \times C}$ are well-defined structures of left $A_C$-module and right $C$-comodule respectively, because $\Omega_A$ is a morphism of left $A_C$-modules and right $C$-comodules. Also, we have a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\omega_A} & A_C \otimes C \\
\downarrow{\Omega_A} & & \downarrow{\Omega_A} \\
A_C \otimes C & \xrightarrow{\omega'_A} & A_C \otimes C \\
\downarrow{r_A} & & \downarrow{s_A} \\
A \times C & & \end{array}
\]

where $\omega_A = b_A^{-1} \circ r_A$ and $\omega'_A = s_A \circ b_A$ are morphisms of left $A_C$-modules and right $C$-comodules.

If we work with entwining structures, and $\Omega_A = id_{A_C \otimes C}$, i.e. $\otimes = \times$, we obtain the normality condition introduced in [1, Definition 4.6] for entwining structures. In [15] a definition of normal basis was introduced in the context of entwining structures, but in this case the existence of a grouplike element $x$ is assumed and the entwined module structure of $A$ is determined by $x$.

**Lemma 1.9.** Let $A$ be a weak $C$-Galois extension that satisfies the normal basis condition. There exists an unique left $A$-module morphism $m_A : A \otimes_{A_C} A \to A$ such that

\[m_A \circ q(A) = \mu_A \circ (A \otimes ((i_C^A \otimes \varepsilon_C) \circ \omega'_A))\]

where $\omega'_A$ is the morphism defined in 1.8.

**Proof.** Take $m'_A = \mu_A \circ (A \otimes ((i_C^A \otimes \varepsilon_C) \circ \omega'_A)) : A \otimes A \to A$. Then,

\[m'_A \circ (A \otimes \phi_A^1) = \mu_A \circ (A \otimes ((i_C^A \otimes \varepsilon_C) \circ (\mu_{AC} \otimes C) \circ (A_C \otimes \omega'_A))))\]

\[= \mu_A \circ (\phi_A^2 \otimes i_C^A \otimes \varepsilon_C) \circ (A \otimes A_C \otimes \omega'_A)\]

\[= m'_A \circ (\phi_A^3 \otimes A).\]

Therefore, there exists an unique morphism $m_A : A \otimes_{A_C} A \to A$ such that $m_A \circ q(A) = \mu_A \circ (A \otimes ((i_C^A \otimes \varepsilon_C) \circ \omega'_A))$. Finally, $m_A$ is a left $A$-module morphism because

\[m_A \circ \varphi_{A \otimes A_C} \circ (A \otimes q(A)) = m_A \circ q(A) \circ (\mu_A \otimes A)\]

\[= \mu_A \circ (A \otimes m_A) \circ (A \otimes q(A)).\]
2. The characterization of weak cleft extensions

In this section, for a weak entwining structure, we introduce the notion of weak $C$-cleft extension $A$ and we obtain a characterization of these extensions as $C$-Galois extensions with normal basis.

**Definition 2.1.** Let $(A, C, \psi)$ be a weak entwining structure and suppose that $(A, \rho_A)$ is a right $C$-comodule. By $\text{Reg}^{WR}(C, A)$ we denote the set of morphisms $h \in \text{Hom}_C(C, A)$ such that there exists a morphism $h^{-1} \in \text{Hom}_C(C, A)$ (the left weak inverse of $h$) satisfying $h^{-1} \wedge h = e_{RR}$.

Let $A$ be an algebra and $C$ be a coalgebra in $\mathcal{C}$. By $\text{Reg}(C, A)$ we denote the set of morphisms $h : C \to A$ such that there exists a morphism $h^{-1} : C \to A$ (the inverse of $h$) satisfying

$$h^{-1} \wedge h = h \wedge h^{-1} = \varepsilon_C \otimes \eta_A = \eta_A \circ \varepsilon_C.$$  

Of course, if $(A, C, \psi)$ is an entwining structure in $\mathcal{C}$ $e_{RR} = \varepsilon_C \otimes \eta_A$ and $\text{Reg}(C, A) \subset \text{Reg}^{WR}(C, A)$.

**Remark 2.2.** Suppose that $(A, C, \psi)$ be a weak entwining structure such that there exists a coaction $\rho_A$ satisfying that $(A, \mu_A, \rho_A)$ belongs to $\mathcal{M}^C_A(\psi)$. Then if $h \in \text{Hom}_C(C, A)$ is a morphism of right $C$-comodules $h \wedge e_{RR} = h$.

**Definition 2.3.** Let $(A, C, \psi)$ be a weak entwining structure and suppose that $(A, \mu_A, \rho_A) \in \mathcal{M}^C_A(\psi)$. We will say that $A_C \hookrightarrow A$ is a weak $C$-cleft extension if there exists a morphism $h : C \to A$ in $\text{Reg}^{WR}(C, A)$ of right $C$-comodules such that

$$\psi \circ (C \otimes h^{-1}) \circ \delta_C = \xi_A \circ (e_{RR} \wedge h^{-1}),$$

where $\xi_A = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \eta_A))$ is the morphism defined in 1.3.

This definition was introduced in [4] and it is a generalization of the one used by Brzeziński [10] (see [26] for the braided case or [21,22,24,30] for the classical definitions) in the context of entwined modules. Note that, while in the case of a cleft extension for an entwining structure $h$ is required to be a comodule morphism and convolution invertible, here both conditions are replaced by weaker ones, quoted in the two last definitions.

**Remarks 2.4.** (i) Let $A_C \hookrightarrow A$ be a weak $C$-cleft extension with morphism $h$ of $C$-comodules in $\text{Reg}^{WR}(C, A)$. Then the interwining $\psi$ is completely determined in the following form:

$$\psi = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ ((h^{-1} \otimes h) \circ \delta_C) \otimes A).$$

(ii) Let $(A, C, \psi)$ be an entwining structure and suppose that $(A, \mu_A, \rho_A) \in \mathcal{M}^C_A(\psi)$. If $h \in \text{Reg}(C, A)$ is a morphism of right $C$-comodules we have that

$$\psi \circ (C \otimes h^{-1}) \circ \delta_C = \xi_A \circ h^{-1} = \xi_A \circ (e_{RR} \wedge h^{-1}).$$

Then, as a consequence, a $C$-cleft extension for an entwining structure is an example of weak $C$-cleft extension.
Example 2.5. If $H$ is a Hopf algebra $\text{id}_H$ is an element of $\text{Reg}(H, H)$ with inverse $\text{id}_H^{-1} = \lambda_H$. In the weak Hopf algebra case $\text{id}_H \in \text{Reg}^{\text{WR}}(H, H)$ with left weak inverse $\lambda_H$ and $\text{id}_H \in \text{Reg}(H, H)$ if and only if $H$ is a Hopf algebra.

If $H$ and $B$ are weak Hopf algebras in the same conditions of 1.7(iii), $f \in \text{Reg}^{\text{WR}}(H, B)$ because for $f^{-1} = \lambda_B \circ f$ we obtain that $f^{-1} \circ f = \Pi^R_B \circ f = \text{e}_{RR}$ and $B_H \hookrightarrow B = (B, \mu_B, \rho_B)$ is a weak $H$-cleft extension because $\psi \circ (H \otimes f^{-1}) \circ \delta_H = \zeta_B \circ (\text{e}_{RR} \otimes f^{-1})$.

When $H = B = \text{id}_H$ we have that $HH = \Pi^L_H \hookrightarrow H = (H, \mu_H, \delta_H)$ is a weak $H$-cleft extension.

2.6. Let $A_C \hookrightarrow A$ be a weak $C$-cleft extension. The morphism

$$q_C^A = \mu_A \circ (A \otimes h^{-1}) \circ \rho_A : A \to A$$

factors through the equalizer $i_C^A$ (see [4]). Therefore, there exists a morphism $p_C^A : A \to A_C$ such that $i_C^A \circ p_C^A = q_C^A$.

Also, the morphism $\varphi_A : (\otimes A \to A$ defined by

$$\varphi_A = \mu_A \circ (\mu_A \otimes h^{-1}) \circ (h \otimes \psi) \circ (\delta_C \otimes A)$$

factors through the equalizer $i_C^A$. Moreover, if $\varphi'_A$ is the factorization of $\varphi_A$, we have the following equality:

$$\mu_{A_C} \circ (\varphi'_A \otimes \varphi'_A) \circ (C \otimes (\otimes A) \circ (\delta_C \otimes A) = \varphi'_A \circ (C \otimes A).$$

2.7. Let $A_C \hookrightarrow A$ be a weak $C$-cleft extension with morphism $h \in \text{Reg}^{\text{WR}}(C, A)$. The left $A_C$-module right $C$-comodule ($\varphi_{A_C \otimes C} = \mu_{A_C} \otimes C$, $\rho_{A_C \otimes C} = A_C \otimes \delta_C$) morphisms

$$\omega_A : A_C \otimes C \to A, \quad \omega'_A : A \to A_C \otimes C$$

defined by $\omega_A = \mu_A \circ (i_A^C \otimes h)$ and $\omega'_A = (p_A^C \otimes C) \circ h$ verify the equality $\omega_A \circ \omega'_A = \text{id}_A$ because $\omega_A \circ \omega'_A = \varphi_A \circ (A \otimes \text{e}_{RR}) \circ \rho_A = \text{id}_A$. As a consequence, the morphism $\Omega_A = \omega'_A \circ \omega_A$ is an idempotent and we have a commutative diagram.
where $r_A \circ s_A = id_{A \times C}$. Therefore, the morphism $b_A = r_A \circ \omega'_A$ is an isomorphism of right $C$-comodules and left $A_C$-modules with inverse $b_A^{-1} = \omega_A \circ s_A$. The module and comodule structures of $A_C \times C$ are the ones induced by the isomorphism $b_A$ and they are equal to

$$\varphi_{A_C \times C} = r_A \circ (\mu_{A_C} \otimes C) \circ (A_C \otimes s_A), \quad \rho_{A_C \times C} = (r_A \otimes C) \circ (A_C \otimes \delta_C) \circ s_A$$

respectively.

Also, $b_A$ is an isomorphism of algebras with $b_A$:

$$\eta_{A_C \times C} = b_A \circ \eta_A, \quad \mu_{A_C \times C} = b_A \circ \mu_A \circ (b^{-1}_A \otimes b^{-1}_A).$$

Finally, the product $\mu_{A_C \times C}$ can be identified in following way (see [4]).

$$\mu_{A_C \times C} = \mu_{A_C \circ Z_A C} = r_A \circ (\mu_{A_C} \otimes C) \circ (\mu_{A_C} \otimes \pi_A) \circ (A_C \otimes \chi_A \otimes C) \circ (s_A \otimes s_A)$$

where

$$\pi_A = (\varphi'_A \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes h), \quad \chi_A = (\varphi'_A \otimes C) \circ (C \otimes \psi) \circ (\delta_C \otimes i^A_C).$$

**Lemma 2.8.** Let $(A, C, \psi)$ be a weak entwining structure such that there exists a coaction $\rho_A$ satisfying that $(A, \mu_A, \rho_A)$ belongs to $\mathcal{M}'_{A}(\psi)$.

(i) The morphism $p_{A \otimes C}$ introduced in 1.5 verifies:

$$p_{A \otimes C} \circ \psi \circ (C \otimes \eta_A) = p_{A \otimes C} \circ (\eta_A \otimes C).$$

(ii) $(A \otimes \delta_C) \circ \psi \circ (C \otimes \eta_A) = ((\psi \circ (C \otimes \eta_A)) \otimes C) \circ \delta_C$.

**Proof.** (i) We have that

$$\Delta_{A \otimes C} \circ \psi \circ (C \otimes \eta_A) = (\mu_A \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A) \circ (C \otimes \eta_A \otimes \eta_A)$$

$$= \psi \circ (C \otimes (\mu_A \circ (\eta_A \otimes \eta_A)))$$

$$= \psi \circ (C \otimes \eta_A)$$

$$= \mu_A \circ (A \otimes \psi) \circ (\eta_A \otimes C \otimes \eta_A)$$

$$= \Delta_{A \otimes C} \circ (\eta_A \otimes C).$$

Therefore, we obtain $p_{A \otimes C} \circ \psi \circ (C \otimes \eta_A) = p_{A \otimes C} \circ (\eta_A \otimes C)$ because $i_{A \otimes C}$ is a monomorphism.

(ii) The equality is a consequence of the following computations:

$$(A \otimes \delta_C) \circ \psi \circ (C \otimes \eta_A) = (e_{RR} \otimes \delta_C) \circ \delta_C$$

$$= (e_{RR} \otimes C \otimes \delta_C) \circ (\delta_C \otimes \delta_C) \circ \delta_C$$

$$= ((\psi \circ (C \otimes \eta_A)) \otimes C) \circ \delta_C. \quad \square$$
Lemma 2.9. Let \( A \) be a weak \( C \)-Galois extension and let \( f_1, f_2 \) be morphism in \( \text{Hom}_C(C, A) \). Then the following are equivalent.

(i) \( \mu_A \circ (A \otimes f_1) \circ \rho_A = \mu_A \circ (A \otimes f_2) \circ \rho_A \).

(ii) \( e_{RR} \wedge f_1 = e_{RR} \wedge f_2 \).

Proof. (i) \( \Rightarrow \) (ii). First notice that

\[
\mu_A \circ (A \otimes f_1) \circ i_{A \otimes C} \circ \gamma_A \circ q(A) = \mu_A \circ (A \otimes (A \otimes f_1) \circ \rho_A) = \mu_A \circ (A \otimes (A \otimes f_2) \circ \rho_A) = \mu_A \circ (A \otimes f_2) \circ i_{A \otimes C} \circ \gamma_A \circ q(A).
\]

Then, \( \mu_A \circ (A \otimes f_1) \circ i_{A \otimes C} = \mu_A \circ (A \otimes f_2) \circ i_{A \otimes C} \) and as a consequence we have the equality \( \mu_A \circ (A \otimes f_1) \circ \Delta_{A \otimes C} = \mu_A \circ (A \otimes f_2) \circ \Delta_{A \otimes C} \). Therefore, composing with \( \eta_A \otimes C \) we obtain that \( e_{RR} \wedge f_1 = e_{RR} \wedge f_2 \).

(ii) \( \Rightarrow \) (i). It is similar and we leave the details to the reader. \( \square \)

Remark 2.10. In the conditions of 2.9 if \( f \in \text{Hom}_C(C, A) \) and \( \mu_A \circ (A \otimes f) \circ \rho_A = \mu_A \circ (A \otimes e_{RR}) \circ \rho_A \) we have that

\[
\mu_A \circ (A \otimes f) \circ \psi \circ (C \otimes \eta_A) = e_{RR} \wedge f = e_{RR} \wedge e_{RR} = e_{RR}.
\]

Theorem 2.11. Let \((A, C, \psi)\) be a weak entwining structure such that \( A \otimes - \) preserves coequalizers and there exists a coaction \( \rho_A \) satisfying that \((A, \mu_A, \rho_A)\) belongs to \( M_{AC}^C(\psi) \). The following are equivalent.

(i) \( A_C \hookrightarrow A \) is a weak \( C \)-cleft extension.

(ii) \( A \) is a weak \( C \)-Galois extension and satisfies the normal basis condition.

Proof. (i) \( \Rightarrow \) (ii). Let \( A_C \hookrightarrow A \) be a weak \( C \)-cleft extension and take

\[
\gamma_A' = q(A) \circ (\mu_A \otimes A) \circ (A \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C) \circ i_{A \otimes C} : A \square C \rightarrow A \otimes_{AC} A.
\]

Then \( \gamma_A \circ \gamma_A' = id_{A \otimes C} \) because

\[
i_{A \otimes C} \circ \gamma_A \circ \gamma_A' = t_A \circ (\mu_A \otimes A) \circ (A \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C) \circ i_{A \otimes C}
\]

\[
= (\mu_A \otimes C) \circ (A \otimes \mu_A \otimes C) \circ (A \otimes h^{-1} \otimes h \otimes C) \circ (A \otimes C \otimes \delta_C)
\]

\[
\circ (A \otimes \delta_C) \circ i_{A \otimes C}
\]

\[
= ((\mu_A \circ (A \otimes e_{RR}) \circ \rho_A) \otimes C) \circ i_{A \otimes C}
\]

\[
= i_{A \otimes C}.
\]

On the other hand, \( \gamma_A' \circ \gamma_A = id_{A \otimes AC} \) because
\[\gamma'_A \circ \gamma_A \circ q(A) = q(A) \circ (\mu_A \otimes A) \circ (A \otimes h^{-1} \otimes h) \circ (A \otimes \delta_C) \circ \iota_A\]

\[= q(A) \circ (\mu_A \otimes A) \circ (A \otimes q^C_A \otimes h) \circ (A \otimes \rho_A)\]

\[= q(A) \circ (\mu_A \otimes A) \circ (A \otimes (i^C_A \circ p^C_A) \otimes h) \circ (A \otimes \rho_A)\]

\[= q(A) \circ (A \otimes \mu_A) \circ (A \otimes (i^C_A \circ p^C_A) \otimes h) \circ (A \otimes \rho_A)\]

\[= q(A) \circ (A \otimes (\mu_A \circ (A \otimes e_{RR}) \circ \rho_A))\]

\[= q(A).\]

Therefore, \(A\) is a weak \(C\)-Galois extension. Finally, by 2.7 we obtain that \(A\) satisfies the normal basis condition.

(ii) \(\Rightarrow\) (i). Let \(A\) be a weak \(C\)-Galois extension that \(A\) satisfies the normal basis condition. Then the canonical morphism \(\gamma_A\) is an isomorphism and there exists and idempotent morphism of left \(A_C\)-modules and right \(C\)-comodules \(\Omega_A : A_C \otimes C \to A_C \otimes C\) \((\varphi_{A_C \otimes C} = \mu_{A_C} \otimes C, \rho_{A_C \otimes C} = A_C \otimes \delta_C)\) and an isomorphism of left \(A_C\)-modules and right \(C\)-comodules \(b_A : A \to A_C \times C\) where \(A_C \times C\) is the image of \(\Omega_A\) and \(\varphi_{A_C \times C} = r_A \circ (\mu_{A_C} \otimes C) \circ (A_C \otimes s_A), \rho_{A_C \times C} = (r_A \otimes C) \circ (A_C \otimes \delta_C) \circ s_A\), being \(s_A : A_C \times C \to A_C \otimes C\) and \(r_A : A_C \otimes C \to A_C \times C\) the morphisms such that \(r_A \circ s_A = id_{A_C \otimes C}\) and \(s_A \circ r_A = \Omega_A\).

Notice that, in these conditions, we have that \(r_A, s_A, \omega_A = b_A^{-1} \circ r_A\) and \(\omega'_A = s_A \circ b_A\) are also morphisms of left \(A_C\)-modules and right \(C\)-comodules.

Take \(h = \omega_A \circ (\eta_{A_C} \otimes C) : C \to A\). The morphism \(h\) is a right \(C\)-comodule morphism because \(\omega_A\) is a morphism of right \(A_C\)-comodules.

Define

\[h^{-1} = m_A \circ \gamma_A^{-1} \circ p_{A \otimes C} \circ (\eta_A \otimes C) : C \to A,\]

where \(m_A\) is the morphism introduced in 1.9. Then, using the equality

\[m_A \circ \gamma_A^{-1} \circ p_{A \otimes C} \circ \rho_A = (i^A_C \otimes \varepsilon_C) \circ \omega'_A,\]

we obtain:

\[\mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \rho_A = \mu_A \circ ((m_A \circ \gamma_A^{-1} \circ p_{A \otimes C} \circ \rho_A) \otimes h) \circ \rho_A\]

\[= \mu_A \circ (((i^A_C \otimes \varepsilon_C) \circ \omega'_A) \otimes h) \circ \rho_A\]

\[= \mu_A \circ (A \otimes \varepsilon_C \otimes h) \circ (i^A_C \otimes \delta_C) \circ \omega'_A\]

\[= \mu_A \circ (A \otimes \omega_A) \circ (i^A_C \otimes \eta_{AC} \otimes C) \circ \omega'_A\]

\[= \omega_A \circ \omega'_A = id_A\]

\[= \mu_A \circ (A \otimes e_{RR}) \circ \rho_A.\]
Thus, by 2.10,

\[ \mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \psi \circ (C \otimes \eta_A) = e_{RR} \]

and as a consequence \( h^{-1} \wedge h = e_{RR}, \) i.e. \( h \in \text{Reg}^{WR}(C, A), \) because

\[ \mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \psi \circ (C \otimes \eta_A) = h^{-1} \wedge h. \]

Indeed, by (i) and (ii) of 2.8 we have the following equalities:

\[ \mu_A \circ (A \otimes (h^{-1} \wedge h)) \circ \psi \circ (C \otimes \eta_A) = h^{-1} \wedge h. \]

For to finish the proof we only need to show that

\[ \psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ (e_{RR} \wedge h^{-1}). \]

Notice that

\[ h \wedge h^{-1} = (i_C^A \otimes \varepsilon_C) \circ \Omega_A \circ (\eta_{AC} \otimes C) \quad \text{and} \]

\[ \psi = (\mu_A \otimes C) \circ (A \otimes (\rho_A \circ \mu_A)) \circ ((h^{-1} \otimes h) \circ \delta_C) \otimes A \]

(h is a right \( C \)-comodule morphism in \( \text{Reg}^{WR}(C, A) \)). Then

\[ \psi \circ (C \otimes h^{-1}) \circ \delta_C = \zeta_A \circ (e_{RR} \wedge h^{-1}). \]
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