Monotone matrix functions of two variables

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Abstract

Monotone operator functions were studied by Korányi who generalised Löwner’s theorem on monotone matrix functions of arbitrary high order \(n\) to two variables. An alternate proof of Korányi’s representation of monotone operator functions is provided. © 2001 Published by Elsevier Science Inc. All rights reserved.

0. Introduction

For each positive integer \(n\) let \(\mathbb{C}^n\) be the standard \(n\)-dimensional complex linear space equipped with its natural inner product \(\langle \cdot, \cdot \rangle\). Let \(L(\mathbb{C}^n)\) denote the space of linear operators on \(\mathbb{C}^n\) and let \(S(\mathbb{C}^n) = S\) be the real linear space of self-adjoint linear operators on \(\mathbb{C}^n\). If \(J\) denotes a subinterval of \(\mathbb{R}\), let \(S_J(\mathbb{C}^n) = S_J\) denote the set of operators in \(S\) with spectrum in \(J\). To each open subinterval \(J\) there corresponds a convex open subset \(S_J\) of \(S\).

Definition. To each real-valued function \(f: J \rightarrow \mathbb{R}\) defined on \(J\) there corresponds an operator function \(f: S_J \rightarrow S\) defined in the following way. An operator \(A \in S_J(\mathbb{C}^n)\) has spectral decomposition

\[ A = \sum \lambda_k E_k \]

in which \(\lambda_1, \lambda_2, \ldots\) are the distinct eigenvalues of \(A\) and \(E_1, E_2, \ldots\) are the corresponding orthogonal projections onto eigenspaces of \(A\). An operator denoted by \(f(A)\) in \(S\) is defined by

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PII: S 0 0 2 4 - 3 7 9 5 ( 0 0 ) 0 0 3 0 8 - 6
\[
\begin{align*}
f(A) &= \sum f(\lambda_k)E_k. \\
\end{align*}
\]

**Definition.** Let \( J \) and \( J' \) be the subintervals of \( \mathbb{R} \) and let \( m, n \in \mathbb{N} \). To a function \( f : J \times J' \rightarrow \mathbb{R} \) defined on \( J \times J' \), there is an associated operator function

\[
f = Of : S_J(C^m) \times S_{J'}(C^n) \rightarrow S(C^m \otimes C^n)
\]

defined in the following way. If \( A \in S_J(C^m) \), \( B \in S_{J'}(C^n) \) have spectral resolutions:

\[
A = \sum \lambda_k E_k, \quad B = \sum \mu_k F_k
\]

in which \( \lambda_1, \lambda_2, \ldots \) (respectively, \( \mu_1, \mu_2, \ldots \)) are the distinct eigenvalues of \( A \) (respectively, \( B \)) and \( E_1, E_2, \ldots \) (respectively, \( F_1, F_2, \ldots \)) are the corresponding orthogonal projections onto eigenspaces of \( A \) (respectively, \( B \)), then an operator denoted by \( f(A, B) \), in \( S(C^m \otimes C^n) \) is defined by

\[
f(A, B) = \sum_j \sum_k f(\lambda_j, \mu_k) E_j \otimes F_k.
\]

There is a substantial literature concerning these operator functions and their properties. A function \( f : J \rightarrow \mathbb{R} \) is said to be operator monotone if \( f(A) \leq f(B) \) whenever the terms are defined and \( A \leq B \). A function \( f : J \times J' \rightarrow \mathbb{R} \) is said to be operator monotone of order \((m, n)\) if for \( A, A' \in S_J(C^m), B, B' \in S_{J'}(C^n) \) satisfying \( A' \geq A, B' \geq B \), the inequality \( f(A', B') - f(A, B') - f(A', B) + f(A, B) \geq 0 \) holds (formal definitions are given in subsequent sections). The present paper is concerned with the Fréchet differentiability and operator monotonicity of operator functions of two variables and their representations using Krein–Milman Theorem.

In 1934, Löwner [10] in a celebrated paper characterised those functions \( f : J \rightarrow \mathbb{R} \) which are operator monotone; they are, in particular, analytic. Several proofs of Löwner’s central result are presented in a monograph by Donoghue [5] (also see [1]). More recently Hansen and Pedersen [8] have obtained yet another very interesting proof of Löwner’s central result. A very readable account of all this may be found in a monograph by Bhatia [3]. Korányi [9] generalised Löwner’s theorem on monotone matrix functions of arbitrary high order \( n \) to two variables. Vasudeva [13] developed a theory of monotone matrix functions of two variables analogous to that developed by Löwner and showed that a complete analogue to that theory exists in two dimensions. Here we give an alternate proof of results obtained in [13] by exploiting Fréchet calculus and also provide an alternate proof of the representation theorem of Korányi [9, Theorem 4] following methods developed by Hansen and Pedersen [8]. We insist on dealing with functions of two variables in order to avoid great notational complications. This approach reveals all the essential features and the generalization to any number of variables will be apparent.

In Section 1, we obtain a differential characterisation of operator monotonicity of functions of two variables and obtain some regularity properties of the class. A class of functions \( \Omega \) is introduced in Section 2, whereas, a representation theorem for matrix monotone functions of two variables is obtained in Section 3.
1. The Fréchet differential

Let \( J \) and \( J' \) denote open subintervals of \( \mathbb{R} \). A function \( f \) defined on \( S_J(\mathbb{C}^m) \subset S(\mathbb{C}^m) \) is Fréchet differentiable at \( A \in S_J(\mathbb{C}^m) \) if there exists a bounded linear operator \( d f(A) \in L(S(\mathbb{C}^m), S(\mathbb{C}^m)) \) such that

\[
\lim_{H \to 0} \| H \|^{-1} \| (f(A + H) - f(A) - d f(A)(H)) \| = 0, \quad H \in S(\mathbb{C}^m).
\]

Likewise \( f \) is said to be Fréchet differentiable in \( S_J(\mathbb{C}^m) \), if \( f \) is Fréchet differentiable at every point \( A \in S_J(\mathbb{C}^m) \). We say that \( f \) is continuously Fréchet differentiable if the differential mapping \( A \to d f(A) \) from \( S_J(\mathbb{C}^m) \) into \( L(S(\mathbb{C}^m), S(\mathbb{C}^m)) \) is continuous. It is easy to see that, if \( f \) is differentiable at \( A \), then for every \( H \in S(\mathbb{C}^m) \),

\[
d f(A)(H) = \frac{d}{dt} \bigg|_{t=0} f(A + tH).
\]

Let \( f : S_J(\mathbb{C}^m) \times S_J'(\mathbb{C}^n) \to S(\mathbb{C}^m \otimes \mathbb{C}^n) \) be defined on \( S_J(\mathbb{C}^m) \times S_J'(\mathbb{C}^n) \). The function \( f \) is said to have partial Fréchet differential at \( A \), where \( (A, B) \in S_J(\mathbb{C}^m) \times S_J'(\mathbb{C}^n) \), if there exists \( T_1 \in L(S(\mathbb{C}^m), S(\mathbb{C}^m \otimes \mathbb{C}^n)) \) such that for \( H \in S(\mathbb{C}^m) \) the following limit

\[
\lim_{H \to 0} \| H \|^{-1} \| (f(A + H, B) - f(A, B) - T_1(H)) \| = 0
\]

holds and \( d_1 f(A, B) = T_1 \). In other words, the partial differential \( d_1 f(A, B) \) exists and equals \( T_1 \) if and only if the function \( H \to f(A + H, B) \) has a differential at \( H = 0 \) equal to \( T_1 \). The mixed partial differential of \( f \) with respect to the first and second variables denoted \( d_2 d_1 f \) is defined to be partial Fréchet differential of the function \( d_1 f \) with respect to the second coordinate so that \( d_2 d_1 f = d_2 (d_1 f) \). The value \( d_2 d_1 f(A, B) \) of \( d_2 d_1 f \) at the point \( (A, B) \) of its domain is an element of \( L(S(\mathbb{C}^n), L(S(\mathbb{C}^m), S(\mathbb{C}^m \otimes \mathbb{C}^n))) \) which can be identified with a continuous bilinear map from \( S(\mathbb{C}^m) \times S(\mathbb{C}^n) \) into \( S(\mathbb{C}^m \otimes \mathbb{C}^n) \). If \( f \) is two-times differentiable at \( (A, B) \), then

\[
d_2 d_1 f(A, B) = d_1 d_2 f(A, B).
\]

The present definitions and various results from the theory of Fréchet differentiable functions that will be subsequently used are taken from [6].

Let \( C(J) \) denote the space of continuous functions \( f : J \to \mathbb{R} \) and \( C^L(J) \) the space of functions \( f \in C(J) \) such that the derivatives \( f^{(1)}, \ldots, f^{(L)} \) exist and are continuous on \( J \). Observe that \( C^L(J) \) is a Fréchet space (see [12]). For each \( L \), we denote the space of functions \( F : S_J(\mathbb{C}^m) \to S(\mathbb{C}^m) \) such that \( F \) and its Fréchet derivatives \( d f, \ldots, d^L F \) exist and are continuous on \( S_J(\mathbb{C}^m) \) by \( C^L(S_J(\mathbb{C}^m), S(\mathbb{C}^m)) \).

The \( k \)th divided difference of a function \( f \) on points \( \lambda_0, \lambda_1, \ldots, \lambda_k \) (not necessarily distinct) will be denoted by \( f^{[k]}(\lambda_0, \lambda_1, \ldots, \lambda_k) \) (see [5]).

**Definition 1.1.** A function \( f : S_J(\mathbb{C}^m) \to S(\mathbb{C}^m) \) defined on \( S_J(\mathbb{C}^m) \) is said to be matrix monotone of order \( m \) if \( A, B \in S_J, A \leq B \) implies \( f(A) \leq f(B) \).
The function $f$ is said to be operator monotone if $f$ is matrix monotone of order $m$ for each $m \in \mathbb{N}$.

Let $C(J \times J')$ be the space of continuous functions $f : J \times J' \to \mathbb{R}$ defined on $J \times J'$. For any positive integer $L$, let $C^L(J \times J')$ be the Fréchet space of $L$-times continuously differentiable functions defined on the open subset $J \times J' \subseteq \mathbb{R}^2$ (see [12]). If $f \in C^L(J \times J')$ and $k + m \leq L$, $f^{(k,m)}$ denote the partial derivatives of $f$ in which $f$ is differentiated $k$-times with respect to first variable and $m$-times differentiated with respect to second variable. For such a function we will denote by

$$f^{[k,m]}(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; \mu_1, \mu_2, \ldots, \mu_{m+1})$$

the repeated divided difference of $f$ with respect to the first variable on the points $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ in $J$ and then with respect to the second variable on the points $\mu_1, \mu_2, \ldots, \mu_{m+1}$ in $J'$. It follows immediately from the definition of operator function that if $f(s, t) = g(s)h(t)$ for $(s, t) \in J \times J'$, then $f(A, B) = g(A) \otimes h(B)$ for all $(A, B) \in S_J(\mathbb{C}^m) \times S_{J'}(\mathbb{C}^n)$. It is easily seen that for such a function $f \in C^L(J \times J')$, if $k + m \leq L$

$$f^{[k,m]}(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; \mu_1, \mu_2, \ldots, \mu_{m+1}) = g^{[k]}(\lambda_1, \ldots, \lambda_{k+1})h^{[m]}(\mu_1, \ldots, \mu_{m+1}).$$

The following lemma is required in the sequel.

**Lemma 1.2.** Suppose $f \in C^L(J \times J')$ and $k + m \leq L$. If $J_1$ and $J_1'$ are closed subintervals of $J$ and $J'$, respectively, and $\lambda_1, \lambda_2, \ldots, \lambda_{k+1} \in J_1$, $\mu_1, \mu_2, \ldots, \mu_{m+1} \in J_1'$, then for some $(\xi, \varsigma) \in J_1 \times J_1'$

$$f^{[k,m]}(\lambda_1, \lambda_2, \ldots, \lambda_{k+1}; \mu_1, \mu_2, \ldots, \mu_{m+1}) = \frac{1}{k!m!} f^{(k,m)}(\xi, \varsigma).$$

For a proof, see [4].

For each integer $L \geq 0$, we denote the space of functions $f : S_J(\mathbb{C}^m) \times S_{J'}(\mathbb{C}^n) \to S(\mathbb{C}^m \otimes \mathbb{C}^n)$ such that $f$ and its Fréchet derivatives $df, d^2f, \ldots, d^L f$ exist and are continuous on $S_J \times S_{J'}$ by $C^L(S_J(\mathbb{C}^m) \times S_{J'}(\mathbb{C}^n), S(\mathbb{C}^m \otimes \mathbb{C}^n))$.

For the following theorem and its proof, the reader may refer [4].

**Theorem 1.3.** If $f \in C^L(J \times J')$, then the operator function

$$f = O f \in C^L(S_J(\mathbb{C}^m) \times S_{J'}(\mathbb{C}^n), S(\mathbb{C}^m \otimes \mathbb{C}^n)).$$

The mapping

$$O : C^L(J \times J') \to C^L(S_J(\mathbb{C}^m) \times S_{J'}(\mathbb{C}^n), S(\mathbb{C}^m \otimes \mathbb{C}^n))$$

is continuous.

**Definition 1.4.** Let $f : J \times J' \to \mathbb{R}$ be defined on $J \times J'$ and let $f : S_J(\mathbb{C}^m) \times S_{J'}(\mathbb{C}^n) \to S(\mathbb{C}^m \otimes \mathbb{C}^n)$ be the associated operator function defined on $S_J(\mathbb{C}^m)$ ×
The operator function is said to be matrix monotone of order \((m, n)\) if for \(A, A' \in S_J(C^m), B, B' \in S_J'(C^n)\) satisfying \(A' \succeq A, B' \succeq B\), the following holds

\[
f(A', B') - f(A, B') - f(A', B) + f(A, B) \geq 0.
\] (1)

The function \(f\) is said to be operator monotone if it is monotone of order \((m, n)\) for each \((m, n) \in \mathbb{N} \times \mathbb{N}\).

The following proposition is an easy consequence of the definition of monotonicity.

**Proposition 1.5.** Let \(f(x, y)\) be a real-valued continuous function of the variables \(x, y\) in \((-1, 1)\). Assume that \(f(x, 0) = f(0, y) = 0\) for all \(x, y\) in \((-1, 1)\). If \(f\) is operator monotone, then the first partial derivatives and the mixed second partial derivative of \(f\) exist and are continuous.

**Proof.** Indeed, for each \(y\) in \((-1, 1)\), the function \(G_y(x)\) defined by \(G_y(x) = yf(x, y)\) is operator monotone; for if \(A, B \in S(-1,1)(C^m), A \succeq B\), then

\[G_y(A) - G_y(B) = y[f(A, y) - f(B, y) - f(A, 0) + f(B, 0)] \geq 0.
\]

So \(G_y(x)\) is continuously differentiable [3, Theorem v.3.6] and hence \(f^{(1,0)}(x, y)\) exists for each \(y \in (-1, 1)\) and is continuous. Similarly, it may be shown that \(f^{(0,1)}(x, y)\) exists for each \(x \in (-1, 1)\) and is continuous.

We next consider the function \(F_h(x)\) defined by

\[F_h(x) = h^{-1}\{f(x, y + h) - f(x, y)\},\]

where \(h > 0\) and small, \(y \in (-1, 1)\). The function \(F_h(x)\) is operator monotone in \(x\). So \(f^{(0,1)}(x, y)\) being the limit of operator monotone function \(F_h(x)\) as \(h \to 0\) is itself operator monotone and hence is continuously differentiable in \(x\) [3, Theorem v.3.6]. \(\square\)

The condition (1) is equivalent to the condition that for \((m, n) \in \mathbb{N} \times \mathbb{N}\) the function

\[\varphi(s, t) = \langle f(A + sH, B + tK)\xi, \xi \rangle\]

of \((s, t)\) is monotone on the set \(\{s \in \mathbb{R} : A + sH \in S_J\} \times \{t \in \mathbb{R} : B + tK \in S_J'\}\) for all \((A, B) \in S_J \times S_J', H, K \in S\) and \(H, K \succeq 0\) and \(\xi \in C^m \otimes C^n\).

**Theorem 1.6.** If \(f \in C^2(J \times J')\), then \(f\) is operator monotone if and only if for each \((m, n) \in \mathbb{N} \times \mathbb{N}\),

\[d_2d_1 f(A, B)(H, K) \geq 0,
\]

whenever \((A, B) \in S_J(C^m) \times S_J'(C^n), H, K \in S, H, K \succeq 0\).
**Proof.** Observe that for \((A, B) \in S_J \times S_{J'}, H, K \in S, H, K \geq 0\) and \(s, t > 0\)

\[
\frac{\partial^2 \varphi(s, t)}{\partial s \partial t}
= \frac{d}{ds} \frac{d}{dt} |_{s=0} \langle f(A + sH, B + tK)(H)\xi, \xi \rangle
= \langle d_1 f(A, B + tK)(H)\xi, \xi \rangle
\]

and

\[
\frac{\partial^2 \varphi(s, t)}{\partial t \partial s}
= \frac{d}{dt} \frac{\partial}{\partial s} |_{s=0} \langle d_1 f(A, B + tK)(H)\xi, \xi \rangle
= \langle d_2 d_1 f(A, B)(H)K\xi, \xi \rangle
= \langle d_2 d_1 f(A, B)(H, K)\xi, \xi \rangle.
\]

Suppose \(\varphi(s, t)\) is monotone on the set \(\{s \in \mathbb{R} : A + sH \in S_J\} \times \{t \in \mathbb{R} : B + tK \in S_{J'}\}\). Then

\[
\frac{\partial^2 \varphi(s, t)}{\partial t \partial s}
\geq 0
\]

and so \(d_2 d_1 f(A, B)(H, K) \geq 0\).

On the other hand suppose that \(d_2 d_1 f(A, B)(H, K) \geq 0\). Then

\[
\frac{\partial^2 \varphi(s, t)}{\partial t \partial s}
\geq 0,
\]

i.e.,

\[
f(A + sH, B + tK) - f(A + sH, B) - f(A, B + tK) + f(A, B) \geq 0
\]

for sufficiently small \(s, t \geq 0\) and hence for \((s, t) \in [0,1] \times [0,1]\). In particular, for \(s = t = 1\), we get

\[
f(A + H, B + K) - f(A + H, B) - f(A, B + K) + f(A, B) \geq 0
\]

for \((A, B) \in S_J(\mathbb{C}^m) \times S_{J'}(\mathbb{C}^n), H, K \in S, H, K \geq 0. \quad \square
\]

If \(A \in S_J(\mathbb{C}^m), B \in S_{J'}(\mathbb{C}^n)\) and \(e_1, e_2, \ldots, e_m\) (respectively, \(f_1, f_2, \ldots, f_n\)) is an orthonormal basis of eigenvectors of \(A\) corresponding to eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_m\) (respectively of eigenvectors of \(B\) corresponding to eigenvalues \(\mu_1, \mu_2, \ldots, \mu_n\)), then \(e_j \otimes f_k, j = 1, 2, \ldots, m, k = 1, 2, \ldots, n\) is an orthonormal basis of \(\mathbb{C}^m \otimes \mathbb{C}^n\).

**Theorem 1.7.** With the notations as in paragraph above, we have

\[
d_2 d_1 f(A, B)(H, K)|_{i,j;k,l} = f^{[1,1]}(\lambda_i \lambda_k; \mu_j \mu_l) h_{i;k} k_{j,l},
\]

(2)
where
\[ h_{ik} = \langle He_k, e_i \rangle, \quad k_{jl} = \langle Kf_l, f_j \rangle. \]

**Proof.** Suppose \( f(s, t) = g(s)h(t) \), where \((s, t) \in J \times J'\). Then
\[
d_2 d_1 f(A, B)(H, K) = d_2(dg(A)(H) \otimes h(B))
= dg(A)(H) \otimes dh(B)(K)
= (g^{[1]}(A) \circ H) \otimes (h^{[1]}(B) \circ K)
= g^{[1]}(A) \otimes h^{[1]}(B) \circ (H \otimes K),
\]
using [3, Theorem v.3.3] or [4, Theorem 2.1]. Formula (2) extends by linearity to the set of all functions \( f \in C^L(J \times J') \) which are finite sums of products of functions of one variable. The proof of the theorem is now completed using the fact that the above set is dense in \( C^L(J \times J') \). □

**Theorem 1.8.** If \( f \in C^2(J \times J') \), then \( f \) is operator monotone on \( J \times J' \) if and only if for Hermitian matrices \( A, B \) whose spectra are contained in \( J \) and \( J' \), respectively, the matrix \( f^{[1,1]}(A, B) \geq 0 \), where
\[
f^{[1,1]}(A, B)_{i,k; j,l} = f^{[1,1]}(\lambda_i, \lambda_j; \mu_k, \mu_l)
\]
and \( \lambda_1, \lambda_2, \ldots, \lambda_m \) are the eigenvalues of \( A \) and \( \mu_1, \mu_2, \ldots, \mu_n \) are the eigenvalues of \( B \).

**Proof.** Let \( f \) be operator monotone, and let \( A \) (respectively, \( B \)) be a Hermitian matrix whose eigenvalues are in \( J \) (respectively, \( J' \)). Let \( H \in S(\mathbb{C}^m), K \in S(\mathbb{C}^n) \) be the matrices all whose entries are 1. Then \( H \) (respectively \( K \)) is positive. So \( A + sH \geq A \) if \( s \geq 0 \) and \( B + tK \geq B \) if \( t \geq 0 \). Hence
\[
f(A + sH, B + tK) - f(A + sH, B) - f(A, B + tK) + f(A, B) \geq 0
\]
for small positive \( s \) and \( t \). This implies
\[
d_2 d_1 f(A, B)(H, K) \geq 0.
\]
So, by Theorem 1.7, \( f^{[1,1]}(A, B) \circ (H \otimes K) \geq 0 \). But for this special choice of \( H \) and \( K \), this just says that \( f^{[1,1]}(A, B) \geq 0 \).

On the other hand assume that \( H \in S(\mathbb{C}^m), K \in S(\mathbb{C}^n) \) and \( H, K \geq 0 \), then \( H \otimes K \geq 0 \). Since the Schur product of positive semidefinite matrices is positive semidefinite, it follows that \( f^{[1,1]}(A, B) \circ (H \otimes K) \geq 0 \), i.e.,
\[
d_2 d_1 f(A, B)(H, K) \geq 0. \quad \Box
\]

The following result [13, Theorem 3.5] can now be easily derived.
Theorem 1.9. If \( f(x, y) \) is a monotone matrix function of order \((m, n)\), \(m, n > 1\), then the distributional derivatives

\[
\frac{\partial^{2p+2q-6}f}{\partial x^{2p-3}\partial y^{2q-3}} \quad (p = 2, \ldots, m; q = 2, \ldots, n)
\]

are convex and positive.

Proof. We shall indicate the proof when \( m = 2, n = 2 \). This has been considered expedient to avoid notational complications. For a detailed proof see [13, Theorem 3.5]. Let \( \lambda_1 < \lambda_2 \) (respectively, \( \mu_1 < \mu_2 \)) be in \( J \) (respectively, \( J' \)). There exists a symmetric matrix \( A \) such that \( \text{Sp}(A) = \{\lambda_1, \lambda_2\} \) (respectively symmetric matrix \( B \) with \( \text{Sp}(B) = \{\mu_1, \mu_2\} \)). Let \( e_1, e_2 \) be the eigenvectors of \( A \) corresponding to the eigenvalues \( \lambda_1, \lambda_2 \) respectively and \( f_1, f_2 \) be the eigenvectors of \( B \) corresponding to the eigenvalues \( \mu_1, \mu_2 \) respectively.

Let \( g(x, y) = f_\varepsilon(x, y) \) be the regularization of \( f \) of order \( \varepsilon \). Then \( g \) is matrix monotone and is smooth. It follows, in view of Theorem 1.8, that \( g^{[1,1]}(A, B) \geq 0 \). Writing the matrix of \( g \) relative to the orthonormal basis \( e_i \otimes f_j, i, j = 1, 2 \), we see that \( (g^{[1,1]}(\lambda_i, \lambda_k; \mu_j, \mu_l))_{i,j,k,l} \) is non-negative definite.

Subtract first column from second column and third column from fourth column. Divide and multiply each element of second and fourth columns by \( \alpha = \mu_2 - \mu_1 \). Subtract first column from third column and second column from fourth column. Divide and multiply each element of third and fourth columns by \( \gamma = \lambda_2 - \lambda_1 \). Apply the same operations to rows as have been applied to columns. We obtain

\[
\begin{pmatrix}
\alpha g^{[1,1]}(\lambda_1\lambda_1; \mu_1\mu_1) & \alpha^2 g^{[1,2]}(\lambda_1\lambda_1; \mu_1\mu_1\mu_2) \\
\alpha g^{[1,2]}(\lambda_1\lambda_1; \mu_1\mu_1\mu_2) & \alpha^2 g^{[1,3]}(\lambda_1\lambda_1; \mu_1\mu_1\mu_2) \\
\gamma g^{[2,1]}(\lambda_1\lambda_1\lambda_2; \mu_1\mu_2) & \alpha \gamma g^{[2,2]}(\lambda_1\lambda_1\lambda_2; \mu_1\mu_1\mu_2) \\
\alpha \gamma g^{[2,2]}(\lambda_1\lambda_1\lambda_2; \mu_1\mu_1\mu_2) & \alpha^2 \gamma g^{[2,3]}(\lambda_1\lambda_1\lambda_2; \mu_1\mu_1\mu_2) \\
\alpha \gamma g^{[2,2]}(\lambda_1\lambda_1\lambda_2; \mu_1\mu_1\mu_2) & \alpha^2 \gamma g^{[2,3]}(\lambda_1\lambda_1\lambda_2; \mu_1\mu_1\mu_2) \\
\gamma^2 g^{[3,1]}(\lambda_1\lambda_1\lambda_2\lambda_2; \mu_1\mu_1) & \alpha^2 \gamma g^{[3,2]}(\lambda_1\lambda_1\lambda_2\lambda_2; \mu_1\mu_1\mu_2) \\
\alpha^2 \gamma g^{[3,2]}(\lambda_1\lambda_1\lambda_2\lambda_2; \mu_1\mu_1\mu_2) & \alpha^4 g^{[3,3]}(\lambda_1\lambda_1\lambda_2\lambda_2; \mu_1\mu_1\mu_2)
\end{pmatrix}
\]

This implies that the \( 2 \times 2 \) submatrix

\[
\begin{pmatrix}
\alpha g^{[1,1]}(\lambda_1\lambda_1; \mu_1\mu_1\mu_2) & \alpha^2 g^{[1,2]}(\lambda_1\lambda_1; \mu_1\mu_1\mu_2) \\
\alpha^2 g^{[1,2]}(\lambda_1\lambda_1; \mu_1\mu_1\mu_2) & \alpha^4 g^{[2,3]}(\lambda_1\lambda_1\lambda_2; \mu_1\mu_1\mu_2)
\end{pmatrix}
\]

since \( \alpha > 0 \) and \( \gamma > 0 \). Letting \( \lambda_1, \lambda_2 \rightarrow x, \mu_1, \mu_2 \rightarrow y \), we get

\[
\begin{pmatrix}
\frac{1}{3!} \frac{\partial^4 g}{\partial x \partial y^3} & \frac{1}{2!2!} \frac{\partial^4 g}{\partial x^2 \partial y^2} \\
\frac{1}{2!2!} \frac{\partial^4 g}{\partial x^2 \partial y^2} & \frac{1}{3!} \frac{\partial^4 g}{\partial x^3 \partial y}
\end{pmatrix}
\]

This says \( \partial^2 g/\partial x \partial y \) is positive and convex.
Since $\partial^2 f_\varepsilon/\partial x \partial y$ is convex for every $\varepsilon$, and $\partial^2 f_\varepsilon/\partial x \partial y$ converges to $\partial^2 f/\partial x \partial y$ as a distribution, using the lemma in [13, p. 316] there exists an infinite sequence which converges to $\partial^2 f/\partial x \partial y$ uniformly on compacts. $\partial^2 f/\partial x \partial y$ being the limit of a sequence of convex functions, is itself convex. □

2. The class $\Omega$

Let $J, J'$ be the intervals of $\mathbb{R}$. Suppose that $0 \in J \cap J'$. Let $f(x, y)$ be a real-valued function of two variables $x, y$, where $x \in J, y \in J'$. The symbol $f$ shall also denote the operator function. For such functions the first partial derivatives and second mixed partial derivative exist and are continuous (Proposition 1.5). In what follows we shall show that if a real-valued function $f(x, y)$ defined on $(-1, 1) \times (-1, 1)$ is operator monotone, then so is the function $(1 + \lambda x^{-1}) f(x, y), |\lambda| \leq 1$. With this in mind, we introduce a class of functions $\Omega$, provide a characterization of the class and establish its connection with operator monotone functions. The desired objective is achieved in Theorem 2.9.

**Definition 2.1.** The operator function $f$ is in $\Omega$ if for all Hermitian operators $A_1, A_2, B, B_1, B \triangleright B_1, \text{Sp}(A_i) \subset J, i = 1, 2, \text{Sp}(B), \text{Sp}(B_1)$ are in $J'$ and for all $\lambda, 0 \leq \lambda \leq 1$, the following inequality

$$f(\lambda A_1 + (1 - \lambda) A_2, B) - f(\lambda A_1 + (1 - \lambda) A_2, B_1)$$

$$\leq \lambda (f(A_1, B) - f(A_1, B_1)) + (1 - \lambda) (f(A_2, B) - f(A_2, B_1))$$

holds.

The following properties of $\Omega$ are immediate:

1. $\Omega \neq \phi$; indeed, if $f(x, y) = g(x)h(y)$, where $g(x)$ is operator convex and $h(y)$ is operator monotone, then $f \in \Omega$;
2. $\Omega$ is a convex cone;
3. If $f \in \Omega$, then so does the function $f(x, y) + \alpha h(x)$, where $\alpha \in \mathbb{R}$ and $h$ is any real-valued function of $x$ only.

The following characterization of the class $\Omega$ follows [8, Theorem 2.1] and [2].

**Theorem 2.2.** Let $f : J \times J' \to \mathbb{R}$ be a real-valued function, where $J$ and $J'$ are subintervals of $\mathbb{R}$ and $0 \in J \cap J'$. The following are equivalent:

(i) $f \in \Omega$.

(ii) $f(K^*AK, B) - f(K^*AK, B_1) \leq (K^* \otimes I)(f(A, B) - f(A, B_1))(K \otimes I)$

where $K$ is a contraction and $A, B, B_1$ are Hermitian operators, $B \triangleright B_1, \text{Sp}(A) \subset J, \text{Sp}(B), \text{Sp}(B_1)$ are in $J'$. 


(iii) \( f(K^*_1 A_1 K_1 + K^*_2 A_2 K_2, B) - f(K^*_1 A_1 K_1 + K^*_2 A_2 K_2, B_1) \)

\[
\leq (K^*_1 \otimes I)(f(A_1, B) - f(A_1, B_1))(K_1 \otimes I) + (K^*_2 \otimes I)(f(A_2, B) - f(A_2, B_1))(K_2 \otimes I)
\]

for all \( K_1, K_2 \) with \( K^*_1 K_1 + K^*_2 K_2 \leq I \) and all Hermitian operator \( A_1, A_2, B, B_1 \geq B_1, \text{Sp}(A_i) \subset J, i = 1, 2 \) and \( \text{Sp}(B), \text{Sp}(B_1) \) are in \( J' \).

(iv) \( f(P A P, B) - f(P A P, B_1) \leq (P \otimes I)(f(A, B) - f(A, B_1))(P \otimes I) \) for every projection \( P, A, B, B_1 \) are Hermitian operators, \( B \geq B_1, \text{Sp}(A) \subset J, \text{Sp}(B), \text{Sp}(B_1) \) are in \( J' \).

**Proof.** (i)\( \Rightarrow \) (ii): Consider the operators on \( H \oplus H \) given by

\[
T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} K^* & L \\ M & -K^* \end{pmatrix}, \quad V = \begin{pmatrix} K^* & -L \\ M & K^* \end{pmatrix},
\]

where \( K \) is a contraction, \( L = (I - K K^*)^{1/2}, M = (I - K^* K)^{1/2} \).

Calculations show that \( U \) and \( V \) are unitaries and that

\[
U^* T U = \begin{pmatrix} K^* A K & K^* A L \\ L^* A K & L^* A L \end{pmatrix}, \quad V^* T V = \begin{pmatrix} K^* A K & -K^* A L \\ -L^* A K & L^* A L \end{pmatrix}.
\]

So

\[
\begin{pmatrix} K^* A K & 0 \\ 0 & L^* A L \end{pmatrix} = \frac{U^* T U + V^* T V}{2}.
\]

Set \( B' = \text{diag}(B, 0) \) and \( B'_1 = \text{diag}(B_1, 0) \). Then

\[
f \left( \begin{pmatrix} K^* A K & 0 \\ 0 & L^* A L \end{pmatrix}, B' \right) - f \left( \begin{pmatrix} K^* A K & 0 \\ 0 & L^* A L \end{pmatrix}, B'_1 \right)
\]

\[
= f \left( \frac{U^* T U + V^* T V}{2}, B' \right) - f \left( \frac{U^* T U + V^* T V}{2}, B'_1 \right)
\]

\[
\leq \frac{1}{2} (f(U^* T U, B') - f(U^* T U, B'_1)) + \frac{1}{2} (f(V^* T V, B') - f(V^* T V, B'_1))
\]

\[
= \frac{1}{2} (U^* \otimes I)(f(T, B') - f(T, B'_1))(U \otimes I)
\]

\[
+ \frac{1}{2} (V^* \otimes I)(f(T, B') - f(T, B'_1))(V \otimes I).
\]

Now

\[
(U^* \otimes I) = \begin{pmatrix} K^* \otimes I & 0 & M^* \otimes I & 0 \\ 0 & K^* \otimes I & 0 & M^* \otimes I \\ L^* \otimes I & 0 & -K \otimes I & 0 \\ 0 & L^* \otimes I & 0 & -K \otimes I \end{pmatrix}
\]

and \( U \otimes I = (U^* \otimes I)^* \). So
\[
\frac{1}{2}(U^* \otimes I)(f(T, B') - f(T, B'_1))(U \otimes I)
\]
\[
= \frac{1}{2} \begin{pmatrix}
(K^* \otimes I)(f(A, B) - f(A, B_1))(K \otimes I) & 0 \\
0 & 0 \\
(L^* \otimes I)(f(A, B) - f(A, B_1))(K \otimes I) & 0 \\
0 & 0 \\
(K^* \otimes I)(f(A, B) - f(A, B_1))(L \otimes I) & 0 \\
0 & 0 \\
(L^* \otimes I)(f(A, B) - f(A, B_1))(L \otimes I) & 0
\end{pmatrix}.
\]
Similarly
\[
\frac{1}{2}(V^* \otimes I)(f(T, B') - f(T, B'_1))(V \otimes I)
\]
\[
= \frac{1}{2} \begin{pmatrix}
(K^* \otimes I)(f(A, B) - f(A, B_1))(K \otimes I) & 0 \\
0 & 0 \\
-(L^* \otimes I)(f(A, B) - f(A, B_1))(K \otimes I) & 0 \\
0 & 0 \\
-(K^* \otimes I)(f(A, B) - f(A, B_1))(L \otimes I) & 0 \\
0 & 0 \\
(L^* \otimes I)(f(A, B) - f(A, B_1))(L \otimes I) & 0
\end{pmatrix}.
\]
We thus have
\[
\text{diag}\left(f(K^* AK, B) - f(K^* AK, B_1), 0, f(L^* AL, B) - f(L^* AL, B_1), 0\right)
\leq \text{diag}\left((K^* \otimes I)(f(A, B) - f(A, B_1))(K \otimes I), 0, (L^* \otimes I)(f(A, B) - f(A, B_1))(L \otimes I), 0\right),
\]
which in turn implies (ii).

(ii)\Rightarrow(iii). Set
\[
T = \text{diag}(A_1, A_2), \quad K = \begin{pmatrix} K_1 & 0 \\ K_2 & 0 \end{pmatrix}.
\]
Clearly, \(K\) is a contraction and \(K^* TK = \text{diag}(K_1^* A_1 K_1 + K_2^* A_2 K_2, 0)\).

Put \(B' = \text{diag}(B, 0)\) and \(B'_1 = \text{diag}(B_1, 0)\). Then
\[
f(K^* TK, B') - f(K^* TK, B'_1)
\]
\[
= \text{diag}(f(K_1^* A_1 K_1 + K_2^* A_2 K_2, B) - f(K_1^* A_1 K_1 + K_2^* A_2 K_2, B_1), 0, 0, 0)
\]
and
\[
(K^* \otimes I)(f(T, B') - f(T, B'_1))(K \otimes I)
\]
\[
= \text{diag}\left((K_1^* \otimes I)(f(A_1, B) - f(A_1, B_1))(K_1 \otimes I)\right).
\]
\[ + (K_2^* \otimes I)(f(A_2, B) - f(A_2, B_1))(K_2 \otimes I), 0, 0, 0). \]

Consequently,
\[
\begin{align*}
& f(K_1^* A_1 K_1 + K_2^* A_2 K_2, B) - f(K_1^* A_1 K_1 + K_2^* A_2 K_2, B_1) \\
& \leq (K_1^* \otimes I)(f(A_1, B) - f(A_1, B_1))(K_1 \otimes I) \\
& + (K_2^* \otimes I)(f(A_2, B) - f(A_2, B_1))(K_2 \otimes I).
\end{align*}
\]

(iii) \(\Rightarrow\) (iv) is obvious.

(iv) \(\Rightarrow\) (i): Given Hermitian operators \(A_1, A_2\) with spectra in \(J\) and Hermitian operators \(B, B_1\) with spectra in \(J'\) and \(B \geq B_1\). Set \(T = \text{diag}(A_1, A_2)\), \(P = \text{diag}(I, 0)\)

and
\[
W = \begin{pmatrix}
\lambda^{1/2} I & -(1 - \lambda)^{1/2} I \\
(1 - \lambda)^{1/2} I & \lambda^{1/2} I
\end{pmatrix},
\]

where \(0 < \lambda < 1\). Then \(PW^*TW P = \text{diag}(\lambda A_1 + (1 - \lambda)A_2, 0)\). Define \(B' = \text{diag}(B, 0)\) and \(B_1' = \text{diag}(B_1, 0)\). Observe that
\[
\begin{align*}
f(PW^*TW P, B') - f(PW^*TW P, B_1') \\
= \text{diag}(f(\lambda A_1 + (1 - \lambda)A_2, B) - f(\lambda A_1 + (1 - \lambda)A_2, B_1), 0, 0, 0)
\end{align*}
\]

and
\[
\begin{align*}
(PW^* \otimes I)(f(T, B') - f(T, B_1'))(WP \otimes I) \\
= \text{diag}(\lambda(f(A_1, B) - f(A_1, B_1)) \\
+ (1 - \lambda)(f(A_2, B) - f(A_2, B_1)), 0, 0, 0).
\end{align*}
\]

Since
\[
\begin{align*}
f(PW^*TW P, B') - f(PW^*TW P, B_1') \\
\leq (P \otimes I)(f(W^*TW, B') - f(W^*TW, B_1'))(P \otimes I) \\
= (PW^* \otimes I)(f(T, B') - f(T, B_1'))(WP \otimes I),
\end{align*}
\]

the inequality being the consequence of assuming (iv), (i) follows. \(\square\)

**Lemma 2.3.** Let \(f\) be a real-valued function defined on \(J \times J'\), where \(J\) and \(J'\) are intervals such that \(0 \in J \cap J'\). Assume \(f(x, 0) = f(0, y) = 0\) for all \((x, y) \in J \times J'\). Then for any projection \(P\) commuting with \(A\), we have
\[
(P \otimes I) f(A, B) = f(PAP, B),
\]
where \(A, B\) are Hermitian operators with \(\text{Sp}(A) \subset J\) and \(\text{Sp}(B) \subset J'\), respectively.
Proof. Let $A = \sum_i \lambda_i P_i$ and $B = \sum_j \mu_j Q_j$ be the spectral decompositions of $A$ and $B$, respectively. Then

$$ PAP = \sum_i \lambda_i P_i P = \sum \lambda_i P_i P. $$

So

$$ f(PAP, B) = \sum_{i,j} f(\lambda_i, \mu_j) P_i \otimes Q_j = (P \otimes I) \sum_{i,j} f(\lambda_i, \mu_j) P_i \otimes Q_j = (P \otimes I) f(A, B). \quad \Box $$

Lemma 2.4. Let $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a continuous function. Then

$$ f(A^{1/2}PA^{1/2}, B)(A^{1/2}P \otimes I) = (A^{1/2}P \otimes I) f(PAP, B), $$

where $P$ is any projection, $A$ and $B$ are Hermitian operators with spectrum in $[0, \infty)$.

Proof. The result can easily be checked for polynomials of the form $x^r y^s$, using induction on $r$ and $s$. The result then follows for all continuous functions on appealing to Weierstrass theorem. \quad \Box

Theorem 2.5. Let $f : [0, \alpha) \times [0, \alpha) \to \mathbb{R}$ be a continuous function satisfying $f(x, 0) = f(0, y) = 0$ for all $x, y \in [0, \infty)$. Then the following two conditions are equivalent:

(i) $f \in \Omega$,

(ii) $x^{-1} f(x, y)$ is operator monotone on $(0, \alpha) \times (0, \alpha)$.

Proof. (i)⇒(ii): Let $0 < A_1 \leq A$ and $0 < B_1 \leq B$. Then $0 < A_1^{1/2} \leq A^{1/2}$, i.e., $A^{-1/2}A_1^{1/2}$ is a contraction. Since $f \in \Omega$, using Theorem 2.2 (ii), we obtain

$$ f(A_1, B) - f(A_1, B_1) = f(A_1^{1/2}A^{-1/2}A_1^{-1/2}A_1^{1/2}, B) - f(A_1^{1/2}A^{-1/2}A_1^{-1/2}A_1^{1/2}, B_1) \leq (A_1^{1/2}A^{-1/2} \otimes I)(f(A, B) - f(A, B_1))(A^{-1/2}A_1^{1/2} \otimes I), $$

i.e.,

$$ (A_1^{-1} \otimes I) f(A_1, B_1) - (A_1^{-1} \otimes I) f(A_1, B) - (A^{-1} \otimes I) f(A, B_1) + (A^{-1} \otimes I) f(A, B) \geq 0. $$

This proves (ii).

(ii)⇒(i): Let $P$ be a projection and $A$ be a positive operator. Choose $\varepsilon > 0$, small such that $\text{Sp}((1 + \varepsilon)A) \subseteq (0, \alpha)$. Since $(P + \varepsilon I) \leq (1 + \varepsilon)I$, it follows that
\[ A^{1/2}(P + \varepsilon I)A^{1/2} \leq (1 + \varepsilon)A. \]

Let \( B \geq B_1 > 0. \) Since \( x^{-1}f(x, y) \) is operator monotone, we have

\[
\begin{align*}
(A^{-1/2}(P + \varepsilon I)^{-1}A^{-1/2} \otimes I)(f(A^{1/2}(P + \varepsilon I)A^{1/2}, B) \\
- f(A^{1/2}(P + \varepsilon I)A^{1/2}, B_1)) - ((1 + \varepsilon)^{-1}A^{-1} \otimes I) \\
\times (f((1 + \varepsilon)A, B) - f((1 + \varepsilon)A, B_1)) \leq 0.
\end{align*}
\]

Multiplying on the right by \( A^{1/2}(P + \varepsilon I) \otimes I \) and on the left by \( (P + \varepsilon I)A^{1/2} \otimes I \) and transposing the second factor to the right hand side of the inequality, we obtain

\[
\begin{align*}
(A^{-1/2} \otimes I)(f(A^{1/2}(P + \varepsilon I)A^{1/2}, B) \\
- f(A^{1/2}(P + \varepsilon I)A^{1/2}, B_1))(A^{1/2}P \otimes I) \\
\leq (P \otimes I)(f(A, B) - f(A, B_1))(P \otimes I).
\end{align*}
\]

Letting \( \varepsilon \to 0, \) we get

\[
\begin{align*}
(A^{-1/2} \otimes I)(f(AP, B) - f(AP, B_1)) \\
\leq (P \otimes I)(f(A, B) - f(A, B_1))(P \otimes I).
\end{align*}
\]

Using Lemmas 2.4, and 2.3, we have

\[
\begin{align*}
f(PAP, B) - f(PAP, B_1) \\
\leq (P \otimes I)(f(A, B) - f(A, B_1))(P \otimes I).
\end{align*}
\]

This, in view of Theorem 2.2 (iv), proves the result for \( A > 0. \) The general case follows by continuity argument. \( \square \)

**Lemma 2.6.** Let \( f(x, y) \) be a continuous operator monotone function on \((-1, 1) \times (-1, 1). \) Then for each \( \lambda, \vert \lambda \vert \leq 1, \) the function \((x + \lambda)f(x, y) \in \Omega. \)

**Proof.** We shall first prove the result for the function defined on \([-1, 1] \times [-1, 1]. \) The lemma will then follow by considering the function \( f((1 - \varepsilon)x, (1 - \varepsilon)y) \) instead of function \( f(x, y). \)

If \( f(x, y) \) is continuous and operator monotone on \([-1, 1] \times [-1, 1], \) then \( f(x - 1, y - 1) \) is operator monotone on \([0, 2) \times [0, 2). \) Set

\[ G(x, y) = xf(x - 1, y - 1) - xf(x - 1, -1). \]

Observe that

\[ G(x, 0) = 0 = G(0, y) \]

and \( x^{-1}G(x, y) \) is operator monotone on \((0, 2) \times (0, 2). \) Hence, by Theorem 2.5, \( G \in \Omega \) on \([0, 2) \times [0, 2) \) and so

\[ H(x, y) = G(x + 1, y + 1). \]
belongs to $\Omega$ on $[-1, 1) \times [-1, 1)$. So $H_1(x, y) = (x + 1) f(x, y) \in \Omega$ on $[-1, 1) \times [-1, 1)$.

Applying the above arguments to the operator monotone function $-f(-x, y)$ on $[-1, 1] \times [-1, 1]$, we obtain

$$H_2(x, y) = -(x + 1) f(-x, y)$$

belongs to $\Omega$ on $[-1, 1] \times [-1, 1)$.

It may be noticed that if $H_2(x, y) \in \Omega$ so does $H_2(-x, y)$. It then follows $H_3(x, y) = H_2(-x, y) = (x - 1) f(x, y) \in \Omega$ on $[-1, 1] \times [-1, 1)$.

Since $\Omega$ is convex, it then follows that

$$(x + \lambda) f(x, y) = \left(\frac{1 + \lambda}{2}\right) H_1(x, y) + \left(\frac{1 - \lambda}{2}\right) H_3(x, y) \in \Omega. \quad \square$$

**Lemma 2.7.** Let $f(x, y)$ be a real-valued function defined on $(-1, 1) \times (-1, 1)$ satisfying $f(x, 0) = f(0, y) = 0$ for all $x, y \in (-1, 1)$ and whose first order partial derivatives, mixed second order partial derivative and the partial derivative $f^{(2,1)}(x, y)$ exist and are continuous. If $\{e_{ij} : 0 \leq i, j \leq m\}$ and $\{f_{ij} : 1 \leq i, j \leq n\}$ are the system of matrix units for $M_{m+1}$ (Algebra of $(m + 1) \times (m + 1)$ matrices) and $M_n$, respectively, then with

$$H = \sum_{i=1}^{m} (e_{i0} + e_{oi}), \quad P = I - e_{00}, \quad A = \sum_{i=1}^{m} \lambda_i e_{ii}, \quad B = \sum_{i=1}^{n} \mu_i f_{ii},$$

where $-1 < \lambda_i, \mu_j < 1, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, we have

$$\lim_{\varepsilon_1 \to 0, \varepsilon_2 \to 0} (P \otimes I) \varepsilon_2^{-1} \varepsilon_1^{-2} [f(A + \varepsilon_1 H, B + \varepsilon_2 K) - f(A, B + \varepsilon_2 K)]$$

$$-\varepsilon_1 f^{[1,0]}(A, B + \varepsilon_2 K) \circ (H \otimes I) - f(A + \varepsilon_1 H, B) + f(A, B)$$

$$+ \varepsilon_1 f^{[1,0]}(A, B) \circ (H \otimes I)](P \otimes I) = g^{[1,1]}(A, B) \circ (H \otimes K).$$

Here $g(x, y) = x^{-1} f(x, y)$ and $K$ is a non-negative matrix.

**Proof.** If $f(x, y) = x^p y^q$, then the second order term in $\varepsilon_1$ and linear in $\varepsilon_2$ in the expansion of $f(A + \varepsilon_1 H, B + \varepsilon_2 K)$ is

$$C = \sum A^m H A^s H A^t \otimes B^u K B^v$$

the summation being over all integers $m, r, s \geq 0$ with $m + r + s = p - 2$ and $u, v \geq 0$ with $u + v = q - 1$. However, $A H A = 0$, by our choice of $H$. Hence

$$(P \otimes I) C (P \otimes I) = (P (H A^{p-2} H) P + \sum A^m H^2 A^{p-2-m} P) P$$

$$\otimes \left( \sum_{u+v=q-1} B^u K B^v \right)$$

$$= P \left( \sum A^m \left( \sum e_{ij} + e_{00} e_{j0} \right) A^{p-2-m} \right) P.$$
\[ \bigotimes \sum_{u+v=q-1} B^u K B^v = \sum A^m e_{ij} A^{p-2-m} \bigotimes \sum_{u+v=q-1} \mu_i^u \mu_j^v k_{kl} f_{kl} = \sum \lambda_i^m \lambda_j^{p-2-m} e_{ij} \bigotimes \sum \mu_k^u \mu_l^v q^{-1-k} k_{kl} f_{kl} = \sum \frac{(\lambda_i - \lambda_j)^{-1}(\lambda_i^{p-1} - \lambda_j^{p-1})}{(\lambda_i - \lambda_j)^{-1}} e_{ij} \bigotimes \sum (\mu_k - \mu_l)^{-1} \times (\mu_k^q - \mu_l^q) k_{kl} f_{kl} = g^{[1,1]}(A, B) \circ (H \otimes K). \]

The general case follows by the approximation theorem. \( \square \)

**Proposition 2.8.** Let \( f(x, y) \) be a real-valued function defined on \((-1, 1) \times (-1, 1)\) satisfying \( f(x, 0) = f(0, y) = 0 \) for all \( x, y \in (-1, 1) \) and whose first order partial derivatives, mixed second order partial derivative and the partial derivative \( f^{(2,1)}(x, y) \) exist and are continuous. If \( f \in \Omega \), then \( x^{-1} f(x, y) \) is operator monotone of two variables.

**Proof.** If \( A \) (respectively, \( B \)) is a self-adjoint matrix of order \( m \) (respectively, \( n \)) with \( \text{Sp}(A) \) (respectively, \( \text{Sp}(B) \)) in \((-1, 1)\), then for \( \varepsilon_1, \varepsilon_2 > 0 \), \( H, K \) non-negative self-adjoint matrices and \( f \in \Omega \), we have

\[
\begin{align*}
f(A + \delta \varepsilon_1 H, B + \varepsilon_2 K) - f(A + \delta \varepsilon_1 H, B) &= f((1 - \delta)A + \delta(A + \varepsilon_1 H), B + \varepsilon_2 K) - f((1 - \delta)A + \delta(A + \varepsilon_1 H), B) \\
&\leq (1 - \delta)(f(A, B + \varepsilon_2 K) - f(A, B)) + \delta(f(A + \varepsilon_1 H, B + \varepsilon_2 K) - f(A + \varepsilon_1 H, B)) \]
\end{align*}
\]

i.e.,

\[
\begin{align*}
f(A + \varepsilon_1 H, B + \varepsilon_2 K) - f(A + \varepsilon_1 H, B) - f(A, B + \varepsilon_2 K) + f(A, B) &- \frac{1}{\delta} (f(A + \delta \varepsilon_1 H, B + \varepsilon_2 K) - f(A + \delta \varepsilon_1 H, B) - f(A, B + \varepsilon_2 K) + f(A, B)) \geq 0.
\end{align*}
\]

Letting \( \delta \to 0 \), we obtain

\[
\begin{align*}
\left[ f(A + \varepsilon_1 H, B + \varepsilon_2 K) - f(A, B + \varepsilon_2 K) - \varepsilon_1 f^{[1,0]}(A, B + \varepsilon_2 K) \circ (H \otimes I) - f(A + \varepsilon_1 H, B) + f(A, B) + \varepsilon_1 f^{[1,0]}(A, B) \circ (H \otimes I) \right] \geq 0.
\end{align*}
\]

It follows from Lemma 2.7, with \( H \) as in Lemma 2.7, that \( g^{[1,1]}(A, B) \circ (H \otimes K) \geq 0 \), where \( g(x, y) = x^{-1} f(x, y) \). \( \square \)
**Theorem 2.9.** Let \( f(x, y) \) be a real-valued function defined on \((-1, 1) \times (-1, 1)\) satisfying \( f(x, 0) = f(0, y) = 0 \) for all \( x, y \in (-1, 1) \). If \( f \) is operator monotone of two variables, then so is \((1 + \lambda x^{-1}) f(x, y), |\lambda| \leq 1\).

**Proof.** Using Lemma 2.6, and Proposition 2.8, it follows that if \( f(x, y) \) is operator monotone, then so is \((1 + \lambda x^{-1}) f(x, y), |\lambda| \leq 1\), at least in the case in which \( f \) is sufficiently smooth.

In the general case, define \( f_\varepsilon \)-regularization of order \( \varepsilon \) of \( f \). Since \( f \in C^1((-1, 1) \times (-1, 1)) \) and the mixed second partial derivative is convex (Theorem 1.9), we know that \( f_\varepsilon \to f \) and

\[
\frac{\partial^i+j f_\varepsilon}{\partial x^i \partial y^j} \to \frac{\partial^i+j f}{\partial x^i \partial y^j},
\]

where \( i + j < 4 \), uniformly on compact subsets of \((-1, 1) \times (-1, 1)\). In particular,

\[
(1 + \lambda x^{-1})(f_\varepsilon(x, y) - f_\varepsilon(x, 0) - f_\varepsilon(0, y) + f_\varepsilon(0, 0))
\to (1 + \lambda x^{-1}) f(x, y),
\]

which is operator monotone, being limit of operator monotone functions. \( \square \)

3. **Korányi’s theorem**

Let \( f \) be a real-valued continuous function of two variables \( x, y \) in \((-1, 1)\). Assume \( f(x, 0) = f(0, y) = 0 \) for all \( x, y \) in \((-1, 1)\). Suppose that the operator function \( f \) is operator monotone in two variables. For such an \( f \), the first partial derivatives and the mixed second partial derivative exist and are continuous (Proposition 1.5). Suppose that the function \( f \) satisfies the requirement \( f(1, 1)(0, 0) = 1 \).

Let \( K \) be the collection of all such functions. Clearly \( K \) is a convex set. We will show that this set is compact in the topology of point-wise convergence and will determine its extreme points. This will enable us to write an integral representation for functions in \( K \). This alternate approach to Korányi’s theorem [9, Theorem 4] follows closely Hansen and Pedersen proof of Löwner’s theorem [8, Theorem 4.4].

The following observations about functions \( f \) in \( K \) will be needed:

If \( f \in K \), then \( F_y(x) = y^{-1} f(x, y), (y \neq 0) \) is operator monotone in \( x \). As the class of operator monotone functions is closed under point-wise limits, it follows that

\[
\lim_{y \to 0} F_y(x) = \lim_{y \to 0} \frac{f(x, y) - f(x, 0)}{y} = f^{(0,1)}(x, 0)
\]

is operator monotone in \( x \). Moreover,

\[
f^{(0,1)}(0, 0) = 0 \quad \text{and} \quad f^{(1,1)}(0, 0) = 1.
\]

In view of Lemma 4.1 of [8] it follows that
\[ f^{(0,1)}(x, 0) \leq \frac{x}{1-x} \quad \text{for } 0 \leq x < 1, \quad (3) \]

\[ f^{(0,1)}(x, 0) \geq \frac{x}{1+x} \quad \text{for } -1 < x < 0 \quad (4) \]

and

\[ |f^{(2,1)}(0, 0)| \leq 2. \quad (5) \]

It may be similarly checked that \( f^{(1,0)}(0, y) \) is operator monotone function of \( y \) satisfying \( f^{(1,0)}(0, 0) = 0 \) and \( f^{(1,1)}(0, 0) = 1 \). Appealing to Lemma 4.1 in [8], we have

\[ f^{(1,0)}(0, y) \leq \frac{y}{1-y} \quad \text{for } 0 \leq y < 1, \quad (6) \]

\[ f^{(1,0)}(0, y) \geq \frac{y}{1+y} \quad \text{for } -1 < y < 0 \quad (7) \]

and

\[ |f^{(1,2)}(0, 0)| \leq 2. \quad (8) \]

**Lemma 3.1.** If \( f \in K \), then

\[ f(x, y) \leq \frac{x}{1-x} \frac{y}{1-y} \quad \text{for } 0 \leq x < 1, \ 0 \leq y < 1, \]

\[ f(x, y) \leq \frac{x}{1+x} \frac{y}{1+y} \quad \text{for } -1 < x < 0, \ -1 < y < 0, \]

\[ f(x, y) \geq \frac{x}{1-x} \frac{y}{1+y} \quad \text{for } 0 \leq x < 1, \ -1 < y < 0, \]

\[ f(x, y) \geq \frac{x}{1+x} \frac{y}{1-y} \quad \text{for } -1 < x < 0, \ 0 \leq y < 1. \]

**Proof.** The result is trivially true if \( x = 0 \) or \( y = 0 \).

Set \( F_y(x) = y^{-1} f(x, y) \), \( y \not= 0 \). It is an operator monotone function of \( x \), which is defined on \((-1, 1)\). Moreover, \( F_y(0) = y^{-1} f(0, y) = 0 \), \( F_y'(x) = y^{-1} f^{(1,0)}(x, y) \) and \( F_y'(0) = y^{-1} f^{(1,0)}(0, y) \).

Observe that \( f^{(1,0)}(0, y) \not= 0 \) for \( y \not= 0 \). For if \( f^{(1,0)}(0, y) = 0 \) for some \( y \), then in view of the fact that \( f^{(1,0)}(0, 0) = 0 \) and the operator monotonicity of the function \( f^{(1,0)}(0, y) \), it follows that \( f^{(1,0)}(0, y) \equiv 0 \) which implies \( f^{(1,1)}(0, 0) = 0 \). This is a contradiction.

Thus the function

\[ G_y(x) = \frac{y^{-1} f(x, y)}{y^{-1} f^{(1,0)}(0, y)}, \quad x \in (-1, 1) \]
is operator monotone. Moreover, $G_y(0) = 0$ and $G'_y(0) = 1$. By [8], we have
\[
\frac{f(x, y)}{f^{(1,0)}(0, y)} \leq \frac{x}{1-x} \quad \text{for } 0 \leq x < 1,
\]
\[
\frac{f(x, y)}{f^{(1,0)}(0, y)} \geq \frac{x}{1+x} \quad \text{for } -1 < x < 0.
\]
Using (6) and (7), the desired inequalities follow. \hfill \square

**Proposition 3.2.** The set $K$ is compact in the topology of point-wise convergence.

**Proof.** Let $\{f_i\}$ be any net in $K$. By Lemma 3.1, the set $\{f_i(x, y)\}$ is bounded for each $x, y \in (-1, 1)$. So, by Tychonoff’s theorem there exists a subnet $\{f_i\}$ that converges point-wise to a bounded function $f$. The limit function $f$ is operator monotone. Moreover $f(x, 0) = 0 = f(0, y)$. We shall show that $f^{(1,1)}(0, 0) = 1$.

Observe that
\[
\lim_{y \to 0} \lim_{x \to 0} (1 + x^{-1})y^{-1} f_i(x, y) = f_i^{(1,1)}(0, 0) = 1.
\]
Also, in view of discussions preceding Lemma 3.1, $f_i^{(0,1)}(x, 0)$ being the limit of monotone functions in $x$ is itself monotone. Therefore, by Theorem 3.9 of [8],
\[
(1 + x^{-1})f_i^{(0,1)}(x, 0) \geq 1 \quad \text{if } x \geq 0 \quad \text{and} \quad (1 + x^{-1})f_i^{(0,1)}(x, 0) \leq 1 \quad \text{if } x \leq 0.
\]
Since $f_i$ is operator monotone, for each $y$
\[
y^{-1} f_i(x, y) = y^{-1}\{f_i(x, y) - f_i(x, 0) - f_i(0, y) + f_i(0, 0)\} \geq 0 \quad \text{for } x \geq 0
\]
\[
y^{-1} f_i(x, y) = y^{-1}\{f_i(x, y) - f_i(x, 0) - f_i(0, y) + f_i(0, 0)\} \leq 0 \quad \text{for } x \leq 0.
\]
Thus
\[
\lim_{y \to 0} y^{-1} f_i(x, y) = f_i^{(0,1)}(x, 0) \geq 0 \quad \text{for } x \geq 0,
\]
\[
\lim_{y \to 0} y^{-1} f_i(x, y) = f_i^{(0,1)}(x, 0) \leq 0 \quad \text{for } x \leq 0.
\]
Consequently,
\[
(1 + x^{-1})y^{-1} f_i(x, y) \geq 1 \quad \text{for } 0 \leq x < 1,
\]
\[
(1 + x^{-1})y^{-1} f_i(x, y) \leq 1 \quad \text{for } -1 < x \leq 0
\]
for all $y$ in a neighbourhood of zero. This implies
\[
(1 + x^{-1})y^{-1} f(x, y) \geq 1 \quad \text{for } 0 \leq x < 1,
\]
\[
(1 + x^{-1})y^{-1} f(x, y) \leq 1 \quad \text{for } -1 < x \leq 0
\]
for all $y$ in a neighbourhood of zero. Hence
\[
f^{(1,1)}(0, 0) = \lim_{x \to 0} \lim_{y \to 0} (1 + x^{-1})y^{-1} f(x, y) = 1,
\]
as \( f \) being operator monotone, possesses continuous partial derivatives. \( \square \)

**Proposition 3.3.** All extreme points of the set \( K \) have the form

\[
f(x, y) = \frac{x}{1 - \alpha x} \frac{y}{1 - \beta y}
\]

where \( \alpha = \frac{1}{2} f^{(2,1)}(0, 0) \) and \( \beta = \frac{1}{2} f^{(1,2)}(0, 0) \).

**Proof.** Let \( f \in K \). For each \( \lambda, -1 < \lambda < 1 \), let

\[
g(\lambda, 0)(x, y) = (1 + \lambda x^{-1}) f(x, y) - \lambda f^{(1,0)}(0, y).
\]

By Theorem 2.9, \( g(\lambda, 0)(x, y) \) is an operator monotone function of two variables. Moreover,

\[
g(\lambda, 0)(0, y) = 0 = g(\lambda, 0)(x, 0)
\]

and

\[
g^{(1,1)}(0, 0) = 1 + \frac{\lambda}{2} f^{(2,1)}(0, 0).
\]

So, the function \( G(\lambda, 0)(x, y) \) defined by

\[
G(\lambda, 0)(x, y) = \frac{1}{1 + \frac{\lambda}{2} f^{(2,1)}(0, 0)} \left[ (1 + \lambda x^{-1}) f(x, y) - \lambda f^{(1,0)}(0, y) \right]
\]

is in \( K \). Since \( |f^{(2,1)}(0, 0)| \leq 2 \), it follows that \( |\frac{1}{2} f^{(2,1)}(0, 0)| \leq 1 \).

We may write

\[
f(x, y) = \frac{1}{2} \left( 1 + \frac{\lambda}{2} f^{(2,1)}(0, 0) \right) G(\lambda, 0)(x, y) + \frac{1}{2} \left( 1 - \frac{\lambda}{2} f^{(2,1)}(0, 0) \right) G(-\lambda, 0)(x, y).
\]

If \( f(x, y) \) is an extreme point of \( K \), then we must have

\[
f = G(\lambda, 0) = G(-\lambda, 0).
\]

This says that

\[
\left( 1 + \frac{\lambda}{2} f^{(2,1)}(0, 0) \right) f(x, y) = \left( 1 + \frac{\lambda}{x} \right) f(x, y) - \lambda f^{(1,0)}(0, y),
\]
i.e.,

\[
f(x, y) = f^{(1,0)}(0, y) \frac{x}{1 - \frac{1}{2} x f^{(2,1)}(0, 0)}.
\]
Similarly, we may conclude that
\[ f(x, y) = f^{(0,1)}(x, 0) \frac{y}{1 - \frac{1}{2}y f^{(1,2)}(0, 0)}. \] (10)

Differentiating (9) w.r.t. \( y \) and then putting \( y = 0 \), we obtain
\[ f^{(0,1)}(x, 0) = \frac{x}{1 - \frac{1}{2}x f^{(2,1)}(0, 0)}. \] (11)

Substituting (11) in (10), we get the desired result. \( \square \)

**Theorem 3.4.** For each \( f \in K \), there exists a unique probability measure \( \mu \) on \([-1, 1] \times [-1, 1] \) such that
\[ f(x, y) = \int \int_{-1}^{1} \frac{x}{1 - sx} \frac{y}{1 - ty} d\mu(s, t). \] (12)

**Proof.** For \(-1 \leq s, t \leq 1\), consider the functions
\[ h(s, t)(x, y) = \frac{x}{1 - sx} \frac{y}{1 - ty}. \]

By Proposition 3.3, the extreme points of \( K \) are included in the family \( \{ h(s, t) \} \). Since \( K \) is compact and convex, it must be the closed convex hull of its extreme points, Krein–Millman theorem [11]. Finite convex combinations of elements of the family \( \{ h(s, t) : -1 \leq s, t \leq 1 \} \) can also be written as \( \int \int h(s, t) d\nu(s, t) \), where \( \nu \) is a finitely supported probability measure on \([-1, 1] \times [-1, 1] \). Since \( f \) is in the closure of these combinations, there exists a net \( \{ \nu_i \} \) of finitely supported probability measures on \([-1, 1] \times [-1, 1] \) such that the net \( f_i(x, y) = \int \int h(s, t) d\nu_i(s, t) \) converges to \( f \). Since the space of probability measures is weak* compact, the net \( \nu_i \) has an accumulation point \( \mu \). In other words a subnet of \( \int \int h(s, t) d\nu_i(s, t) \) converges to \( \int \int h(s, t) d\mu(s, t) \).

So
\[ f(x, y) = \int \int h(s, t) d\mu(s, t) = \int \int \frac{x}{1 - sx} \frac{y}{1 - ty} d\mu(s, t). \]

Now suppose that there are two measures \( \mu_1 \) and \( \mu_2 \) for which representation (12) is valid. Observe that
\[ \frac{x}{1 - sx} = \sum_{n=0}^{\infty} x^{n+1} s^n. \]
The series on the right is uniformly convergent in \(|s| < 1\) and for every fixed \( x \) with \(|x| < 1\). Similarly
\[ \frac{y}{1 - ty} = \sum_{n=0}^{\infty} y^{n+1} t^n, \]
the series, being uniformly convergent in \(|t| < 1\) and for every fixed \( y \) with \(|y| < 1\).
This shows that
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^{n+1} y^{m+1} \int \int s^n t^m \, d\mu_1(s, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^{n+1} y^{m+1} \int \int s^n t^m \, d\mu_2(s, t)
\]
for all \(|x| < 1, |y| < 1\). The identity theorem for power series shows that
\[
\int \int s^n t^m \, d\mu_1(s, t) = \int \int s^n t^m \, d\mu_2(s, t), \quad n, m = 0, 1, 2 \ldots,
\]
and this is possible if and only if \(\mu_1 = \mu_2\). □

**Remark 3.5.** The uniqueness of the measure \(\mu\) in the representation implies that every
\[
h_{(s_0, t_0)}(x, y) = \frac{x}{1 - s_0 x} \frac{y}{1 - t_0 y}
\]
is extreme point of \(K\) because it can be represented as an integral like this with \(\mu\) concentrated at \((s_0, t_0)\).

**References**