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SOME PROPERTIES OF $C(X)$, I

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By a result of A.V. Arhangel'skiĭ and E.G. Pytkeiev, the space $C(X)$ of the continuous real functions on X with the topology of pointwise convergence has tightness ω iff X^n is Lindelöf for every $n\in\omega$. In this paper we describe other convergence properties of $C(X)$ (e.g. the Fréchet–Urysohn property) in terms of covering properties of X.

In some cases the equivalences between these properties turn out to be dependent on the set theory we choose. Some open problems are also stated.

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space of continuous functions	topology of pointwise convergence
Fréchet–Urysohn property	topological games

1. The neighbourhood-point game

In this paper by a space we shall always mean a Tychonoff space.

Definition (G. Gruenhage [4]). Let E be a topological space, $q\in E$. The *neighbourhood-point game* $G_{np}(q,E)$ is defined as follows. It is played by two players, I and II. In the nth step ($n\in\omega$) I chooses a neighbourhood U_n of q and II selects a point $q_n\in U_n$. I wins if the sequence $\langle q_n : n\in\omega\rangle$ converges to q, otherwise II wins.

Definition. Let E be a topological space, $q\in E$.

E is strictly Fréchet at q if $A_n\subset E$, $q\in \bar{A}_n$ ($n\in w$) implies the existence of a sequence $q_n\in A_n$ with $\lim q_n = q$. *E is strictly Fréchet* if it is strictly Fréchet at each point.

E is *Fréchet at q* if $A\subset E$, $q\in\bar{A}$ implies $\lim q_n = q$ for a suitable sequence $\langle q_n\rangle$ with $q_n\in A$. E is *Fréchet* if it is Fréchet at each point.

E is *sequential* if for any non-closed set $A\subset E$ there is a sequence $\langle q_n\rangle$ with $q_n\in A$, $\lim q_n = q$ and $q\notin A$.

Finally, the *tightness* of E is ω (denoted by $t(E)=\omega$) if $q\in E$, $A\subset E$, $q\in\bar{A}$ implies the existence of a set $M\in[A]^{\le\omega}$ with $q\in\bar{M}$.

By a ‘convergence property’ of a topological space E we shall mean one of the following properties:

(i) E is first-countable.

(ii) For any $q \in E$, $I \uparrow G_{np}(q, E)$ (I has a winning strategy in $G_{np}(q, E)$, i.e. E is a W-space in the sense of [4]).

(iii) E is strictly Fréchet.

(iv) E is Fréchet.

(v) E is sequential.

(vi) $t(E) = \omega$.

It is very easy to see that each property implies the next one. Only (v)$\Rightarrow$(vi) is not quite trivial; for its proof see [2, p. 87].

We prove now by examples that none of these implications is reversible.

(ii)$\not\Rightarrow$(i). Take the one-point compactification of an uncountable discrete space [4, p. 341].

(vi)$\not\Rightarrow$(v). Let $\mathbb{N}$ denote a countable discrete space, $\beta\mathbb{N}$ its Stone–Čech compactification, $p \in \beta\mathbb{N} - \mathbb{N}$. If E is the subspace $\mathbb{N} \cup \{p\}$, then E is a suitable example [2, p. 229].

Note that no compact Hausdorff space of this kind is known [1].

(v)$\not\Rightarrow$(iv). A compact Hausdorff example is given in [2, 3.6.I].

(iv)$\not\Rightarrow$(iii). Example 1.4.17 in [2] is a suitable space. We now give a compact Hausdorff counter-example. Let X be the ‘two arrows space“ [2, 3.10.C]. Let E be the quotient of $X \times X$ defined by the equivalence relation, the only non-trivial element of which is the diagonal Δ.

E is Fréchet [2, p. 134]; we prove it is not strictly Fréchet.

Let δ denote the image of Δ by the quotient mapping and choose an enumeration $\langle r_n : n \in \omega \rangle$ of the rationals in the interval $(0, 1)$. For $n \in \omega$ put

$$A_n = \{\langle (a, 0), (b, 0) \rangle : 0 < a, b < 1,\ r_n - 2^{-n} < a < b < r_n\}.$$

Evidently $A_n \subset E$, $\delta \in \bar{A}_n$ $(n \in \omega)$. If $p_n = \langle (a_n, 0), (b_n, 0) \rangle \in A_n$ for $n \in \omega$, then it is easy to find a subsequence $\langle n_k : k \in \omega \rangle$ with

$$a_{n_k} < a_{n_{k+1}} < b_{n_{k+1}} < b_{n_k}, \qquad \lim_k a_{n_k} = \lim_k b_{n_k} = x.$$

However, this means that $\lim_k p_{n_k} = ((x, 0), (x, 1)) \neq \delta$ so $\lim p_n = \delta$ does not hold.

Before we proceed to the example for (iii)$\not\Rightarrow$(ii) we mention a result of P.L. Sharma [8].

Theorem. *II $\uparrow G_{np}(q, E)$ iff there are subsets $A_n \subset E$, $q \in \bar{A}_n$ $(n \in \omega)$ such that for any sequence $q_n \in A_n$, $\lim q_n = q$ does not hold. (II $\uparrow G_{np}(q, E)$ means that II has a winning strategy in G_{np}, i.e. q is not a w-point).*

Hence, to get an example for (iii)$\not\Rightarrow$(ii) we have to produce an undecided game $G_{np}(q, E)$. We present here an unpublished result of A. Hajnal and I. Juhász (1977).

Example. Let E be the one-point compactification of an Aronszajn-tree with the tree-topology (see [7] for the necessary notions and notations) and denote by q the compactifying point.

It is folklore that E is Fréchet. (*Hint*: a tree either contains infinitely many pairwise incomparable elements or can be covered with finitely many branches). Using now that any countable subspace of E is first-countable we get that E is strictly Fréchet. On the other hand player I has no WS in $G_{np}(q, E)$, either. Assume that S is a strategy of I. We may assume without loss of generality that each move of I has the form $U(F)$ where F is a finite subset of $E-\{q\}$ and $U(F)=\{x \in E: x \lhd y$ does not hold for any $y \in F\}$. Using now that each level of an Aronszajn-tree is countable we get a limit ordinal $\alpha < \omega_1$ such that if player II picks points always below the αth level, then the finite sets F determining the responses $U(F)$ of player I according to the strategy S are also below the αth level.

If now II selects any point x from the αth level and in any step he chooses a $q_n \lhd x$, then $\lim q_n = q$ does not hold.

2. The point-open game

Definition (F. Galvin [3], R. Telgársky [10]). Let x be a topological space. The *point-open game* $G_{po}(X)$ is defined as follows. It is played by two players, I and II. In the nth step ($n \in \omega$) I chooses a finite subset F_n of X and II selects an open set G_n in X, $F_n \subset G_n$. I wins if $\bigcup\{G_n: n \in \omega\} = X$, otherwise II wins.

Definition. A family of subsets $\mathscr{A}$ of a set X is said to be an *ω-cover of X* if for any finite subset F of X there is an $A \in \mathscr{A}$ with $F \subset A$.

Definition. If $\langle A_n: n \in \omega \rangle$ is a sequence of subsets of a set X,

$$\underline{\mathrm{Lim}}\, A_n = \{x \in X: \exists n_0 \in \omega\ \forall n \geq n_0\ x \in A_n\}$$

If $\mathscr{A}$ is a family of subsets of a set X, then $L(\mathscr{A})$ denotes the smallest family of subsets of X containing $\mathscr{A}$ and closed under $\underline{\mathrm{Lim}}$.

Consider now the following list of properties of a topological space X.

(α) X is countable.

(β) $\mathrm{I} \uparrow G_{po}(X)$.

(γ) If $\mathscr{G}$ is an open ω-cover of X, then there is a sequence $G_n \in \mathscr{G}$ with $\underline{\mathrm{Lim}}\, G_n = X$.

(δ) If $\mathscr{G}$ is an open ω-cover of X, then $X \in L(\mathscr{G})$.

(ε) Any open ω-cover of X contains a countable ω-subcover.

We prove now that any of these properties implies the next one. Here $(\alpha) \Rightarrow (\beta)$ and $(\gamma) \Rightarrow (\delta)$ are trivial.

$(\delta)\Rightarrow(\varepsilon)$ Let $\mathcal{G}$ be an open ω-cover of X and let $\mathcal{A}$ denote the family of those $A\subset X$ for which there is a countable $\mathcal{G}_0\subset\mathcal{G}$ such that $\mathcal{G}_0(\cap)\{A\}$ $(=\{G\cap A: G\in\mathcal{G}_0\})$ is an ω-cover of the subspace A. It is easily seen that $\mathcal{G}\subset\mathcal{A}$ and $\mathcal{A}=L(\mathcal{A})$, hence $X\in L(\mathcal{G})\subset L(\mathcal{A})=\mathcal{A}$.

For the proof of $(\beta)\Rightarrow(\gamma)$ we need a modification of the point-open game, the *strict point-open game* $G^s_{po}(X)$. Its rules are the same as those of the original game, i.e. in the nth step I chooses a finite set $F_n\subset X$ and II an open set G_n, $F_n\subset G_n\subset X$, but I wins if $\underline{\mathrm{Lim}}\, G_n=X$.

Theorem 1. *I* $\uparrow G_{po}(X)$ *iff* $I\uparrow G^s_{po}(X)$.

Proof. To prove the non-trivial part, assume S is a WS of I in $G_{po}(X)$. We shall say that a sequence $\langle\langle F_i, G_i\rangle: i<\omega\rangle$ is compatible with S if F_i is finite, G_i is open, $F_i\subset G_i$ for $i<\omega$ and for any $k<\omega$, $F_k\supset S(\langle G_i: i<k\rangle)$. Evidently if $\langle\langle F_i, G_i\rangle: i<\omega\rangle$ is compatible with S, then $\langle\langle F_i, G_i\rangle: i<\omega\rangle$ is a win for I. We now give a WS for I in $G^s_{po}(X)$. Assume it is I's turn after the moves $\langle\langle F_i, G_i\rangle: i<n\rangle$. Choose a subsequence $\langle i_j: j\le k\rangle$ with $0\le i_0<\cdots<i_k<n$ and put $F(i_0,\cdots,i_k)=S(\langle G_{i_j}: j\le k\rangle)$. Finally, let F_n be the union of all such finite sets. It is easily seen that if $\langle\langle F_i, G_i\rangle: i<\omega\rangle$ is a game we get, by using this strategy and $0\le i_0<i_1<\cdots<i_n<\cdots$ is any infinite subsequence, then also the game $\langle\langle F_{i_k}, G_{i_k}\rangle: k<\omega\rangle$ is compatible with S and hence a win for I; consequently $\bigcup\{G_{i_k}: k<\omega\}=X$. This means just that $\underline{\mathrm{Lim}}\, G_n=X$.

Problem. Are the games G_{po} and G^s_{po} equivalent also for player II?[1]

We are now ready to prove the implication $(\beta)\Rightarrow(\gamma)$. Indeed, if $\mathcal{G}$ is an open ω-cover witnessing that (γ) does not hold, then II has a WS in $G^s_{po}(X)$; in the nth step he simply chooses a $G_n\in\mathcal{G}$ with $F_n\subset G_n$.

Note that exactly as in the proof of Sharma's theorem in Section 1, it can be shown that II has a WS in $G^s_{po}(X)$ iff (γ) does not hold.

We now formulate the main theorem of the paper.

Theorem 2. *Let X be a Tychonoff-space. If $E=C(X)$, then the following implications are valid.*

$$\begin{array}{lccccccccccc}
E=C(X) & \text{(i)} & \Rightarrow & \text{(ii)} & \Rightarrow & \text{(iii)} & \Leftrightarrow & \text{(iv)} & \Rightarrow & \text{(v)} & \Rightarrow & \text{(vi)} \\
 & \Updownarrow & & \Updownarrow & & \searrow & & \nearrow & & \Downarrow & & \Updownarrow \\
X & (\alpha) & \Rightarrow & (\beta) & \Rightarrow & & (\gamma) & & \Rightarrow & (\delta) & \Rightarrow & (\varepsilon)
\end{array}$$

Proof. (i)$\Leftrightarrow(\alpha)$ $\mathbb{R}^X$ is regular and $C(X)$ is dense in it (cf. [2, 2.1.C]).

(ii)$\Leftrightarrow(\beta)$ Assume first that player I has a WS for $G^s_{op}(X)$; we describe a WS of I in the game $G_{np}(\mathbf{0}, C(X))$, where $\mathbf{0}$ denotes the identically zero function on X.

[1] F. Galvin has shown that in some models of ZFC there is a subspace X of the reals such that II has a winning strategy in $G^s_{po}(X)$ but not in $G_{po}(X)$.

(It is enough to prove only this because $C(X)$ is a topological group.) Let S be a fixed winning strategy of I for $G^s_{po}(X)$. Now the strategy of player I is the following: He mentally plays also another game on the 'board' X according to S. After a move of player II on the genuine board $C(X)$ he 'translates' this move to a move of II on the board X, responds to it according to S and then translates his own response to a move on the board $C(X)$.

Now, if the winning move of player I in the game on the board X is F_n, then let his move on the board $C(X)$ be

$$U_n = U(F_n, 2^{-n}) = \{f \in C(X): |f(x)| < 2^{-n} \text{ for any } x \in F_n\}.$$

If II's response is $f_n \in U_n$, then let the move of the imaginary player II on the board X be

$$G_n = \{x \in X: |f_n(x)| < 2^{-n}\}.$$

As $f_n \in U(F_n, 2^{-n})$, G_n is open and $F_n \subset G_n$, hence this is a correct move.

Using that S is a winning strategy of I in the game $G^s_{po}(X)$, $\underline{\mathrm{Lim}}\ G_n = X$; hence for any $x \in X$ there is an $n_0 \in \omega$ such that for $n \geq n_0$, $x \in G_n$. Consequently $|f_n(x)| < 2^{-n}$ for $n \geq n_0$ and so the sequence $\langle f_n : n \in \omega \rangle$ converges to $\mathbf{0}$.

The proof of the other half of the proposition is similar. If S is a WS for I on the board $C(X)$ (in the game $G_{np}(\mathbf{0}, C(X))$), our scheme for the translations is as follows. If the winning move of I on the board $C(X)$ is $U(F_n, \varepsilon)$, then his move on the board X is to be F_n. If II's response is the open set G_n, $F_n \subset G_n$, choose any $f_n \in C(X)$ with $f_n | F_n \equiv 0$, $f_n | (X - G_n) \equiv 1$. As X is Tychonoff, there exists such a function f_n; interpret it as II's response on the board $C(X)$. This is a correct move because $f_n \in U(F_n, \varepsilon)$.

Using now that S is a winning strategy for I on the board $C(X)$, $f_n \to \mathbf{0}$. Consequently for any $x \in X$ there is an $n_0 \in \omega$ such that $f_n(x) < 1$ for $n \geq n_0$ but then $x \in G_n$ for $n \geq n_0$, i.e. $\underline{\mathrm{Lim}}\ G_n = X$.

To prove (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (γ) we prove (iii) $\Rightarrow$ (iv) $\Rightarrow$ (γ) $\Rightarrow$ (iii).

(iv) $\Rightarrow$ (γ). Let $\mathscr{G}$ be an open ω-cover of X and put

$$\Phi = \{f \in C(X): \exists G \in \mathscr{G}\ \{x \in X: |f(x)| < 1\} \subset G\}.$$

Note that $\mathbf{0} \in \bar{\Phi}$ (the closure taken in $C(X)$). Indeed, if $U(F, \varepsilon)$ is a basic neighbourhood of $\mathbf{0}$ in $C(X)$, choose a $G \in \mathscr{G}$ with $F \subset G$ and an $f \in C(X)$, $0 \leq f \leq 1$ with $f|F \equiv 0$, $f|(X - G) \equiv 1$. Evidently then $f \in U(F, \varepsilon) \cap \Phi$. Now, as $C(X)$ is assumed to be Fréchet, there is a sequence $f_n \in \Phi$ with $f_n \to \mathbf{0}$. Choose a set $G_n \in \mathscr{G}$ with $\{x \in X: |f_n(x)| < 1\} \subset G_n$; then $\underline{\mathrm{Lim}}\ G_n = X$.

The proof of $(\gamma) \Rightarrow$ (iii) will be carried out via a new property for X.

(γ') If $\langle \mathscr{G}_n : n \in \omega \rangle$ is a sequence of open ω-covers of X, then there is a sequence $G_n \in \mathscr{G}_n$ with $\underline{\mathrm{Lim}}\ G_n = X$.

$(\gamma) \Rightarrow (\gamma')$. Let $\langle \mathscr{G}_n : n \in \omega \rangle$ be a sequence of open ω-covers of X. As we can suppose that $\mathscr{G}_{n+1}$ is a refinement of $\mathscr{G}_n$ for each $n \in \omega$, it is enough to prove that there is an infinite subsequence $\langle n_k : k \in \omega \rangle$ and a sequence $G_k \in \mathscr{G}_{n_k}$ with $\underline{\mathrm{Lim}}\ G_k = X$.

If X is finite, then this is certainly true. Choose now a sequence $(x_n : n \in \omega)$, $x_n \in X$, $x_n \neq x_m$ if $n \neq m$ and put

$$\mathcal{U}_n = \{G - \{x_n\} : G \in \mathcal{G}_n\}, \qquad \mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}.$$

Evidently $\mathcal{U}$ is an open ω-cover of X hence there is a sequence $U_k \in \mathcal{U}$, $\underline{\text{Lim}}\, U_k = X$. For any $k \in \omega$ there is an $n_k \in \omega$ and a set G_k with $U_k \subset G_k \in \mathcal{G}_{n_k}$. Now if $n \in \omega$ and $\{x_i : i \leq n\} \subset U_k$, then $n_k > n$ so $\{n_k : k \in \omega\}$ is infinite.

(γ') $\Rightarrow$ (iii). Let $\Phi_n \subset C(X)$, $\mathbf{0} \in \bar{\Phi}_n$ $(n \in \omega)$. Put

$$\mathcal{G}_n = \{\{x \in X : |f(x)| < 2^{-n}\} : f \in \Phi_n\} \quad (n \in \omega).$$

As $\mathbf{0} \in \bar{\Phi}_n$, $\Phi_n \subset C(X)$, $\mathcal{G}_n$ is an open ω-cover of X for any $n \in \omega$. Choose a $G_n \in \mathcal{G}_n$ with $\underline{\text{Lim}}\, G_n = X$. If $G_n = \{x \in X : |f_n(x)| < 2^{-n}\}$, where $f_n \in \Phi_n$, then $f_n \to \mathbf{0}$.

(v) $\Rightarrow$ (δ). Assume $C(X)$ is sequential and let $\mathcal{G}$ be an open ω-cover of X. Put

$$\Phi = \{f \in C(X) : \exists L \in L(\mathcal{G})\, \{x \in X : |f(x)| < 1\} \subset L\}.$$

Using that $\mathcal{G}$ is an open ω-cover of X and $\mathcal{G} \subset L(\mathcal{G})$, we get that $\mathbf{0} \in \bar{\Phi}$. Moreover, Φ is sequentially closed because if $f_n \in \Phi$ and $f_n \to f \in C(X)$, choose a set $L_n \in L(\mathcal{G})$ with $\{x \in X : |f_n(x)| < 1\} \subset L_n$.

If $L = \underline{\text{Lim}}\, L_n$, then $L \in L(\mathcal{G})$ and $\{x \in X : |f(x)| < 1\} \subset L$. Consequently $\mathbf{0} \in \Phi$ so $X \in L(\mathcal{G})$.

Problem. Is (δ) $\Rightarrow$ (v) true?

We shall show in Section 3 that in a suitable model of ZFC the answer is yes.

For the proof of (vi) $\Leftrightarrow$ (ε) I remark that Arhangel'skiĭ and Pytkeiev proved [1, Theorem 4.1.2] that $t(C(X)) = \omega$ iff X^n is Lindelöf for each $n \in \omega$. Consequently the equivalence follows from the following Proposition.

Proposition. *X^n is Lindelöf for each $n \in \omega$ iff X satisfies* (ε).

Proof. If X^n is Lindelöf for each $n \in \omega$ and $\mathcal{G}$ is an open ω-cover of X, it is easily seen that

$$\mathcal{G}^n = \{G^n : G \in \mathcal{G}\}$$

is an open cover of X^n for $n \in \omega$. If $\mathcal{G}_n \subset \mathcal{G}$ is countable and $\mathcal{G}_n^n$ covers X^n for each n, then $\mathcal{G}_\omega = \bigcup\{\mathcal{G}_n : n \in \omega\}$ is a countable ω-subcover of $\mathcal{G}$. Conversely, if X satisfies (ε) and $\mathcal{U}$ is an open cover of X^n, put

$$\mathcal{G} = \{G \subset X : G \text{ is open in } X,\ G^n \text{ can be covered with finitely many sets of } \mathcal{U}\}.$$

It is immediate that $\mathcal{G}$ is an open ω-cover of X and if $\mathcal{G}_0 \subset \mathcal{G}$ is a countable ω-subcover, then $\mathcal{G}_0^n$ is a cover of X^n and the assertion follows.

In the sequel we study the relations between the properties (α)–(ε).

(β) ⇏ (α) Take the one-point compactification of an uncountable discrete space.

(γ) ⇏ (β) For a subspace of the reals both (γ) ⇔ (β) and (γ) ⇏ (β) are consistent (see models 1 resp. 2 or 3 at the end of the paper).

If we do not restrict ourselves to the subspaces of the reals there is an example in ZFC for (γ) ⇏ (β). Indeed, recently E. van Douwen and R. Telgársky gave an example for a P-space in which the point-open game is undecided [11]. Such a space necessarily satisfies (γ) because of the following lemma.

Lemma (F. Galvin). *If X is a Lindelöf P-space, then X satisfies* (γ).

Proof. We have shown in the previous Proposition that a space Y satisfies (ε) iff Y^n is Lindelöf for $n \in \omega$. As the product of finitely many Lindelöf P-space is again Lindelöf, X satisfies (ε). Let now $\mathscr{G}$ be an open ω-cover of X. We can assume that $\mathscr{G}$ is countable. Put for $x \in X$

$$U_x = \bigcap\{G \in \mathscr{G} : x \in G\}.$$

$\{U_x ; x \in X\}$ is an open cover of the Lindelöf P-space X. Choose a countable subcover $\{U_{x_n} : n \in \omega\}$ and let $G_n \in \mathscr{G}$ contain $\{x_1, \ldots, x_n\}$.

(δ) ⇏ (γ) This problem will be discussed in Section 3.

(ε) ⇏ (δ) Simple example is the closed interval [0, 1]; see Lemma 1 at the beginning of Section 3 and the Proposition.

3. Properties (β), (γ) and (δ)

In this section we study properties (β), (γ) and (δ).

Theorem 3. *Any of the properties* (α)–(ε) *are hereditary to closed subspaces and continuous images.*

Proof. Routine.

A certain converse holds for (γ) and (δ). We begin with a lemma.

Lemma 1. *The interval $I = [0, 1]$ does not satisfy* (δ).

Proof. Let $\mathscr{G}$ denote the family of open sets of I having Lebesgue-measure $\leq \frac{1}{2}$. Then $L(\mathscr{G}) \subset \mathscr{L}$, where $\mathscr{L}$ is the family of measurable subsets of I having Lebesgue-measure $\leq \frac{1}{2}$, because $\mathscr{G} \subset \mathscr{L}$ and $\mathscr{L}$ is closed under Lim.

As $\mathscr{G}$ is an open ω-cover of I and $I \notin L(\mathscr{G})$, I does not satisfy (δ).

Corollary. *If X satisfies* (δ), *then X is zero-dimensional.*

Proof. If $\operatorname{ind} X \neq 0$, there is a point $x \in X$ and a neighbourhood U of x in X such that there is no clopen set V with $x \in V \subset U$. Choose a continuous real function f on X with $0 \leq f \leq 1$, $f(x) = 0$, $f|(X - U) \equiv 1$. Now $f''X = [0, 1]$ because if $0 < \varepsilon < 1$, then $f^{-1}([0, \varepsilon)) \subset U$ is not closed and there is therefore a $y \in X$ with $f(y) = \varepsilon$.

Theorem 4. *Let X be Čech-complete. Then we have three possibilities.*

(a) *If X is not Lindelöf, then $t(C(X)) > \omega$.*

(b) *If X is Lindelöf and not scattered, then $t(C(X)) = \omega$ and $C(X)$ is not sequential.*

(c) *If X is Lindelöf and scattered, then $C(X)$ satisfies* (ii).

Proof. (c) If X is Lindelöf and scattered, then by a result of R. Telgársky [10] X satisfies (β).

(b) As the product of countably many Lindelöf Čech-complete spaces is again Lindelöf [2, 3.9.F], X satisfies (ε). On the other hand, it is easy to see that a non-scattered Čech-complete space contains a compact subspace which can be continuously mapped onto the Cantor-set, hence onto the closed interval I so, by Theorem 3 and Lemma 1, X does not satisfy (δ).

Corollary. *Let X be a compact T_2-space. $C(X)$ is Fréchet iff X is scattered.*

Theorem 5. *The space X satisfies (γ) (resp. (δ)) iff X satisfies (ε), and each of its continuous images on the real line satisfies (γ) (resp. (δ)).*

Proof. The necessity is obvious. Assume now that X satisfies (ε) but does not satisfy (γ). Let $\mathscr{G}$ be an open ω-cover of X witnessing that X does not satisfy (γ). Using that X can be assumed to be zero-dimensional (see the argument of the corollary to Lemma 1) and satisfies (ε) we can suppose that $\mathscr{G}$ is countable and consists of clopen sets. The members of $\mathscr{G}$ and their complements define a coarser zero-dimensional topology on X; it has also a countable base. In general it is not a T_0-space but identifying the points with identical closures [2, 2.4.A] we get a continuous mapping $f: X \to M$ where M is a zero-dimensional separable metrizable space and hence homeomorphic to a subset of the real line. It is immediate that $f(X) = M$ does not satisfy (γ). The proof of the case for (δ) is perfectly analogous.

We shall now prove that (δ) is a very strict restriction for a subset of the real line; indeed (δ) implies property C″:

A space X satisfies C″, [6], if for any sequence $\langle \mathscr{G}_n : n \in \omega \rangle$ of open covers of X there is a sequence $G_n \in \mathscr{G}_n$ with $\bigcup \{G_n : n \in \omega\} = X$. Let $\phi = \langle \mathscr{G}_n : n \in \omega \rangle$ be a sequence of open covers of the space X. A set $A \subset X$ is said to be ϕ-small if for any $n \in \omega$ there are a $k \in \omega$ and sets $G_i \in \mathscr{G}_{n+i}$ ($i < k$) with $A \subset \bigcup \{G_i : i < k\}$.

Let now (*) be the following property:

(*) If $\phi = \langle \mathscr{G}_n : n \in \omega \rangle$ is a sequence of open covers of X, then X is the union of countably many ϕ-small sets.

Theorem 6. *The property* (δ) *implies* $(*)$.

Proof. Assume X satisfies (δ) and $\phi = \langle \mathscr{G}_n : n \in \omega \rangle$ is a sequence of open covers of X. Using now that X must be a Lindelöf-space we can assume that $\mathscr{G}_n$ is locally finite for any $n \in \omega$. For $n \in \omega$ put now

$$\mathscr{H}_n = \{\bigcup\{G_i : i < 2n+1\} : G_i \in \mathscr{G}_{n^2+i}\}, \qquad \mathscr{H} = \bigcup\{\mathscr{H}_n : n \in \omega\}$$

$\mathscr{H}$ is then an open ω-cover of X. Put

$$\mathscr{A} = \{A \subset X : \exists H \in \mathscr{H}\, A \subset H\},$$

$$\mathscr{B} = \{\bigcup\{S_n : n \in \omega\} : S_n \subset X \text{ is } \phi\text{-small}\}.$$

Evidently $\mathscr{H} \subset \mathscr{A} \cup \mathscr{B}$; we assert that $\mathscr{A} \cup \mathscr{B}$ in closed under $\underline{\mathrm{Lim}}$, hence $L(\mathscr{H}) \subset \mathscr{A} \cup \mathscr{B}$. Indeed, let $T_n \in \mathscr{A} \cup \mathscr{B}$ $(n \in \omega)$, $T = \underline{\mathrm{Lim}}\, T_n$. If for infinitely many n's $T_n \in \mathscr{B}$, then T is contained in the union of these T_n's, hence $T \in \mathscr{B}$. So we can assume that $T_n \in \mathscr{A}$ for each $n \in \omega$. Consequently $T_n \subset H_n \in \mathscr{H}$ for a suitable H_n. For each $n \in \omega$ there is a $k(n) \in \omega$ with $H_n \in \mathscr{H}_{k(n)}$. If now the set $\{k(n) : n \in \omega\}$ is infinite, then T is, evidently, the union of countably many ϕ-small sets. Otherwise for infinitely many indices n $k(n) = k$ is fixed, hence $T \subset \underline{\mathrm{Lim}}\, K_n$, $K_n \in \mathscr{H}_k$ $(n \in \omega)$. Using now that any of the systems $\mathscr{G}_{k^2+i}$ $(i \leq 2k)$ is point-finite, it is not difficult to see that T can be covered with a member of $\mathscr{H}_k$.

As X satisfies (δ), $X \in L(\mathscr{H}) \subset \mathscr{A} \cup \mathscr{B}$. If $X \notin \mathscr{B}$, then $X \in \mathscr{A}$, hence a suitable member H of $\mathscr{H}$ covers X; let $H \in \mathscr{H}_n$. Drop out $\mathscr{H}_n$; repeat the above argument for $\mathscr{H}' = \bigcup\{\mathscr{H}_k : n < k < \omega\}$. Then again a suitable member H' of $\mathscr{H}'$ covers X; let $H' \in \mathscr{H}_{n'}$. Put $\mathscr{H}'' = \bigcup\{\mathscr{H}_k : n' < k < \omega\}$. Etc.

We get in this manner that X is indeed ϕ-small.

Corollary. *Property* (δ) *implies property* C″.

Proof. Let $\phi = \langle \mathscr{G}_n : n \in \omega \rangle$, $X = \bigcup\{S_n : n \in \omega\}$, let S_n be ϕ-small. Choose an $n_0 \in \omega$, $G_i \in \mathscr{G}_i$, $i < n_0$ with $S_0 \subset \bigcup\{G_i : i < n_0\}$. Then choose an $n_1 \in \omega$ and for any i with $n_0 \leq i < n_1$ a set $G_i \in \mathscr{G}_i$ with $S_1 \subset \bigcup\{G_i : n_0 \leq i < n_1\}$ etc.

We get thus a sequence $\{n_k : k < \omega\}$ and sets $G_i \in \mathscr{G}_i$ with $S_k \subset \bigcup\{G_i : n_k \leq i < n_{k+1}\}$. Now $X = \bigcup\{G_i : i < \omega\}$.

Note that property $(*)$ is strictly stronger than C″ even on the real line. Indeed, a standard example for an uncountable linear set satisfying C″ is a Lusin-set [9]. However, a Lusin-set does not have property $(*)$.

Definition [9]. Let $X \subset \mathbb{R}$; X is said to be *always first category* if for any perfect set $P \subset \mathbb{R}$, $P \cap X$ is first category in P.

Evidently a Lusin-set is *not* always first category. However, if $X \subset \mathbb{R}$ satisfies $(*)$, then it is always first category. Choose a perfect subset $P \subset \mathbb{R}$ and let μ be a continuous Borel measure on P such that for G open in P, $\mu(G) \neq 0$. Put now for $n \in \omega$

$$\mathcal{G}_n = \{G \subset X : G \text{ is open in } X, \mu(\overline{G \cap P}) < 2^{-n}\}$$

(closure in P). Evidently $\phi = \langle \mathcal{G}_n : n \in \omega \rangle$ is a sequence of open covers of X. If $A \subset X$ is ϕ-small, then A is nowhere dense in P, hence $X \cap P$ is first category in P.

In view of Theorem 5 it would be very important to know if there are non-trivial (i.e. uncountable) subspaces of the reals with property (δ) (or (γ)).

The answer depends on the set theory we choose.

Model 1. R. Laver constructed a model of ZFC [6] in which every subset of the reals satisfying C″ is countable. In this model, by Theorem 5, (δ) implies (γ). Hence

Theorem 7. *It is consistent with* ZFC *to assume that for any space X, $C(X)$ is sequential iff it is Fréchet.*

Problem. Is $(\delta) \Rightarrow (\gamma)$ true (in ZFC)? Is there a model of ZFC in which $(\delta) \Rightarrow (\gamma)$ does not hold?

Model 2. Assume MA $+ 2^{\omega} > \omega_1$ and take a subspace X of size ω_1 of the reals. Then X does not satisfy (β) (by a result of R. Telgársky, a metrizable space satisfies (β) iff it is countable) but it has the property (γ). Indeed, let $\mathcal{G}$ be a countable open ω-cover of X. We construct now a partially ordered set P. Its elements are pairs $p = \langle F, \phi \rangle$ where $F \in [X]^{<\omega}$ and ϕ is a function from a finite subset of ω into $\mathcal{G}$. If $p = \langle F, \phi \rangle$, $p' = \langle F', \phi' \rangle$ are members of P we put: $p' < p$ iff $F \subset F'$, $\phi \subset \phi'$ and for any $n \in \operatorname{Dom} \phi' - \operatorname{Dom} \phi$, $F \subset \phi'(n)$ holds.

It is very easy to check that P is indeed a partially ordered set, it is ccc and $|P| = |X| = \omega_1 < 2^{\omega}$. Moreover, if the dense sets we take into account are

$$D_x = \{\langle F, \phi \rangle \in P : x \in F\} \quad (x \in X),$$

$$D'_n = \{\langle F, \phi \rangle \in P : n \in \operatorname{Dom} \phi\} \quad (n \in \omega),$$

then a generic set over P gives rise to a sequence $G_n \in \mathcal{G}$ with $\underline{\operatorname{Lim}}\, G_n = X$.

Model 3. Assuming ZFC + CH a construction was given by F. Galvin of an uncountable subspace of the reals which satisfies property (γ).

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