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Hydrodynamic limit for a nongradient system in infinite volume

Anne Perrut

Université de Rouen, Site Colbert – UFR des Sciences, LAMS, UPRES-A 6085, F-76821 Mont-Saint-Aignan cedex, France

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Abstract

The hydrodynamic limit of the symmetric generalized exclusion process on the torus $[0,1)$ has previously been proved to be a nonlinear diffusive equation. We consider in this paper this model in infinite volume. We prove that the H_{-1} norm of the difference between the process and the solution of the hydrodynamic equation goes to zero.

Keywords: Infinite interacting particle systems; Hydrodynamic limits; Nongradient techniques

1. Introduction

The interacting particle systems introduced by Spitzer at the beginning of the 1970s may model the behavior of the molecules of a gas. The hint of the theory of hydrodynamic limits is to bound this microscopic dynamics to the macroscopic evolution of the gas. For a large survey on hydrodynamic limits, one may refer to the books of Spohn (1991), De Masi and Presutti (1989), or Kipnis and Landim (1999). In 1988, Guo et al. (1988) introduced large deviations techniques to obtain rigorously this limiting behavior for a large class of gradient systems, by controlling the entropy production. They proved a law of large numbers for the empirical measure, that is the convergence in probability of the density of particles at time t in a small macroscopic neighborhood to the solution ρ_t of a PDE, namely the hydrodynamic equation.

This had been extended to a nongradient Ginzburg–Landau model by Varadhan (1998). The generalized symmetric exclusion process, which is also nongradient, had been studied in Kipnis et al. (1994, 1995), where the hydrodynamic limits had been obtained, first on the finite torus and then in a finite box with reservoirs. Our purpose is to extend the result presented in Kipnis et al. (1994) in a stronger version in infinite volume: We would like to prove, by rescaling time and space, that the macroscopic behavior of this model is governed by a nonlinear diffusive PDE. For this, we study

E-mail address: Anne.Perrut@univ-rouen.fr (A. Perrut)

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the H_{-1} norm of the difference of the process and the solution of this PDE, under regularity conditions on the diffusion coefficient. A proof based on H_{-1} norm techniques was made by Chang and Yau (1992) for the Ginzburg–Landau model, and by Landim and Yau (1995): The authors extend the standard method to prove uniqueness of the solution of a parabolic PDE to the stochastic context. Other papers treat the use of this norm: For instance Yau (1994) and Landim and Vares (1996), where an exponential estimate for the H_{-1} norm of the difference of a reaction-diffusion process and the solution of its hydrodynamic equation in infinite volume is obtained.

The main ingredients in our proof are an estimate of the relative entropy and of the Dirichlet form, the *replacement lemma* and the study of time evolution of the H_{-1} norm. To obtain the estimate of entropy production, we follow the approach of Fritz (1990) and Yau (1994) (one could refer to Landim and Mourragui (1997) too). This is done in Section 2. Then, in Section 3, we have to prove that the so-called replacement lemma holds. In fact, this lemma allows the replacement of the current (which is not a gradient quantity) by a gradient plus a negligible term of the form $LF(\eta)$ where L is the generator of the process. This term $LF(\eta)$ turns out to be irrelevant because its time fluctuations are orthogonal to the ones of the gradient part. For an overview on these techniques, see Kipnis et al. (1994) or Kipnis and Landim (1999). With these results, we actually may study the time evolution of the H_{-1} norm by deriving it. The Gronwall lemma allows us to conclude. In the appendix, the reader will find some tools on the H_{-1} norm used throughout this paper.

We now describe the generalized symmetric simple exclusion process on $\mathbb{Z}$, denoted by $(\eta_t)_{t\geqslant 0}$: The particles jump to a nearest-neighbor site with a random rate, but at most two particles per site are allowed. The space of configurations is $\mathscr{X}=\{0,1,2\}^{\mathbb{Z}}$. The generator of the process is defined, for all cylinder function f, for all $\eta\in\mathscr{X}$, by

$$(L^N f)(\eta) = N^2(Lf)(\eta),$$

$$(Lf)(\eta) = \frac{1}{2}\sum_{\substack{x\in\mathbb{Z}\\ |y|=1}} r_{x,x+y}(\eta)[f(\eta^{x,x+y}) - f(\eta)],$$

where the jump rate is given by $r_{x,x+y}(\eta)=\mathbb{1}_{\{\eta(x)>0,\ \eta(x+y)<2\}}$ for $y=\pm 1$, and $\eta^{x,y}$ is the configuration obtained from η by letting one particle jump from x to y:

$$\eta^{x,y}(z) = \begin{cases} \eta(x)-1 & \text{if } z=x,\\ \eta(y)+1 & \text{if } z=y,\\ \eta(z) & \text{otherwise.} \end{cases}$$

For an integer x, we define the *current* $W_{x,x+1}$ by

$$W_{x,x+1}(\eta) = r_{x,x+1}(\eta) - r_{x+1,x}(\eta).$$

We denote by τ_x the translation operator defined on $\mathscr{X}$ by $\tau_x\eta(y)=\eta(x+y)$ and on the space of functions by $(\tau_x f)(\eta)=f(\tau_x\eta)$. The system is said to be *gradient* if there exists a function h such that $W_{0,1}=\tau_1 h - h$. Here, the system is not gradient: Consider the configuration η with two particles at site 0 and one at site 1. Then $\sum_x \tau_x W_{0,1}(\eta)=1$ which is inconsistent with $\sum_x \tau_x(\tau_1 h - h)(\eta)=0$.

Let $S^N(t)$ be the semi-group associated to the generator and E_η^N be the expectation with respect to the law of the Markov process with generator L^N when it starts from η. The process $(\eta_t)_{t\geqslant 0}$ is self-adjoint with respect to $\bar{\nu}_\varphi$ ($\varphi > 0$), the product measure on $\mathscr{X}$ with marginals

$$\bar{\nu}_\varphi\{\eta\colon \eta(x) = k\} = \frac{\varphi^k}{Z(\varphi)}$$

for $k=0,1,2$, where $Z(\varphi)=1+\varphi+\varphi^2$ is the normalization constant. We will denote by ν_ρ the measure $\bar{\nu}_\varphi$ when $\bar{\nu}_\varphi[\eta(0)] = \rho$ is the mean occupation number per site. Notice that $\rho=(\varphi+2\varphi^2)/(1+\varphi+\varphi^2)$ and $0\leqslant\rho\leqslant 2$. Fix once for all an invariant measure ν_ρ.

Before stating the main result of this paper, we have to define in details the H_{-1} norm on $\mathscr{X}$. As in [1], let $K_N(.,.)$ be the kernel associated to $(I - N^2\Delta)^{-1}(.,.)$ where Δ is the discrete Laplacian:

$$(\Delta f)(x) = f(x+1) + f(x-1) - 2f(x).$$

By its definition, K_N is perfectly adapted to our parabolic case. A calculation based on Fourier transforms and valid only in dimension 1 shows that

$$K_N(x,y) = \frac{1-a}{1+a}\, a^{|x-y|} \tag{1}$$

with a solution of $N=\sqrt{a}/(1-a)$ such that $0 < a < 1$ (see Landim and Vares (1996)). As in Yau (1994), we consider $K_{N,\theta}(.,.)$ the kernel $K_N(.,.)$ multiplied by $\exp(-\theta(.))$:

$$K_{N,\theta}(x,y) = \mathrm{e}^{-\theta(x/N)} K_N(x,y) \mathrm{e}^{-\theta(y/N)}.$$

We define the H_{-1} norm and the H_0 norm of a configuration η by

$$\| \eta \|_{-1} = \frac{1}{N} \sum_{x,\, y\in\mathbb{Z}} \eta(x) K_{N,\theta}(x,y)\eta(y) \quad \text{and} \quad \| \eta \|_0 = \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta(x)^2 \mathrm{e}^{-2\theta(x/N)}.$$

The properties of the kernel K_N are postponed at the end of the paper.

To obtain the hydrodynamic limit, we rescale time and space. We set that the distance between two neighboring sites is $1/N$ and then accelerate the displacements of particles by N^2, because they are symmetric. In the limit as N goes to infinity, the process satisfies the nonlinear parabolic differential equation

$$\partial_t \rho = \partial_{xx} d(\rho) = \partial_x (D(\rho)\partial_x \rho),$$

where $d(\rho)=\int_0^\rho D(\alpha)\,\mathrm{d}\alpha$. Spohn (1991) proved that the diffusion coefficient D is given by the Green–Kubo formula which involves equilibrium expectation of the space–time correlation of the current. Its variational formula is

$$D(\rho) = \frac{1}{2\chi(\rho)} \inf_h \hat{a}(\rho, h),$$

where the infimum is taken on the set of cylinder functions on $\mathscr{X}$, $\langle\,.\,\rangle_\rho$ is the expectation with respect to ν_ρ,

$$\hat{a}(\rho,h) = \tfrac{1}{2}\langle r_{0,1}(\eta)(1 - [\Gamma_h(\eta^{0,1}) - \Gamma_h(\eta)])^2\rangle_\rho$$
$$+ \tfrac{1}{2}\langle r_{1,0}(\eta)(1 - [\Gamma_h(\eta^{1,0}) - \Gamma_h(\eta)])^2\rangle_\rho,$$

$\chi(\rho) = \langle \eta(0)^2 \rangle_\rho - \langle \eta(0) \rangle_\rho^2$ and $\Gamma_h = \sum_{x\in\mathbb{Z}}(\tau_x h)$ (see Kipnis and Landim (1999) for details). From Theorem 8.1 in the book [8], the unique classical solution $\rho(t,x)$ of this equation with initial condition $\rho(0,.) = \rho_0(.)$ is of class C^2 when the diffusion coefficient is continuous, differentiable, its derivative is a continuous Lipschitz function and when the initial condition ρ_0 is of class C^3. It has been shown only in dimension 1 that $D(.)$ is continuous (see Kipnis and Landim (1999)), and we are not able to show more.

We state here the main theorem of this paper.

Theorem 1.1. *Let* (η^N) *be a sequence of initial configurations. Then, under regularity assumptions on* $D(.)$

$$\lim_{N\to\infty} E^N_{\eta^N}[\ \| \eta_t(.) - u_t(./N) \|_{-1}^2\ ] = 0$$

when u_t *is the solution of the equation*

$$\partial_t u = \partial_{xx} d(u) = \partial_x (D(\rho)\partial_x \rho)$$

with the initial condition $u_0(.)$ *being of class* C^3 *and satisfying*

$$\lim_{N\to\infty} \| \eta^N(.) - u_0(./N) \|_{-1}^2 = 0.$$

Remark. The initial distribution had been chosen deterministic, but we can extend the result to any sequences of initial distributions μ^N.

Corollary 1.2. *Let* G *be a continuous function on* $\mathbb{R}$ *with compact support and* μ^N *a sequence of initial measures on* $\mathscr{X}$. *Under the assumptions of Theorem* 1.1, *for all* $\delta > 0$ *and* $t \geqslant 0$,

$$\lim_{N\to\infty} \mu^N S^N(t) \left\{ \eta\colon \left| \frac{1}{N} \sum_{x\in\mathbb{Z}} \eta(x) G(x/N) - \int G(r) u_t(r)\,\mathrm{d}r \right| > \delta \right\} = 0.$$

We shall now try to explain the strategy of the proof. To control the time evolution of $E^N_{\eta^N}[\ \| \eta_t - u_t \|_{-1}^2\ ]$, we derive it. Then we can perform one spatial integration by parts, instead of two for a gradient system. So we obtain an expression involving the currents multiplied by a factor N. To get rid of this factor N, we wish we could perform another summation by parts, which will be possible once the replacement of the current $W_{x,x+1}(\eta)$ by a term proportional to a local average of $\eta(x+1) - \eta(x)$ (which is a gradient) is done. More precisely, we denote by $\mathscr{C}_0$ the space of cylinder functions on $\mathscr{X}$ with mean zero with respect to all canonical measures $\nu_{l,K}$ defined in Section 3. For example, the currents $W_{x,x+1}$ belong to $\mathscr{C}_0$. By providing this space with an inner product $\langle\langle .,. \rangle\rangle_\rho$, we obtain a Hilbert space $\mathscr{H}_\rho$ such that the gradient $\eta(1) - \eta(0)$ is orthogonal to the space $L\mathscr{C}_o$ and these two spaces generate $\mathscr{H}_\rho$. Then there exists a D depending on the density satisfying

$$W_{0,1} + D(\rho)[\eta(1) - \eta(0)] \in \overline{L\mathscr{C}_0},$$

where $\overline{L\mathscr{C}_0}$ stands for the closure of $L\mathscr{C}_0$ in $\mathscr{C}_0$. Let $f \in \mathscr{C}_0$ such that $\langle\langle W_{0,1} + D(\rho)[\eta(1) - \eta(0)] - Lf \rangle\rangle_\rho$ is small. The first proof in Section 3 consists in showing that the term Lf may be introduced in our expression. In fact, the function f depends on the density because the inner product does. Moreover, the rest of this section is devoted to the proof of the replacement of $W_{0,1}$ by $D(\rho)(\eta(1)-\eta(0))+Lf$. Indeed, we first simplify the problem, using the entropy inequality. Then Feynman–Kac formula reduces the problem to a static one. It is the place where we need the estimate on the Dirichlet form proved in Section 2. And we use a series of lemmas to localize all the terms. Afterwards, we project the new expression on the hyperplanes with a fixed total number of particles, and an estimate on small perturbations of the generator in finite volume reduces the problem to the computation of a central limit variance, which achieves the proof of the replacement lemma. Once this done, we can prove Theorem 1.1 (Section 4): We perform a second summation by parts and calculations involving the properties of the kernel K_N allow us to conclude with Gronwall lemma.

2. Entropy estimate

We study here the most useful objects in the proof of the hydrodynamic limits: The entropy and the Dirichlet form of a measure with respect to the reference measure ν_ρ.

For each subset Λ of $\mathbb{Z}$, denote by ν_ρ^Λ the product measure on $\mathbb{N}^\Lambda$ with the same marginals as ν_ρ. We set: $\Lambda_n=\{-n,\dots,n\}$ and $\mathscr{X}_n=\{0,1,2\}^{\Lambda_n}$. We shall denote $\nu_\rho^{\Lambda_n}$ simply by ν_ρ^n. More generally, for a measure μ on $\mathscr{X}$, we denote by μ^n the restriction of μ on $\mathscr{X}_n:\mu^n(\xi)=\mu\{\eta\colon \eta(x)=\xi(x),\ \forall |x|\leqslant n\}$ for all $\xi\in\mathscr{X}_n$. For all $n\geqslant 1$, for all measure λ on $\mathscr{X}_n$, we define the relative entropy of λ with respect to ν_ρ^n by

$$H_n(\lambda)=\sup_{f\in C_b(\mathscr{X}_n)}\left\{\int f\,d\lambda-\operatorname{Log}\int e^f\,d\nu_\rho^n\right\}=\int \operatorname{Log}\frac{d\lambda}{d\nu_\rho^n}\,d\lambda,$$

where $C_b(\mathscr{X}_n)$ is the set of bounded continuous functions on $\mathscr{X}_n$. We also define the Dirichlet form of the measure λ:

$$D_n(\lambda)=-\sum_{x\in\Lambda_n}\int\sqrt{\frac{d\lambda}{d\nu_\rho^n}}\left(L_{x,x+1}\sqrt{\frac{d\lambda}{d\nu_\rho^n}}\right)d\nu_\rho^n,$$

where $L_{x,x+1}$ is the generator of the process restricted to the sites x and $x+1$, that is

$$L_{x,x+1}f(\eta)=\tfrac{1}{2}r_{x,x+1}(\eta)[f(\eta^{x,x+1})-f(\eta)]+\tfrac{1}{2}r_{x+1,x}(\eta)[f(\eta^{x+1,x})-f(\eta)].$$

We are now able to define the entropy of a measure μ on $\mathscr{X}$ with respect to ν_ρ. Let $\theta\colon\mathbb{R}\to\mathbb{R}^+$ be a positive function, three times differentiable, such that: $\theta(a)=|a|$ when $|a|>1$. The entropy of μ with respect to the reference measure ν_ρ is defined by

$$\mathscr{H}(\mu)=\frac{1}{N}\sum_{n\geqslant 1}H_n(\mu^n)\,e^{-\theta(n/N)}$$

and the Dirichlet form is

$$\mathscr{D}(\mu)=\frac{1}{N}\sum_{n\geqslant 1}D_n(\mu^n)\,e^{-\theta(n/N)}.$$

We state now a first result concerning a bound of the entropy and the time integral of the Dirichlet form.

Theorem 2.1. *There exists a positive and finite constant C depending only on the function θ and on the parameterization ρ such that, for all sequence of initial measures μ_N on $\mathscr{X}$,*

$$\mathscr{H}(S^N(t)\mu_N) + \frac{N^2}{2}\int_0^t \mathscr{D}(S^N(s)\mu_N)\,\mathrm{d}s \leqslant CN. \tag{2}$$

This theorem provides a bound for the time integral of the Dirichlet form, that is $\int_0^t \mathscr{D}(S^N(s)\mu_N)\,\mathrm{d}s \leqslant \frac{C}{N}$.

Proof. It follows the one of Fritz (1990) for the Ginzburg–Landau model.

We consider a sequence of initial measures μ_N. For each configuration η, we denote by δ_η the measure on $\mathscr{X}$ which only charges η:

$$H_n(\delta_\eta^n) = \int \mathrm{Log}\frac{\mathrm{d}\delta_\eta^n}{\mathrm{d}\nu_\rho^n}\,\mathrm{d}\delta_\eta^n = \mathrm{Log}\frac{1}{\nu_\rho^n(\eta)} = \sum_{x\in\Lambda_n}\mathrm{Log}\frac{1+\varphi+\varphi^2}{\varphi^{\eta(x)}} \leqslant c(\varphi)n.$$

Moreover, $\mu_N(\eta) = \sum_\eta \mu_N(\eta)\delta_\eta$, therefore, $H_n[\mu_N^n] \leqslant \sum_\eta \mu_N(\eta)H[\delta_\eta^n] \leqslant c(\varphi)n$. We then deduce, for N large enough,

$$\mathscr{H}(\mu_N^n) \leqslant \frac{c(\varphi)}{N}\sum_n n\,\mathrm{e}^{-\theta(n/N)} \leqslant C_0N, \tag{3}$$

where C_0 is a positive constant. (We used $\sum_n n\mathrm{e}^{-n/N} \sim N^2$, when N is large.)

Once this done, we have to consider the process on large finite volumes, i.e. on volumes of length M for some $M = M(N) \gg N$. In this context, we prove a bound for the entropy and for the time integral of the Dirichlet form, uniformly in M. Indeed, we follow the approach of Landim and Mourragui (1997). Up to this point, it will be easy to deduce Theorem 2.1, by letting M go to infinity (see the end of this section).

We fix a positive integer M, large compared to N and consider the restriction of the process on $\mathscr{X}_M = \{0,1,2\}^{\Lambda_M}$. Its generator L_M is given by

$$N^2L_M = N^2\sum_{x,x+1\in\Lambda_M} L_{x,x+1}.$$

More generally, the generator L_n ($1\leqslant n\leqslant M$) will be the restriction of L on Λ_n. The semi-group of the process with generator L_M, accelerated by N^2 will be denoted by $S^{M,N}(t)$. Consider a measure μ on $\mathscr{X}_M$ and let $\mu(t) = S^{M,N}(t)\mu$ be the law of the process at time t, starting from μ. The density of $\mu(t)$ with respect to ν_ρ^M will be denoted by $f(t)$ and the one of $\mu^n(t)$ with respect to ν_ρ^n by $f_n(t)$ for $n\leqslant M$.

Let $R:\mathbb{R}^+ \to \mathbb{R}^+$ be a continuous positive function with support contained in $[0, (M-N)/N]$. The entropy of a measure μ on $\mathscr{X}_M$ and the Dirichlet form are defined by

$$H_{M,R}(\mu) = \frac{1}{N}\sum_{n=1}^{M} H_n(\mu^n)R(n/N),$$

$$D_{M,R}(\mu) = \frac{1}{N}\sum_{n=1}^{M} D_n(\mu^n)R(n/N).$$

Fix an integer $1\leqslant n\leqslant M-N$. For two subsets Ω and Λ of $\mathbb{Z}$, such that $\Omega\subset\Lambda$, for a function g with support included in Λ, $\langle g\rangle_\Omega$ indicates that we integrate g over the coordinates $\{\eta(x),\, x\in\Omega\}$ with respect to ν_ρ^Λ. When $\Omega=\Lambda_{n+1}\backslash\Lambda_n$, we simply denote $\langle g\rangle_\Omega$ by $\langle g\rangle_{n+1}$. Since ν_ρ^{n+1} is a product measure, $f_n(t)=\langle f_{n+1}(t)\rangle_{n+1}$.

To estimate the entropy production, we shall compute the time derivative of the entropy. As in Landim and Mourragui (1997), we have

$$\partial_t f_n(t)=N^2\langle L_{n+1}f_{n+1}(t)\rangle_{n+1}$$

because L_{n+1} is self-adjoint with respect to ν_ρ^{n+1}. As a consequence,

$$\begin{aligned}\partial_t H_n[f_n(t)]&=\partial_t\int f_n(t)\mathrm{Log}\, f_n(t)\,\mathrm{d}\nu_\rho^n\\ &=N^2\int L_{n+1}f_{n+1}(t)\mathrm{Log}\, f_n(t)\,\mathrm{d}\nu_\rho^{n+1}+\int\partial_t f_n(t)\,\mathrm{d}\nu_\rho^n.\end{aligned}$$

But the second term is null. Besides, $L_{n+1}=L_n+L_{n,n+1}+L_{-n-1,-n}$, thus

$$\begin{aligned}\partial_t H_n[f_n(t)]=N^2&\int f_{n+1}(t)L_n\mathrm{Log}\, f_n(t)\,\mathrm{d}\nu_\rho^{n+1}\\ &+N^2\int f_{n+1}(t)L_{n,n+1}\mathrm{Log}\, f_n(t)\,\mathrm{d}\nu_\rho^{n+1}\\ &+N^2\int f_{n+1}(t)L_{-n-1,-n}\mathrm{Log}\, f_n(t)\,\mathrm{d}\nu_\rho^{n+1}.\end{aligned}$$

By a classical computation which can be found in the proofs using this relative entropy method (see Fritz (1990) for example), the first term is bounded above by $-N^2D_n(f(t))$. And we denote the two last terms by I_1 and I_2, respectively. Now, we will simply write f_n instead of $f_n(t)$:

$$\begin{aligned}I_1=&\frac{N^2}{2}\int r_{n,n+1}(\eta)f_{n+1}(\eta)\mathrm{Log}\frac{f_n(\eta^{n,n+1})}{f_n(\eta)}\,\mathrm{d}\nu_\rho^{n+1}(\eta)\\ &+\frac{N^2}{2}\int r_{n+1,n}(\eta)f_{n+1}(\eta)\mathrm{Log}\frac{f_n(\eta^{n+1,n})}{f_n(\eta)}\,\mathrm{d}\nu_\rho^{n+1}(\eta).\end{aligned}$$

We make a change of variables in the second term. The rate $r_{n+1,n}(\eta)$ insures that $\eta(n+1)>0$ and $\eta(n)<2$, so we are allowed to set $\xi=\eta^{n+1,n}$. Moreover, we have: $r_{n+1,n}(\xi^{n,n+1})=r_{n,n+1}(\xi)$. Therefore

$$\begin{aligned}I_1&=\frac{N^2}{2}\int r_{n,n+1}(\eta)[f_{n+1}(\eta)-f_{n+1}(\eta^{n,n+1})]\mathrm{Log}\frac{f_n(\eta^{n,n+1})}{f_n(\eta)}\,\mathrm{d}\nu_\rho^{n+1}(\eta)\\ &=\frac{N^2}{2}\int \mathbb{1}_{\{\eta(n)>0\}}<\mathbb{1}_{\{\eta(n+1)<2\}}[f_{n+1}(\eta)-f_{n+1}(\eta^{n,n+1})]>_{n+1}\\ &\quad\times\mathrm{Log}\frac{f_n(\eta-e_n)}{f_n(\eta)}\,\mathrm{d}\nu_\rho^n(\eta),\end{aligned}$$

where e_n is the configuration with only one particle at site n: $e_n(n)=1$ and $e_n(x)=0$ if $x\neq n$. In this integral, we can add $\mathbb{1}_{A_n}$ and $\mathbb{1}_{B_n}$ the indicator functions of the sets

A_n and B_n defined by

$$A_n = \{\eta\colon \langle \mathbb{1}_{\{\eta(n+1)<2\}} f_{n+1}(\eta)\rangle_{n+1} \geqslant \langle \mathbb{1}_{\{\eta(n+1)<2\}} f_{n+1}(\eta^{n,n+1})\rangle_{n+1}$$
$$\text{and } f_n(\eta - e_n) \geqslant f_n(\eta)\},$$
$$B_n = \{\eta\colon \langle \mathbb{1}_{\{\eta(n+1)<2\}} f_{n+1}(\eta^{n,n+1})\rangle_{n+1} \geqslant \langle \mathbb{1}_{\{\eta(n+1)<2\}} f_{n+1}(\eta)\rangle_{n+1}$$
$$\text{and } f_n(\eta) \geqslant f_n(\eta - e_n)\}$$

since outside of these sets, I_1 is nonpositive. So we rewrite I_1 as

$$\begin{aligned}
I_1 \leqslant{} & \frac{N^2}{2}\int r_{n,n+1}(\eta)\mathbb{1}_{A_n}(\eta)[f_{n+1}(\eta) - f_{n+1}(\eta^{n,n+1})]\mathrm{Log}\frac{f_n(\eta-e_n)}{f_n(\eta)}\,\mathrm{d}\nu_\rho^{n+1}(\eta)\\
&+\frac{N^2}{2}\int r_{n,n+1}(\eta)\mathbb{1}_{B_n}(\eta)[f_{n+1}(\eta) - f_{n+1}(\eta^{n,n+1})]\mathrm{Log}\frac{f_n(\eta-e_n)}{f_n(\eta)}\,\mathrm{d}\nu_\rho^{n+1}(\eta)\\
\leqslant{} & \frac{AN^2}{4}\int r_{n,n+1}(\eta)[\sqrt{f_{n+1}(\eta)} - \sqrt{f_{n+1}(\eta^{n,n+1})}]^2\,\mathrm{d}\nu_\rho^{n+1}(\eta)\\
&+\frac{N^2}{4A}\int r_{n,n+1}(\eta)(\mathbb{1}_{A_n}(\eta)+\mathbb{1}_{B_n}(\eta))^2[\sqrt{f_{n+1}(\eta)} + \sqrt{f_{n+1}(\eta^{n,n+1})}]^2\\
&\times\left(\mathrm{Log}\frac{f_n(\eta^{n,n+1})}{f_n(\eta)}\right)^2\,\mathrm{d}\nu_\rho^{n+1}(\eta).
\end{aligned} \tag{4}$$

To obtain the last inequality, we used

$$ab \leqslant \frac{A}{2}a^2 + \frac{1}{2A}b^2 \quad \text{for all } a,b\in\mathbb{R} \text{ and } A>0. \tag{5}$$

We may remove the square of $(\mathbb{1}_{A_n}(\eta)+\mathbb{1}_{B_n}(\eta))$ because the term containing the product $\mathbb{1}_{A_n}(\eta)\mathbb{1}_{B_n}(\eta)$ vanishes. We choose $A=\varepsilon N$. Since $(\sqrt{a}+\sqrt{b})^2\leqslant 2(a+b)$ for all positive real numbers a and b, the second line of the last expression is, after a change of variables,

$$\begin{aligned}
&\leqslant \frac{N}{2\varepsilon}\int \mathbb{1}_{\{\eta(n)>0\}}\mathbb{1}_{\{f_n(\eta-e_n)\geqslant f_n(\eta)\}} f_n(\eta)\left(\mathrm{Log}\frac{f_n(\eta-e_n)}{f_n(\eta)}\right)^2\,\mathrm{d}\nu_\rho^n(\eta)\\
&\quad+\frac{N}{2\varepsilon}\int \mathbb{1}_{\{\eta(n)<2\}}\mathbb{1}_{\{f_n(\eta+e_n)\geqslant f_n(\eta)\}} f_n(\eta)\left(\mathrm{Log}\frac{f_n(\eta+e_n)}{f_n(\eta)}\right)^2\,\mathrm{d}\nu_\rho^n(\eta).
\end{aligned}$$

Using the elementary inequality

$$x\left(\mathrm{Log}\,\frac{y}{x}\right)^2 = 4\left[\sqrt{x}\,\mathrm{Log}\sqrt{\frac{y}{x}}\right]^2 \leqslant 4(\sqrt{x}-\sqrt{y})^2 \quad \text{when } y\geqslant x>0,$$

we obtain that the last integral is

$$\begin{aligned}
&\leqslant \frac{2N}{\varepsilon}\int \mathbb{1}_{\{\eta(n)>0\}}[\sqrt{f_n(\eta)} - \sqrt{f_n(\eta-e_n)}]^2\,\mathrm{d}\nu_\rho^n(\eta)\\
&\quad+\frac{2N}{\varepsilon}\int \mathbb{1}_{\{\eta(n)<2\}}[\sqrt{f_n(\eta)} - \sqrt{f_n(\eta+e_n)}]^2\,\mathrm{d}\nu_\rho^n(\eta)
\end{aligned}$$

$$\leqslant \frac{4N}{\varepsilon}\int [2f_n(\eta)+\mathbb{1}_{\{\eta(n)>0\}}f_n(\eta-e_n)+\mathbb{1}_{\{\eta(n)<2\}}f_n(\eta+e_n)]\,\mathrm{d}\nu_\rho^n(\eta)$$

$$\leqslant \frac{8N}{\varepsilon}+\frac{4N\varphi}{\varepsilon}+\frac{4N}{\varepsilon\varphi}:=\frac{C(\varphi)N}{\varepsilon}.$$

Notice that $C(\varphi)$ is a positive constant depending only on φ. Coming back to inequality (4), we have

$$I_1 \leqslant \frac{C(\varphi)N}{\varepsilon}+\frac{\varepsilon N^3}{4}\int r_{n,n+1}(\eta)[\sqrt{f_{n+1}(\eta)}-\sqrt{f_{n+1}(\eta^{n,n+1})}]^2\,\mathrm{d}\nu_\rho^{n+1}(\eta).$$

Since ν_ρ is a product measure, $f_{n+1}(\eta)=\langle f_m(\eta)\rangle_{\{n+2,\dots,m\}}$ for all $m\geqslant n+2$, then

$$I_1 \leqslant \frac{C(\varphi)N}{\varepsilon}+\frac{\varepsilon N^2}{4}\sum_{m=n+2}^{n+N+1}\int r_{n,n+1}(\eta)[\sqrt{\langle f_m(\eta)\rangle_{\{n+2,\dots,m\}}}$$

$$-\sqrt{\langle f_m(\eta^{n,n+1})\rangle_{\{n+2,\dots,m\}}}]^2\,\mathrm{d}\nu_\rho^{n+1}(\eta).$$

By Schwarz inequality, $(\sqrt{\int f}-\sqrt{\int g})^2=\int f+\int g-2\sqrt{\int fg}\leqslant \int(\sqrt{f}-\sqrt{g})^2$. So it comes

$$I_1 \leqslant \frac{C(\varphi)N}{\varepsilon}+\frac{\varepsilon N^2}{4}\sum_{m=n+2}^{n+N+1}\int r_{n,n+1}(\eta)\langle[\sqrt{f_m(\eta)}$$

$$-\sqrt{f_m(\eta^{n,n+1})}]^2\rangle_{\{n+2,\dots,m\}}\,\mathrm{d}\nu_\rho^{n+1}(\eta)$$

$$= \frac{C(\varphi)N}{\varepsilon}+\frac{\varepsilon N^2}{4}\sum_{m=n+2}^{n+N+1}\int r_{n,n+1}(\eta)[\sqrt{f_m(\eta)}-\sqrt{f_m(\eta^{n,n+1})}]^2\,\mathrm{d}\nu_\rho^{n+N+1}(\eta).$$

The same calculation for I_2 allows us to conclude

$$\partial_t H_n[f_n(t)] \leqslant \frac{C(\varphi)N}{\varepsilon}-N^2 D_n(f_n(t))$$

$$+\frac{\varepsilon N^2}{4}\sum_{m=n+2}^{n+N+1}\int [(L_{n,n+1}+L_{-n-1,-n})\sqrt{f_m(t)}]\sqrt{f_m(t)}\,\mathrm{d}\nu_\rho. \tag{6}$$

This last integral is a part of the Dirichlet form of f_m. By summation over n, it will be compensated by the Dirichlet form with a negative sign ahead. We choose $R(n/N)$ equal to $\exp(-\theta(n/N))$ for all $n\leqslant M-N-1$, we set $R(n/N)=0$ if $n\geqslant M-N$ and interpolate linearly in between. A computation involving the continuity of θ shows, for $M\geqslant N^2$, that

$$\sum_{n=1}^{M-N}\sum_{m=n+2}^{n+N+1}\int [(L_{n,n+1}+L_{-n-1,-n})\sqrt{f_m(t)}]\sqrt{f_m(t)}\,\mathrm{d}\nu_\rho\,\mathrm{e}^{-\theta(n/N)}$$

$$\leqslant C(\theta)\sum_{m=1}^{M}\int(L_m\sqrt{f_m})\sqrt{f_m}\,\mathrm{d}\nu_\rho\,\mathrm{e}^{-\theta(m/N)}$$

$$\leqslant C/N + C(\theta)\sum_{m=1}^{M-N}\int (L_m\sqrt{f_m})\sqrt{f_m}\,\mathrm{d}\nu_\rho R(m/N)$$

$$\leqslant C/N + C(\theta) N D_{M,R}(f(t)).$$

Let us multiply both sides of (6) by $R(n/N)$ and sum over $1\leqslant n\leqslant M-N$:

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{M,R}(f(t))\leqslant -N^2 D_{M,R}(f(t)) + K_1 N + K_2\varepsilon N^2 D_{M,R}(f(t)),$$

where K_1 depends on R and ε and K_2 depends only on R. Choosing ε small enough, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{M,R}(f(t))\leqslant -\frac{N^2}{2}D_{M,R}(f(t)) + K_3 N.$$

Following the proof in [9], and letting M go to infinity, we conclude

$$\mathscr{H}(S^N(t)\mu_N) + \frac{N^2}{2}\int_0^t \mathrm{d}s\,\mathscr{D}(S^N(s)\mu_N)\leqslant CN,$$

where C is a positive constant. To obtain this final inequality, it suffices to argue of the convexity and the lower semi-continuity of the entropy and of the Dirichlet form (refer to Landim and Mourragui, 1997).

3. Replacement lemma

In order to estimate $E^N_{\eta^N}[\,\|\,\eta_t - u_t\,\|^2_{-1}\,]$, we shall derive this expectation in time and a term containing the current $W_{0,1}$ will appear. In this section, we prove that we can replace the current by a gradient plus a negligible term. Before stating the so-called replacement lemma, we will modify slightly the H_{-1} norm, with a view to removing the diagonal terms that are difficult to treat, and the terms which are far from 0. This is possible thanks to the functions $\mathrm{e}^{-\theta(x/N)}$ and $\mathrm{e}^{-\theta(y/N)}$.

Fix two positive real numbers h and δ with h large and δ small. Let H be a smooth function on $\mathbb{R}^2$ satisfying

$$H(a,b) = \begin{cases} 1 & \text{if } |a|\leqslant h,\ |b|\leqslant h \text{ and } |a-b|>\delta,\\ 0 & \text{if } |a|>h+\delta \text{ or } |b|>h+\delta \text{ or } |a-b|\leqslant\delta/2.\end{cases}$$

Notice that H can be chosen uniformly bounded by 1. We set

$$\phi_N(x,y) = H(x/N, y/N)K_{N,\theta}(x,y).$$

An easy calculation shows that

$$\|\,\eta_t - u_t\,\|^2_{-1} = \frac{1}{N}\sum_{x,y}(\eta_t(x)-u_t(x/N))\phi_N(x,y)(\eta_t(y)-u_t(y/N)) + r^N(\eta_t,u_t),$$

where r^N satisfies $|r^N(\eta_t,u_t)|\leqslant C(\theta)(\mathrm{e}^{-h}+\delta)$. From now on, we will study the sum

$$S_1(t) = \frac{1}{N}\sum_{x,y}(\eta_t(x)-u_t(x/N))\phi_N(x,y)(\eta_t(y)-u_t(y/N)).$$

Let us derive it in time. For this, we compute $(\partial_s + L^N)S_1(s)$. Recall that u_t is solution of the PDE $\partial_t u = \frac{1}{2}\partial_{xx} d(u)$. Thus,

$$(\partial_s + L^N)S_1(s) = -\frac{1}{N}\sum_{x,y} \partial_{xx} d(u_t)(x/N)\phi_N(x,y)(\eta_s(y) - u_s(y/N))$$

$$+N\sum_{x,y}(W_{x-1,x}(\eta_s) - W_{x,x+1}(\eta_s))\phi_N(x,y)(\eta_s(y) - u_s(y/N)).$$

Because d is supposed to be regular, so is the solution u_t. And, by a Taylor expansion,

$$[\Delta d(u_s(./N))](x) = \frac{1}{N^2}[\partial_{xx} d(u_t(.))](x/N) + \frac{1}{N^2}\,\mathrm{o}_N(1). \tag{7}$$

Moreover, $\phi_N(x,y) \leqslant N^{-1}\mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}$ (see Lemma 5.1), and

$$N^2\Delta d(u_s)(x/N) = N^2[\nabla d(u_s)(x/N) - \nabla d(u_s)((x-1)/N)],$$

where the discrete derivative is defined for all function f by

$$\nabla f(x) = f(x+1) - f(x).$$

Therefore by summation by parts,

$$(\partial_s + L^N)S_1(s) = \mathrm{o}_N(1) - \frac{1}{N}\sum_{x,y} N^2\Delta d(u_s)(x/N)\phi_N(x,y)(\eta_s(y) - u_s(y/N))$$

$$+\sum_{x,y} W_{x,x+1}(\eta_s)N[\phi_N(x+1,y) - \phi_N(x,y)](\eta_s(y) - u_s(y/N))$$

$$= \mathrm{o}_N(1) + \sum_{x,y}[W_{x,x+1}(\eta_s) + \nabla d(u_s)(x/N)]$$

$$[N\nabla\phi_N(.,y)](x)(\eta_s(y) - u_s(y/N)).$$

We have to replace the current $W_{x,x+1}$ by a gradient to perform another summation by parts. This is the content of the following lemma. We set

$$V_l(\eta) = W_{0,1}(\eta) + D(\eta^l(0))[\eta^l(1) - \eta^l(0)].$$

As usual, $\eta^l(x)$ stands for the mean number of particles in a box of size l ($l \in \mathbb{N}\backslash\{0\}$), centered at x: $\eta^l(x) = (2l+1)^{-1}\sum_{|y|\leqslant l}\eta(x+y)$.

Lemma 3.1 (Replacement lemma)**.**

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} E^N_{\eta^N}\left[\int_0^t \sum_{x,y}\tau_x V_{\varepsilon N}(\eta_s)N\nabla\phi_N(.,y)(x)(\eta_s(y) - u_s(y/N))\,\mathrm{d}s\right] \leqslant 0.$$

In fact, we shall show that we may decompose the current as a sum of a gradient and a term Lf that turns out to be negligible: We will prove that

$$V_l(\eta) - LF(\eta) = W_{0,1}(\eta) + D(\eta^l(0))(\eta^l(1) - \eta^l(0)) - LF(\eta)$$

is small for each function F that belongs to a set $\mathscr{F}$, whose definition may be mostly guessed from the paper schedule at the end of the introduction. For all $l \in \mathbb{N}\backslash\{0\}$ and

$0 \leqslant K \leqslant 2(2l+1)$, denote by $\nu_{l,K}$ the canonical measures on Λ_l given by

$$\nu_{l,K}(.) = \nu_\rho\left(. \left| \sum_{x \in \Lambda_l} \eta(x) = K \right.\right).$$

Let $\mathscr{F}$ be the set of functions $F : [0,2] \times \mathscr{X} \to \mathbb{R}$ such that

(i) For all $\rho \in [0,2]$, $F(\rho, .)$ are cylinder, with common finite support Λ_{s_F}, of length $2s_F + 1$. Besides,

$$E_{\nu_{s_F,K}}[F(\rho,.)] = 0 \quad \text{for all } 0 \leqslant K \leqslant 2(2s_F+1),$$

(ii) for all $\eta \in \mathscr{X}$, $F(.,\eta)$ is a smooth function. For a function F in $\mathscr{F}$ and a positive integer l, define the cylinder function F_l by $F_l(\eta) = F(\eta^l(0), \eta)$.

The proof of Lemma 3.1 is contained in the statement of the two following lemmas.

Lemma 3.2.

$$E^N_{\eta^N}\left[\int_0^t \sum_{x,y} \tau_x L F_{\varepsilon N}(\eta_s) N \nabla \phi_N(.,y)(x)(\eta_s(y) - u_s(y/N))\, \mathrm{d}s \right] = \mathrm{o}_N(1).$$

We set $V_{F,l}(\eta) = W_{0,1}(\eta) + D(\eta^l(0))(\eta^l(1) - \eta^l(0)) - LF_l(\eta)$.

Lemma 3.3.

$$\inf_{F \in \mathscr{F}} \lim_{\varepsilon \to 0} \lim_{N \to \infty} E^N_{\eta^N}\left[\int_0^t \sum_{x,y} \tau_x V_{F,\varepsilon N}(\eta_s) N \nabla \phi_N(.,y)(x)(\eta_s(y) - u_s(y/N))\, \mathrm{d}s \right] \leqslant 0.$$

Proof of Lemma 3.2. For notational simplicity, we set: $\tilde{\phi}_N(x,y) = N\nabla\phi_N(.,y)(x)$. We consider the following martingale:

$$M(t) = G_t(\eta_t, u_t) - G_0(\eta_0, u_0) - \int_0^t (\partial_s + N^2 L) G_s(\eta_s, u_s)\, \mathrm{d}s,$$

where G_t is nearly the function in the statement of the lemma:

$$G_t(\eta_t, u_t) = \frac{1}{N^2} \sum_{x,y} \tau_x F_{\varepsilon N}(\eta_t) \tilde{\phi}_N(x,y)(\eta_t(y) - u_t(y/N)).$$

We have

$$G_0(\eta_0, u_0) = \frac{1}{N^2} \sum_{x,y} \tau_x F_{\varepsilon N}(\eta^N) \tilde{\phi}_N(x,y)(\eta^N(y) - u_0(y/N)) \leqslant \frac{C(F)}{N}$$

because $\tilde{\phi}_N(x,y) \leqslant C/N$, where C and $C(F)$ are constants. Then $G_0(\eta_0, u_0)$ goes to 0 when N goes to infinity.

Moreover, the function H is present in $\tilde{\phi}_N(x,y)$ and it vanishes when $|x-y| \leqslant \delta N/2$. For N large enough, $\delta N/2 > s_F$ then the generator acts separately on each term $\tau_x F_{\varepsilon N}(\eta_t)$ and $(\eta_t(y) - u_t(y/N))$. Finally, recall (7) to replace the term $\partial_s u_s$ by $N^2 \Delta u_s$ and so we obtain

$$M(t)=\frac{1}{N^2}\sum_{x,y}\tau_xF_{\varepsilon N}(\eta_t)\tilde{\phi}_N(x,y)(\eta_t(y)-u_t(y/N))$$
$$-\int_0^t \mathrm{d}s\sum_{x,y}L\tau_xF_{\varepsilon N}(\eta_s)\tilde{\phi}_N(x,y)(\eta_s(y)-u_s(y/N))$$
$$-\int_0^t \mathrm{d}s\sum_{x,y}\tau_xF_{\varepsilon N}(\eta_s)\tilde{\phi}_N(x,y)(L\eta_s(y)-\tfrac{1}{2}\Delta d(u_s(y/N)))+\mathrm{o}_N(1).$$

The first term of the martingale is of order N^{-1} because $\tilde{\phi}_N$ is uniformly bounded by a constant times N^{-1} (Lemma 5.2). The third term may be decomposed in two integrals

$$\int_0^t \mathrm{d}s\sum_{x,y}\tau_xF_{\varepsilon N}(\eta_s)\tilde{\phi}_N(x,y)L\eta_s(y) \quad \text{and}$$

$$\int_0^t \mathrm{d}s\sum_{x,y}\tau_xF_{\varepsilon N}(\eta_s)\tilde{\phi}_N(x,y)\tfrac{1}{2}\Delta d(u_s(y/N)).$$

This last one is equal to

$$\frac{1}{2N^2}\int_0^t \mathrm{d}s\sum_{x,y}N^2\Delta\tilde{\phi}_N(x,.)(y)d(u_s(y/N))$$

by summation by parts. Since $N^2\Delta\tilde{\phi}_N(x,.)(y)$ is bounded above by a constant times $N^{-1}\mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(x/N)}$ (see Lemma 5.2), this expression is of order N^{-1}. It remains to control the other part of the third term of the martingale. Since $L\eta_s(y)=W_{y-1,y}(\eta_s)-W_{y,y+1}(\eta_s)$, its expectation is equal to

$$E^N_{\eta^N}\left[\int_0^t \mathrm{d}s\,\frac{1}{N}\sum_{x,y}\tau_xF_{\varepsilon N}(\eta_s)N\nabla\tilde{\phi}_N(x,.)(y)W_{y,y+1}(\eta_s)\right].$$

But this sum depends on η only through $\{\eta(x),\ x\in\Lambda_{hN+s_F}\}$ or simplier through the coordinates in $\Lambda_{(h+1)N}$ for N large enough. So we are allowed to take the conditional expectation with respect to the σ-field generated by this set. Let us denote by $f_{t,(h+1)N}$ the density with respect to ν_ρ of the law of the process at time t, projected on this set. Then

$$E^N_{\eta^N}\left[\int_0^t \mathrm{d}s\,\frac{1}{N}\sum_{x,y}\tau_xF_{\varepsilon N}(\eta_s)N\nabla\tilde{\phi}_N(x,.)(y)W_{y,y+1}(\eta_s)\right]$$
$$=\int_0^t \mathrm{d}s\int\frac{1}{N}\sum_{x,y}\tau_xF_{\varepsilon N}(\eta)N\nabla\tilde{\phi}_N(x,.)(y)W_{y,y+1}(\eta)f_{s,(h+1)N}(\eta)\,\mathrm{d}\nu_\rho(\eta)$$
$$=t\int\frac{1}{N}\sum_{x,y}\tau_xF_{\varepsilon N}(\eta)N\nabla\tilde{\phi}_N(x,.)(y)W_{y,y+1}(\eta)\bar{f}_{t,(h+1)N}(\eta)\,\mathrm{d}\nu_\rho(\eta), \tag{8}$$

where $\bar{f}_{t,(h+1)N} = t^{-1}\int_0^t f_{s,(h+1)N}\,\mathrm{d}s$. By convexity of the Dirichlet form, using Theorem 2.1, we obtain

$$\begin{aligned}
D_{(h+1)N}[\bar{f}_{t,(h+1)N}] &\leqslant \frac{1}{t}\int_0^t D_{(h+1)N}[f_{s,(h+1)N}]\,\mathrm{d}s\\
&\leqslant \frac{1}{t}\int_0^t \frac{1}{N}\sum_{n=(h+1)N}^{(h+2)N} D_{(h+1)N}[f_{s,(h+1)N}]\,\mathrm{d}s\\
&\leqslant \mathrm{e}^{\theta(h+2)}\frac{1}{t}\int_0^t \frac{1}{N}\sum_{n=(h+1)N}^{(h+2)N} D_n[f_{s,n}]\mathrm{e}^{-\theta(n/N)}\,\mathrm{d}s\\
&\leqslant \mathrm{e}^{\theta(h+2)}\frac{1}{t}\int_0^t \mathscr{D}(S^N(s)\delta_{\eta^N})\,\mathrm{d}s\\
&\leqslant c(h,\rho)N^{-1}.
\end{aligned}$$

Then we may take the supremum in (8) on the set $\mathscr{S}$ of all probability densities f on $\mathscr{X}_{(h+1)N}$ with $D_{(h+1)N}[f]\leqslant CN^{-1}$ and bound it above by

$$t\sup_{f\in\mathscr{S}}\int \frac{1}{N}\sum_{x,\,y}\tau_x F_{\varepsilon N}(\eta)N\nabla\tilde{\phi}_N(x,.)(y)W_{y,y+1}(\eta)f(\eta)\,\mathrm{d}\nu_\rho(\eta). \tag{9}$$

Let us notice that, by a change of variables, for all function g,

$$\begin{aligned}
\int g(\eta)W_{0,1}(\eta)\,\mathrm{d}\nu_\rho(\eta) &= \frac{1}{2}\int r_{0,1}(\eta)[g(\eta^{0,1}) - g(\eta)]\,\mathrm{d}\nu_\rho(\eta)\\
&\quad -\frac{1}{2}\int r_{1,0}(\eta)[g(\eta^{1,0}) - g(\eta)]\,\mathrm{d}\nu_\rho(\eta).
\end{aligned}\tag{10}$$

Because of the term $\nabla\tilde{\phi}_N(x,.)(y)$, we have $|x-y|\geqslant\delta N$, then $\tau_x F_{\varepsilon N}(\eta^{y,y+1}) = \tau_x F_{\varepsilon N}(\eta)$ and as a consequence of (10)

$$\begin{aligned}
&\int \tau_x F_{\varepsilon N}(\eta)W_{y,y+1}(\eta)f(\eta)\,\mathrm{d}\nu_\rho(\eta)\\
&\quad= \frac{1}{2}\int r_{y,y+1}(\eta)[\tau_x F_{\varepsilon N}(\eta^{y,y+1})f(\eta^{y,y+1}) - \tau_x F_{\varepsilon N}(\eta)f(\eta)]\,\mathrm{d}\nu_\rho(\eta)\\
&\qquad -\frac{1}{2}\int r_{y+1,y}(\eta)[\tau_x F_{\varepsilon N}(\eta^{y+1,y})f(\eta^{y+1,y}) - \tau_x F_{\varepsilon N}(\eta)f(\eta)]\,\mathrm{d}\nu_\rho(\eta)\\
&\quad= \frac{1}{2}\int r_{y,y+1}(\eta)\tau_x F_{\varepsilon N}(\eta)[f(\eta^{y,y+1}) - f(\eta)]\,\mathrm{d}\nu_\rho(\eta).
\end{aligned}$$

Using (5), we obtain

$$\begin{aligned}
&\int \tau_x F_{\varepsilon N} W_{y,y+1} f\,\mathrm{d}\nu_\rho\\
&\quad\leqslant \frac{A}{4}I_{y,y+1}(f) + \frac{1}{4A}C(F) + C(F)(\mathbb{1}_{\{x\in y+\Lambda_{s_F}\}} + \mathbb{1}_{\{x=y-\varepsilon N\}} + \mathbb{1}_{\{x=y+\varepsilon N+1\}}),
\end{aligned}$$

where $C(F)$ is a positive constant depending on F and $I_{y,y+1}(f)$ stands for the piece of the Dirichlet form of the function f, concerning the jumps between y and $y+1$, i.e.

$$I_{y,y+1}(f)=\frac{1}{4}\int r_{y,y+1}(\eta)[\sqrt{f(\eta^{y,y+1})}-\sqrt{f(\eta)}]^2\,\mathrm{d}\nu_\rho(\eta)$$
$$+\frac{1}{4}\int r_{y+1,y}(\eta)[\sqrt{f(\eta^{y+1,y})}-\sqrt{f(\eta)}]^2\,\mathrm{d}\nu_\rho(\eta).$$

After having chosen $A=N$, we deduce that (9) is of order N^{-1}.

Collecting all these results, recalling that $M(t)$ is a mean-zero martingale, we deduce that the second term in the definition of $M(t)$ is a $\mathrm{o}_N(1)$ like the others. □

To prove Lemma 3.3, we shall first reduce the dynamic problem to a static one, using Theorem 2.1. We would like to estimate the expectation

$$E^N_{\eta^N}\left[\int_0^t \sum_{x,\,y}\tau_x V_{F,\varepsilon N}(\eta_s)\tilde{\phi}_N(x,y)(\eta_s(y)-u_s(y/N))\,\mathrm{d}s\right]. \tag{11}$$

We will cut the sum into two parts. Again, we notice that it depends only on $\{\eta(x),\ x\in \Lambda_{hN+s_F+\varepsilon N}\}$, i.e. on $\{\eta(x), x\in\Lambda_{(h+1)N}\}$ for N large enough. As in the last proof, we consider $f_{s,(h+1)N}$ and for all $\gamma>0$, we have

$$E^N_{\eta^N}\left[\int_0^t \sum_{x,\,y}\tau_x V_{F,\varepsilon N}(\eta_s)\tilde{\phi}_N(x,y)\eta_s(y)\,\mathrm{d}s\right]$$
$$=\int\sum_{x,\,y}\tau_x V_{F,\varepsilon N}(\eta)\tilde{\phi}_N(x,y)\eta(y)\bar{f}_{t,(h+1)N}(\eta)\,\mathrm{d}\nu_\rho(\eta)$$
$$-\frac{N}{\gamma}D_{(h+1)N}[\bar{f}_{t,(h+1)N}]+\frac{N}{\gamma}D_{(h+1)N}[\bar{f}_{t,(h+1)N}]$$
$$\leqslant \sup_{f\in\mathscr{S}'}\left[\int\sum_{x,\,y}\tau_x V_{F,\varepsilon N}(\eta)\tilde{\phi}_N(x,y)\eta(y)f(\eta)\,\mathrm{d}\nu_\rho(\eta)-\frac{N}{\gamma}D_{(h+1)N}[f]\right]+\frac{C}{\gamma},$$

where $\mathscr{S}'$ is the set of all densities on $\mathscr{X}_{(h+1)N}$. Moreover, the other part of (11) is equal to

$$E^N_{\eta^N}\left[\int_0^t \sum_{x,\,y}\tau_x V_{F,\varepsilon N}(\eta_s)\tilde{\phi}_N(x,y)(-u_s(y/N))\,\mathrm{d}s\right]$$
$$=\int_0^t\left[\sum_{x,\,y}\tau_x V_{F,\varepsilon N}\tilde{\phi}_N(x,y)(-u_s(y/N))f_{s,(h+1)N}\,\mathrm{d}\nu_\rho-\frac{N}{\gamma}D_{(h+1)N}[f_{s,(h+1)N}]\right]\mathrm{d}s$$
$$+\int_0^t\frac{N}{\gamma}D_{(h+1)N}[f_{s,(h+1)N}]\,\mathrm{d}s$$
$$\leqslant\int_0^t\sup_{f\in\mathscr{S}'}\left[\sum_{x,\,y}\tau_x V_{F,\varepsilon N}(\eta)\tilde{\phi}_N(x,y)(-u_s(y/N))f(\eta)\,\mathrm{d}\nu_\rho(\eta)-\frac{N}{\gamma}D_{(h+1)N}[f]\right]\mathrm{d}s$$
$$+\frac{N}{\gamma}\mathrm{e}^{\theta(h+2)}\int_0^t\mathscr{D}[S^N(s)\delta_{\eta^N}]\,\mathrm{d}s$$

and this last term is bounded by a constant times γ^{-1}. Since γ is arbitrary, then it is enough to prove for all $\gamma > 0$,

$$\inf_{F\in\mathscr{F}} \lim_{\varepsilon\to 0} \lim_{N\to\infty} \int_0^t \mathrm{d}s \sup_{f\in\mathscr{S}'} \left[\int \sum_{x,y} \tau_x V_{F,\varepsilon N}(\eta)\tilde{\phi}_N(x,y)(\eta(y) - u_s(y/N)) f(\eta)\,\mathrm{d}\nu_\rho(\eta) - \frac{N}{\gamma} D_{(h+1)N}[f] \right] \leqslant 0.$$

So we have to localize the function $V_{F,\varepsilon N}$: Roughly speaking, we would like to replace it by $V_{F,l}$ for a fixed integer l (that is to reduce the problem on a small macroscopic box to the same problem on a large microscopic box). First of all, we treat the current, because it is the easiest, and we replace it by its spatial average.

Lemma 3.4. *There exists a positive constant C such that, for all $\eta \in \mathscr{X}$*

$$S := \sum_{x,y} \tau_x \left[W_{0,1}(\eta) - \frac{1}{2l+1} \sum_{|z|\leqslant l} W_{z,z+1}(\eta) \right] \tilde{\phi}_N(x,y)(\eta_s(y) - u_s(y/N)) \leqslant Cl^3 N^{-1}.$$

Proof. It suffices to perform a summation by parts and then to use the following equality:

$$f(x+z) - f(z) = z[f(x) - f(x-1)] + \sum_{a=x}^{x+z} (x+z-a)\Delta f(a).$$

When we sum over z such that $|z| \leqslant l$, the first term of the r.h.s. of the last expression vanishes. We then obtain

$$S = \frac{1}{2l+1} \sum_{|z|\leqslant l} \sum_{x,y} W_{x,x+1}(\eta)(\eta_s(y) - u_s(y/N)) \sum_{a=x}^{x+z} (x+z-a)\Delta[\tilde{\phi}_N(.,y)](a)$$

The properties of the kernel $K_{N,\theta}$ (see appendix) permit to have the bound

$$\Delta[\tilde{\phi}_N(.,y)](a) \leqslant CN^{-3} \mathrm{e}^{-\theta(a/N)} \mathrm{e}^{-\theta(y/N)}$$

and the conclusion of the proof becomes clear. □

We set $V_D^{\varepsilon N,l}(\eta) = D(\eta^{\varepsilon N}(0))(\eta^{\varepsilon N}(1) - \eta^{\varepsilon N}(0)) - D(\eta^l(0))(\eta^l(1) - \eta^l(0))$.

Lemma 3.5. *For all $\delta > 0$,*

$$\lim_{l\to\infty} \lim_{\varepsilon\to 0} \lim_{N\to\infty} \sup_{f\in\mathscr{S}'} \left\{ \int \sum_{x,y} \tau_x V_D^{\varepsilon N,l}(\eta)\tilde{\phi}_N(x,y)(\eta(y) - u_s(y/N)) f(\eta)\,\mathrm{d}\nu_\rho(\eta) - \delta N D_{(h+1)N}[f] \right\} \leqslant 0.$$

Proof. Using the properties of $\tilde{\phi}_N(.,.)$ just like in the proof of Lemma 3.2, it is easy to deduce the proof from the Lemma 4.5 in Kipnis et al. (1994). Actually, the strategy is roughly the same as in the proof of Lemma 3.2. More precisely, after a summation by parts, the idea is to perform a change of variables like in (10) with $\eta(1)-\eta(0)$ instead of the current $W_{0,1}$, and an analogue of the computation which follows (10) reduces the problem to the usual *two blocks estimate*. □

We now deal with the term $LF_{\varepsilon N}(\eta)$. The idea is to replace first $(LF_{\varepsilon N})(\eta)$ by $(LF)(\eta^{\varepsilon N}(0),\eta)$ where the generator acts only on the second coordinate. Then, we introduce the spatial average of $(LF)(\eta^{\varepsilon N}(0),\eta)$ and we may consider the restriction of L on a box Λ which contains the support of F. We obtain instead of $LF_{\varepsilon N}(\eta)$

$$\frac{1}{2l+1}\sum_{|y|\leqslant l}\tau_y(L_\Lambda F)(\eta^{\varepsilon N}(0),\eta)=\frac{1}{2l+1}\sum_{|y|\leqslant l}(L_{y+\Lambda}\tau_y F)(\eta^{\varepsilon N}(0),\eta).$$

Now we perform the replacement of $(L_{y+\Lambda}\tau_y F)(\eta^{\varepsilon N}(0),\eta)$ by $(L_{y+\Lambda}\tau^2_y F)(\eta^{\varepsilon N}(0),\eta)$, where τ^2_y stands for the translation acting on the second coordinate:

$$(\tau^2_y F)(\rho,\eta)=f(\rho,\tau_y\eta)$$

the last point is the substitution of $\eta^{\varepsilon N}(0)$ by a local average. And we obtain the next lemma.

We set: $\tilde{V}^{\varepsilon N,l}_F(\eta)=LF_{\varepsilon N}(\eta)-1/(2l_F+1)\sum_{|y|\leqslant l_F}(L\tau^2_y F)(\eta^l(0),\eta)$, with $l_F=l-s_F+1$.

Lemma 3.6. *For all* $\delta>0$,

$$\inf_{F\in\mathscr{F}}\lim_{l\to\infty}\lim_{\varepsilon\to 0}\lim_{N\to\infty}\sup_{f\in\mathscr{S}'}\left\{\int\sum_{x,\,y}\tau_x\tilde{V}^{\varepsilon N,l}_F(\eta)\tilde{\phi}_N(x,y)(\eta(y)-u_s(y/N))f(\eta)\,\mathrm{d}\nu_\rho(\eta)\right.$$
$$\left.-\delta N D_{(h+1)N}[f]\right\}\leqslant 0.$$

The proof of this lemma, as the previous one, is not so different from those of Lemmas 5.2–5.4 in Kipnis et al. (1995). Then we omit it.

We just have seen that it is enough to study the following supremum:

$$\sup_{f\in\mathscr{S}'}\left\{\int\sum_{x,\,y}\tau_x W^l_F(\eta)\tilde{\phi}_N(x,y)(\eta(y)-u_s(y/N))f(\eta)\,\mathrm{d}\nu_\rho(\eta)-\frac{N}{2\gamma}D_{(h+1)N}[f]\right\}, \tag{12}$$

where W^l_F is given by

$$W^l_F(\eta)=\frac{1}{2l'+1}\sum_{|y|\leqslant l'}W_{y,y+1}(\eta)+D(\eta^{l'}(0))(\eta^{l'}(1)-\eta^{l'}(0))$$
$$-\frac{1}{2l_F+1}\sum_{|y|\leqslant l_F}(L\tau^2_y F)(\eta^l(0),\eta)$$

with $l'=l-1$ and $l_F=l-s_F$, so that $W^l_F(\eta)$ depends on η only through the coordinates $\{\eta(-l),\ldots,\eta(l)\}$. We will cut the sum in (12) into two pieces: One contains $\eta(y)$ and the other contains $-u_s(y/N)$. The first one is a little more difficult to treat, because $\eta(y)$ is integrated with respect to the density f. So we will take care of it first. The other part of the proof being easier, we will not detail it. Precisely, we wish we could estimate the term

$$J_1^{N,l,F}(f) = \sum_{x,y} \tilde{\phi}_N(x,y) \int \tau_x W_F^l(\eta)\eta(y) f(\eta)\,\mathrm{d}\nu_\rho(\eta) - \frac{N}{4\gamma} D_{(h+1)N}[f].$$

We define the density f_y by

$$f_y(\eta) = \frac{1}{Z_y(f)} \eta(y) f(\eta),$$

where the renormalization constant is given by $Z_y(f) = \int \eta(y) f(\eta)\,\mathrm{d}\nu_\rho(\eta)$. Notice that $Z_y(f) \leqslant 2$. Clearly, we have

$$J_1^{N,l,F}(f) = \sum_{x,y} Z_y(f)\tilde{\phi}_N(x,y) \int \tau_x W_F^l(\eta) f_y(\eta)\,\mathrm{d}\nu_\rho(\eta) - \frac{N}{4\gamma} D_{(h+1)N}[f].$$

As usual, we now project the new density f_y on boxes of size $2l+1$. For this, we denote by $f_{y,x}$ the conditional expectation of f_y by the σ-field generated by $\{\eta(x-l), \dots, \eta(x+l)\}$. We obtain

$$J_1^{N,l,F}(f) = \sum_{x,y} Z_y(f)\tilde{\phi}_N(x,y) \int \tau_x W_F^l(\eta) f_{y,x}(\eta)\,\mathrm{d}\nu_\rho(\eta) - \frac{N}{4\gamma} D_{(h+1)N}[f]. \quad (13)$$

A simple calculation allows us to verify that, when $|x-y|>1$,

$$I_{x,x+1}(f_y) \leqslant \frac{2}{Z_y(f)} I_{x,x+1}(f).$$

Moreover, $I_{z,z+1}(f_{y,x}) \leqslant I_{z,z+1}(f_y)$ for $z = x-l, \dots, x+l-1$. Then we deduce

$$\begin{aligned}
\sum_{\substack{|x|,|y|\leqslant hN\\ |x-y|>l+1}} Z_y(f) \sum_{z=x-l}^{x+l-1} I_{z,z+1}(f_{y,x}) &\leqslant \sum_{\substack{|x|,|y|\leqslant hN\\ |x-y|>l+1}} Z_y(f) \sum_{z=x-l}^{x+l-1} I_{z,z+1}(f_y)\\
&\leqslant \sum_{\substack{|x|,|y|\leqslant hN\\ |x-y|>l+1}} 2 \sum_{z=x-l}^{x+l-1} I_{z,z+1}(f)\\
&\leqslant 4l \sum_{\substack{|x|\leq(h+1)N,|y|\leqslant hN\\ |x-y|>1}} I_{x,x+1}(f)\\
&\leqslant 8hNlD_{(h+1)N}(f).
\end{aligned}$$

This inequality together with (13) gives, for N large enough

$$J_1^{N,l,F}(f) \leqslant \sum_{\substack{|x|,|y|\leqslant hN\\ |x-y|>\delta N}} Z_y(f) \left\{ \tilde{\phi}_N(x,y) \int \tau_x W_F^l(\eta) f_{y,x}(\eta)\,\mathrm{d}\nu_\rho(\eta) - \frac{c_0(h)}{l\gamma} \sum_{z=x-l}^{x+l-1} I_{z,z+1}(f_{y,x}) \right\}.$$

Classically, we now project on the hyperplanes $\mathscr{X}_{l,K} = \{\eta: \sum_{x=-l}^{l} \eta(x) = K\}$ for all $0 \leqslant K \leqslant 2(2l+1)$. Recall that the conditional measure is given by $\nu_{l,K}(.) = \nu_\rho(\cdot \mid \sum_{x=-l}^{l} \eta(x) = K)$. We define a new density on these hyperplanes

$$f^{l,K}(\eta) = \frac{f(\eta)}{\int f(\xi) \mathrm{d}\nu_{l,K}(\xi)} \quad \text{for all } \eta \in \mathscr{X}_{l,K}$$

and we denote by $c(x, f, K)$ the normalization coefficient

$$c(x, f, K) = \int \mathbb{1}_{\mathscr{X}_{l,K}}(\eta)(\tau_{xf})(\eta)\,\mathrm{d}\nu_\rho(\eta).$$

Since ν_ρ is translation invariant,

$$\int W_F^l(\eta)\tau_{-x} f_{y,x}(\eta)\,\mathrm{d}\nu_\rho(\eta) = \sum_{K=0}^{2(2l+1)} c(-x, f_{y,x}, K) \int W_F^l(\eta)(\tau_{-x} f_{y,x})^{l,K}(\eta)\,\mathrm{d}\nu_{l,K}(\eta),$$

and in the same manner,

$$\sum_{z=x-l}^{x+l-1} I_{z,z+1}(f_{y,x}) = \sum_{z=-l}^{l-1} \sum_{K=0}^{2(2l+1)} c(-x, f_{y,x}, K) I_{z,z+1}^{l,K}((\tau_{-x} f_{y,x})^{l,K}),$$

where $I_{z,z+1}^{l,K}$ is the Dirichlet form restricted to the sites z and $z+1$ with respect to the measure $\nu_{l,K}$. Then, for N large enough

$$\begin{aligned} &J_1^{N,l,F}(f) \\ &\leqslant \sum_{\substack{|x|,|y| \leqslant hN \\ |x-y| > \delta N}} Z_y(f) \sum_{K=0}^{2(2l+1)} c(-x, f_{y,x}, K) \\ &\quad \times \left[\tilde{\phi}_N(x,y) \int W_F^l(\eta)(\tau_{-x} f_{y,x})^{l,K}(\eta)\,\mathrm{d}\nu_{l,K}(\eta) - \frac{c_0(h)}{l\gamma} \sum_{z=-l}^{l} I_{z,z+1}^{l,K}((\tau_{-x} f_{y,x})^{l,K}) \right] \\ &\leqslant \sum_{\substack{|x|,|y| \leqslant hN \\ |x-y| > \delta N}} Z_y(f) \sup_K \sup_g \left[\tilde{\phi}_N(x,y) \int W_F^l(\eta) g(\eta)\,\mathrm{d}\nu_{l,K}(\eta) - \frac{c_0(h)}{l\gamma} \sum_{z=-l}^{l} I_{z,z+1}^{l,K}(g) \right], \end{aligned}$$

where the last supremum is taken on the set of all densities g with respect to the measure $\nu_{l,K}$. We set $D_{l,K}(g) = \sum_{z=-l}^{l} I_{z,z+1}^{l,K}(g)$ and we rewrite $J_1^{N,l,F}(f)$ as

$$\begin{aligned} J_1^{N,l,F}(f) \leqslant (c_0(h)/l\gamma) &\sum_{\substack{|x|,|y| \leqslant hN \\ |x-y| > \delta N}} Z_y(f) \sup_K \sup_g \left[\tilde{\phi}_N(x,y) \frac{l\gamma}{c_0(h)} \right. \\ &\left. \times \int W_F^l(\eta) g(\eta)\,\mathrm{d}\nu_{l,K}(\eta) - D_{l,K}(g) \right]. \end{aligned}$$

We recognize that

$$\sup_g \left(\beta \int W_F^l\, g\,\mathrm{d}\nu_{l,K} - D_{l,K}(g) \right)$$

is the variational formula for the largest eigenvalue of a small perturbation of the generator, in a box of size $2l+1$. Recall that L_l is the restriction of the generator to Λ_l. Since the symmetric generalized exclusion process is ergodic on a finite box, L_l admits a positive spectral gap, denoted by σ_l (see Kipnis and Landim, 1999). Let λ_β be the largest eigenvalue of $L_l+\beta W_F^l$. Then

$$\lambda_\beta \leqslant \frac{\beta^2}{1-2\parallel W_F^l \parallel_\infty \beta\sigma_l}\langle(-L_l)^{-1}W_F^l, W_F^l\rangle_{l,K}.$$

This bound is uniform in K. Since β vanishes when N goes to infinity, for N large enough, we have $(1-2\parallel W_F^l \parallel_\infty \beta\sigma_l)^{-1}\leqslant 1/2$. Thus,

$$J_1^{N,l,F}(f) \leqslant \frac{c_0(h)}{l\gamma}\sum_{\substack{|x|,|y|\leqslant hN\\ |x-y|>\delta N}} Z_y(f)\sup_K\left[\tilde{\phi}_N(x,y)^2\frac{l^2\gamma^2}{c_0(h)^2}\langle(-L_l)^{-1}W_F^l, W_F^l\rangle_{l,K}\right]$$

$$\leqslant \sum_{|x|,|y|\leqslant hN}\sup_K \frac{l\gamma}{N^2c_0(h)}\langle(-L_l)^{-1}W_F^l, W_F^l\rangle_{l,K}$$

because $\tilde{\phi}_N(x,y)\leqslant N^{-1}$. This expression depends no more on the density f, moreover Theorem 4.6 and Corollary 5.9 of Kipnis and Landim (1999) give that

$$\inf_{F\in\mathscr{F}}\lim_{l\to\infty}\sup_{K\to\infty}(2l)\langle(-L_l)^{-1}W_F^l, W_F^l\rangle_{l,K}=0.$$

Thus $\sup_{f\in\mathscr{S}'}J_1^{N,l,F}(f)$ goes to zero uniformly in F when l goes to infinity. □

4. Proof of Theorem 1.1

At the beginning of Section 3, we started computing $(\partial_s+L^N)S_1(s)$. We obtained

$$(\partial_s+L^N)S_1(s)$$
$$=\mathrm{o}_N(1)+\sum_{x,y}[W_{x,x+1}(\eta_s)+\nabla d(u_s)(x/N)][N\nabla\phi_N(.,y)](x)(\eta_s(y)-u_s(y/N)).$$

After the efforts previously made, we may introduce a gradient and perform a summation by parts:

$$(\partial_s+L^N)S_1(s)$$
$$=\mathrm{o}_N(1)+\frac{1}{N}\sum_{x,y}[d(\eta_s^{\varepsilon N}(x))-d(u_s(x/N))]N^2\Delta\phi_N(.,y)(x)(\eta_s(y)-u_s(y/N))$$
$$+\sum_{x,y}\tau_x[W_{0,1}(\eta_s)+D(\eta_s^{\varepsilon N}(0))(\eta_s^{\varepsilon N}(1)-\eta_s^{\varepsilon N}(0))]$$
$$\times N\nabla\phi_N(.,y)(x)(\eta_s(y)-u_s(y/N))$$
$$+\sum_{x,y}\tau_x[d(\eta_s^{\varepsilon N}(1))-d(\eta_s^{\varepsilon N}(0))-D(\eta_s^{\varepsilon N}(0))(\eta_s^{\varepsilon N}(1)-\eta_s^{\varepsilon N}(0))]$$
$$\times N\nabla\phi_N(.,y)(x)(\eta_s(y)-u_s(y/N)).$$

We denote by $I_1(s)$, $I_2(s)$ and $I_3(s)$ these three sums. First of all, we bound above $I_3(s)$: Since d is differentiable and $|\eta_s^{\varepsilon N}(x+1)-\eta_s^{\varepsilon N}(x)|\leqslant 4(\varepsilon N)^{-1}$ for all x,

$$d(\eta_s^{\varepsilon N}(x+1))-d(\eta_s^{\varepsilon N}(x))=D(\eta_s^{\varepsilon N}(x))(\eta_s^{\varepsilon N}(x+1)-\eta_s^{\varepsilon N}(x))+\frac{1}{N}\mathrm{o}_N(1),$$

where $\mathrm{o}_N(1)\to 0$ when $N\to\infty$. Then

$$I_3(s)\leqslant\frac{C}{N}\sum_{x,\,y}N\nabla\phi_N(.,y)(x)\mathrm{o}_N(1)=\mathrm{o}_N(1)$$

because $N\nabla\phi_N(.,y)(x)$ is of order $N^{-1}\mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}$ (see Lemma 5.2). Furthermore, by the replacement lemma, $(\partial_s+L^N)S_1(s)$ may be rewritten as

$$(\partial_s+L^N)S_1(s)=\tilde{r}_{\varepsilon,N}(s)+\mathrm{o}_N(1)$$
$$+\frac{1}{N}\sum_{x,\,y}[d(\eta_s^{\varepsilon N}(x))-d(u_s(x))]N^2\Delta\phi_N(.,y)(x)(\eta_s(y)-u_s(y)),$$

where $\tilde{r}_{\varepsilon,N}(t)$ is a quantity such that

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}E_{\eta^N}\left[\int_0^t\tilde{r}_{\varepsilon,N}(s)\,\mathrm{d}s\right]=0.$$

Again, we must replace the function $\eta(y)$ by its average $\eta^{\varepsilon N}(y)$:

$$(\partial_s+L^N)S_1(s)=\tilde{r}_{\varepsilon,N}(s)+\mathrm{o}_N(1)$$
$$+\frac{1}{N}\sum_{x,\,y}[d(\eta_s^{\varepsilon N}(x))-d(u_s(x/N))]N^2\Delta\phi_N(.,y)(x)(\eta_s(y)-\eta_s^{\varepsilon N}(y))$$
$$+\frac{1}{N}\sum_{x,\,y}[d(\eta_s^{\varepsilon N}(x))-d(u_s(x/N))]N^2\Delta\phi_N(.,y)(x)(\eta_s^{\varepsilon N}(y)-u_s(y/N)).$$

After a summation by parts, the first sum in the last expression is equal to

$$\frac{1}{N}\sum_{x,\,y}[d(\eta_s^{\varepsilon N}(x))-d(u_s(x/N))]$$
$$\times\left. N^2\Delta\phi_N(.,y)(x)-\frac{1}{2N\varepsilon+1}\sum_{|z|\leqslant\varepsilon N}N^2\Delta\phi_N(.,y+z)(x)\right]\eta_s(y).$$

From the computations in Lemma 5.2, and because $|\phi_N(x,y)|\leqslant N^{-1}$, we obtain

$$|N^2\Delta\phi_N(.,y+z)(x)-N^2\Delta\phi_N(.,y)(x)|\leqslant\frac{|z|}{N^2}\leqslant\frac{\varepsilon}{N}.$$

And finally, this first sum is bounded by a constant depending on θ times ε, which will vanish when ε will go to zero. Thus we reduced the study to

$$(\partial_s+L^N)S_1(s)\leqslant\tilde{r}_{\varepsilon,N}(s)+\mathrm{o}_N(1)+C(\theta)\varepsilon$$
$$+\frac{1}{N}\sum_{x,\,y}[d(\eta_s^{\varepsilon N}(x))-d(u_s(x/N))]N^2\Delta\phi_N(.,y)(x)(\eta_s^{\varepsilon N}(y)-u_s(y/N)).$$

Recall $\phi_N(x,y) = \mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}K_N(x,y)H(x/N,y/N)$. We would like to simplify the last expression, using (19) of Lemma 5.2 and the boundedness of H. So first we write

$$
\begin{aligned}
&(\partial_s + L^N)S_1(s) \leqslant \tilde{r}_{\varepsilon,N}(s) + \mathrm{o}_N(1) + C(\theta)\varepsilon\\
&\quad + \frac{1}{N}\sum_{x,y}[d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]N^2\Delta[\mathrm{e}^{-\theta(./N)}H(./N,y/N)](x)\\
&\quad \times K_N(x,y)\mathrm{e}^{-\theta(y/N)}(\eta_s^{\varepsilon N}(y) - u_s(y/N))\\
&\quad + \frac{1}{N}\sum_{x,y}[d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]N^2\Delta K_N(.,y)(x)\mathrm{e}^{-\theta(x/N)}\\
&\quad \times H(x/N,y/N)\mathrm{e}^{-\theta(y/N)}(\eta_s^{\varepsilon N}(y) - u_s(y/N))\\
&\quad + \frac{1}{N}\sum_{x,y}[d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]N\nabla[\mathrm{e}^{-\theta(./N)}H(./N,y/N)](x)\\
&\quad \times N\nabla K_N(.,y)(x)\mathrm{e}^{-\theta(y/N)}(\eta_s^{\varepsilon N}(y) - u_s(y/N))\\
&\quad + \frac{1}{N}\sum_{x,y}[d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]N\nabla[\mathrm{e}^{-\theta(./N)}H(./N,y/N)](x-1)\\
&\quad \times N\nabla K_N(.,y)(x-1)\mathrm{e}^{-\theta(y/N)}(\eta_s^{\varepsilon N}(y) - u_s(y/N)).
\end{aligned}
$$

We denote by $S_{11}(s)$, $S_{12}(s)$, $S_{13}(s)$, $S_{14}(s)$ these four lines. For the first one, since K_N is a nonnegative self-adjoint operator in $l^2(\mathbb{Z})$, we may apply Schwarz inequality to obtain, for all functions f and g,

$$
\frac{1}{N}\sum_{x,y} f(x)K_{N,\theta}(x,y)g(y) \leqslant \frac{\alpha}{2}\parallel f\parallel_{-1}^2 + \frac{1}{2\alpha}\parallel g\parallel_{-1}^2
$$

and then

$$
S_{11}(s) \leqslant \frac{\alpha}{2}\|[d(\eta_s^{\varepsilon N}) - d(u_s)]N^2\Delta[\mathrm{e}^{-\theta(./N)}H(./N,y/N)]\mathrm{e}^{\theta(./N)}\|_{-1}^2 + \frac{1}{2\alpha}\|\eta_s^{\varepsilon N} - u_s\|_{-1}^2.
$$

Now recall that $\mathrm{e}^{-\theta(./N)}H(./N,y/N)$ is a smooth function so that

$$
\begin{aligned}
N^2\Delta[\mathrm{e}^{-\theta(./N)}H(./N,y/N)](x) &= \frac{\partial^2}{\partial x_1^2}[\mathrm{e}^{-\theta(./N)}H(./N,y/N)](x) + \mathrm{o}_N(1)\\
&\leqslant C_0\mathrm{e}^{-\theta(x/N)}
\end{aligned}
$$

because H, $(\partial/\partial x_1)H$ and $(\partial^2/\partial x_1^2)H$ had been chosen bounded. Then we obtain, for all $\alpha > 0$,

$$
S_{11}(s) \leqslant \frac{\alpha C(\theta)}{2}\parallel d(\eta_s^{\varepsilon N}) - d(u_s)\parallel_{-1}^2 + \frac{1}{2\alpha}\parallel \eta_s^{\varepsilon N}(x) - u_s(x/N)\parallel_{-1}^2.
$$

Furthermore, for the sum $S_{12}(s)$, $1 - H(x/N,y/N)$ vanishes except when x or y is large or when $|x-y|$ is small. Then we replace H by 1. And recall that K_N is the kernel associated to $(I - N^2\Delta)^{-1}$. It becomes

$$S_{12}(s) = \frac{1}{N}\sum_{x,\,y} [d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]N^2\Delta K_N(.\,,y)(x)$$

$$\times \mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}(\eta_s^{\varepsilon N}(y) - u_s(y/N)) + C(\mathrm{e}^{-h} + \delta)$$

$$= \frac{1}{N}\sum_{x,\,y} [d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]K_{N,\theta}(.\,,y)(x)(\eta_s^{\varepsilon N}(y) - u_s(y/N))$$

$$-\frac{1}{N}\sum_{x} [d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))](\eta_s^{\varepsilon N}(x) - u_s(x/N))\mathrm{e}^{-2\theta(x/N)}$$

$$+ C(\mathrm{e}^{-h} + \delta).$$

This last term which comes with a negative sign, will be very useful to control the others. Again, we use (5) for the first line and, for the second one, we claim

$$[d(\eta_s^{\varepsilon N}(x) - d(u_s(x/N))](\eta_s^{\varepsilon N}(x) - u_s(x/N)) \leqslant C_d[d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]^2,$$

where C_d is a constant depending on d. Indeed, d is a nondecreasing function and

$$|d(\eta^{\varepsilon N}(x) - d(u_s(x/N))| \leqslant \| D \|_\infty \, |\eta^{\varepsilon N}(x) - u_s(x/N)|$$

(recall that D is the differential of d). Then it becomes

$$S_{12}(s) \leqslant \frac{\alpha}{2} \| d(\eta_s^{\varepsilon N}) - d(u_s) \|_{-1}^2 + \frac{1}{2\alpha} \| \eta_s^{\varepsilon N}(x) - u_s(x/N) \|_{-1}^2 + C(\mathrm{e}^{-h} + \delta)$$

$$-\frac{C_d}{N}\sum_{x} [d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]^2 \mathrm{e}^{-2\theta(x/N)}$$

$$\leqslant \frac{\alpha}{2} \| d(\eta_s^{\varepsilon N}) - d(u_s) \|_{-1}^2 + \frac{1}{2\alpha} \| \eta_s^{\varepsilon N} - u_s \|_{-1}^2 - C_d \| d(\eta_s^{\varepsilon N}) - d(u_s) \|_0^2$$

$$+ C(\mathrm{e}^{-h} + \delta).$$

Now we take care of the third sum: We use (5):

$$S_{13}(s) \leqslant \frac{1}{N}\sum_{x} ([d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N))]N\nabla(\mathrm{e}^{-\theta(./N)}H(./N, y/N))(x))$$

$$\left(\sum_{y} N\nabla K_N(.\,,y)(x)\mathrm{e}^{-\theta(y/N)}(\eta_s^{\varepsilon N}(y) - u_s(y/N))\right)$$

$$\leqslant \frac{\gamma}{2N}\sum_{x} (d(\eta_s^{\varepsilon N}(x)) - d(u_s(x/N)))^2 (N\nabla(\mathrm{e}^{-\theta(./N)}H(./N, y/N))(x))^2$$

$$+\frac{1}{2\gamma N}\sum_{x}\left(\sum_{y} N\nabla K_N(.\,,y)(x)\mathrm{e}^{-\theta(y/N)}(\eta_s^{\varepsilon N}(y) - u_s(y/N))\right)^2.$$

We set $R_\theta(y) = \mathrm{e}^{-\theta(y/N)}(\eta_s^{\varepsilon N}(y) - u_s(y/N))$ and rewrite the last sum as

$$\frac{1}{2\gamma N}\sum_{x,\,y,\,z} N[K_N(x+1, y) - K_N(x, y)]R_\theta(y)N[K_N(x+1, z) - K_N(x, z)]R_\theta(z).$$

By a summation by parts it is equal to

$$-\frac{1}{2\gamma N}\sum_{x,\,y,\,z} N^2\Delta K_N(.,y)(x)R_\theta(y)K_N(x,z)R_\theta(z). \tag{14}$$

If we add and subtract $(1/2\gamma N)\sum_{x,\,y,\,z}K_N(x,y)R_\theta(y)K_N(x,z)R_\theta(z)$, by definition of the kernel K_N, (14) becomes

$$-\frac{1}{2\gamma N}\sum_x\left(\sum_y K_N(x,y)R_\theta(y)\right)^2+\frac{1}{2\gamma}\parallel \eta_s^{\varepsilon N}-u_s\parallel_{-1}^2 .$$

The first term is nonpositive so we may forget it. Moreover, we can get that

$$N\nabla[\mathrm{e}^{-\theta(./N)}H(./N,y/N)](x)\leqslant C(\theta)\mathrm{e}^{-\theta(x/N)}$$

and then

$$S_{13}(s)\leqslant\gamma C(\theta)\parallel d(\eta_s^{\varepsilon N})-d(u_s)\parallel_0^2+\frac{1}{2\gamma}\parallel\eta_s^{\varepsilon N}-u_s\parallel_{-1}^2+C(\mathrm{e}^{-h}+\delta).$$

The same bound holds for $S_{14}(s)$. Gathering all these computations, we obtain

$$\begin{aligned}(\partial_s+L^N)S_1(s)\leqslant{}&\tilde r_{\varepsilon,N}(s)+C(\theta)\varepsilon+o_N(1)\\&+\alpha\tilde C(\theta)\parallel d(\eta_s^{\varepsilon N})-d(u_s)\parallel_{-1}^2\\&+\frac{3}{2\alpha}\parallel\eta_s^{\varepsilon N}-u_s\parallel_{-1}^2-C_d\parallel d(\eta_s^{\varepsilon N})-d(u_s)\parallel_0^2 .\end{aligned}$$

But the H_{-1} norm is smaller than the H_0 norm, so for α small enough,

$$\alpha\tilde C(\theta)\parallel d(\eta_s^{\varepsilon N})-d(u_s)\parallel_{-1}^2\leqslant C_d\parallel d(\eta_s^{\varepsilon N})-d(u_s)\parallel_0^2 .$$

And there exists a constant β, depending on θ and d such that

$$(\partial_s+L^N)S_1(s)\leqslant\tilde r_{\varepsilon,N}(s)+C(\theta)\varepsilon+o_N(1)+C(\mathrm{e}^{-h}+\delta)+\beta\parallel\eta_s^{\varepsilon N}-u_s\parallel_{-1}^2 .$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\eta^N}^N[\parallel\eta_s-u_s\parallel_{-1}^2]&=E_{\eta^N}^N[L^N\parallel\eta_s-u_s\parallel_{-1}^2+\partial_t\parallel\eta-u_s\parallel_{-1}^2{}_{|\eta=\eta_s}]\\&\leqslant\beta E_{\eta^N}^N[\parallel\eta_s-u_s\parallel_{-1}^2]+E_{\eta^N}[\tilde r_{\varepsilon,N}(s)]+\mathrm{o}_N(1).\end{aligned}$$

Let us integrate this expression

$$\begin{aligned}E_{\eta^N}^N[\parallel\eta_t-u_t\parallel_{-1}^2]\leqslant{}&\parallel\eta^N-u_0\parallel_{-1}^2+\int_0^t\beta E_{\eta^N}^N[\parallel\eta_s-u_s\parallel_{-1}^2]\,\mathrm{d}s\\&+E_{\eta^N}^N\left[\int_0^t\tilde r_{\varepsilon,N}(s)\,\mathrm{d}s\right]+\mathrm{o}_N(1)+C(\theta)\varepsilon\\\leqslant{}&\int_0^t\beta E_{\eta^N}^N[\parallel\eta_s-u_s\parallel_{-1}^2]\,\mathrm{d}s+f(N,\varepsilon),\end{aligned}$$

where f goes to zero when N goes to infinity and then ε goes to zero. Finally, by Gronwall's lemma, we get

$$E_{\eta^N}^N[\parallel\eta_s-u_s\parallel_{-1}^2]\leqslant f(N,\varepsilon)\mathrm{e}^{\beta t}.$$

And we conclude that

$$\lim_{\varepsilon\to0}\lim_{N\to\infty}E_{\eta^N}^N[\parallel\eta_t-u_t\parallel_{-1}^2]=0. \qquad\square$$

Appendix

We present here some results concerning the kernel associated to the operator $(I - N^2\Delta)^{-1}$. One may read the papers by Landim and Vares (1996) and Landim and Yau (1995) for more details.

For a fixed integer N, we endow the space $l^2(\mathbb{Z})$ of functions with summable square on $\mathbb{Z}$, with the inner product

$$\langle f, g\rangle = \frac{1}{N}\sum_x f(x)g(x).$$

And we define the convolution of these functions by

$$(f * g)(x) = \sum_y f(y)g(x-y).$$

The kernel $(I - N^2\Delta)^{-1}$ is defined in the sense

$$(I - N^2\Delta)^{-1} f(x) = (K_N * f)(x)$$

for all function f on $\mathbb{Z}$. From expression (1), we see that $K_N(x, y) = K_N(x - y)$ is an even function regular with a singularity at the origin. Moreover,

Lemma 5.1.

$$\sup_{x\in\mathbb{Z}} K_N(x) \leqslant K_N(0) \leqslant N^{-1}, \qquad \sup_{x\in\mathbb{Z}} |N\nabla K_N(x)| \leqslant N^{-1},$$

$$\sup_{x\in\mathbb{Z}-\{0\}} |N^2\Delta K_N(x)| \leqslant N^{-1}.$$

Proof. Since $0 < a < 1$, from (1), $K_N(x) \leqslant K_N(0)$ and

$$K_N(0) = \frac{1-a}{1+a} = \frac{\sqrt{a}}{1+a} N^{-1} \leqslant N^{-1}.$$

Besides,

$$|N\nabla K_N(x)| = \sqrt{a}|a^{|x+1|} - a^{|x|}| \leqslant \sqrt{a}(1-a)a^{|x|} = aN^{-1} \leqslant N^{-1}.$$

If $x \neq 0$,

$$N^2\Delta K_N(x) = \frac{a}{(1-a)^2}\frac{1-a}{1+a} a^{|x|}\left(a + \frac{1}{a} - 2\right) = K_N(x).$$

But if $x = 0$, we have

$$N^2\Delta K_N(0) = -\frac{2a}{1+a} = \frac{1-a}{1+a} - 1 = K_N(0) - 1.$$

We just verified that

$$K_N(x, y) - N^2\Delta K_N(., y)(x) = \mathbb{1}_{\{x=y\}}. \qquad \square$$

Lemma 5.2. There exists a positive constant C such that, for all x and y in $\mathbb{Z}$ such that $|x - y| > 1$,

$$\tilde{\phi}_N(x, y) = N\nabla\phi_N(., y)(x) \leqslant CN^{-1}\mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}, \tag{15}$$

$$N\nabla\tilde{\phi}_N(., y)(x) \leqslant CN^{-1}\mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}, \tag{16}$$

$$N^2\Delta\tilde{\phi}_N(., y)(x) \leqslant CN^{-1}\mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}. \tag{17}$$

Proof. We will use intensively the two following equalities easy to verify for two functions f and g on $\mathbb{Z}$:

$$\nabla(fg)(a)=(\nabla f(a))g(a+1)+f(a)\nabla g(a), \tag{18}$$

$$\Delta(f.g)(a)=(\Delta f)(a)g(a)+f(a)(\Delta g)(a) + (\nabla f)(a)(\nabla g)(a)+(\nabla f)(a-1)(\nabla g)(a-1). \tag{19}$$

We look first at (15). Recall the definition of ϕ_N:

$$\phi_N(x,y)=K_{N,\theta}(x,y)H(x/N,y/N)=\mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}K_N(x,y)H(x/N,y/N).$$

Then, applying (18), we obtain

$$\begin{aligned}\tilde{\phi}_N(x,y)&=\mathrm{e}^{-\theta(y/N)}N[\mathrm{e}^{-\theta((x+1)/N)}H((x+1)/N,y/N)K_N(x+1,y)\\ &\qquad -\mathrm{e}^{-\theta(x/N)}H(x/N,y/N)K_N(x,y)]\\ &=\mathrm{e}^{-\theta(y/N)}N[\mathrm{e}^{-\theta((x+1)/N)}H((x+1)/N,y/N)-\mathrm{e}^{-\theta(x/N)}H(x/N,y/N)]K_N(x,y)\\ &\qquad +\mathrm{e}^{-\theta(y/N)}\mathrm{e}^{-\theta(x/N)}H(x/N,y/N)N[\nabla K_N(.,y)](x).\end{aligned}$$

But H had been chosen smooth and bounded by 1, so that we can perform a Taylor expansion. We set

$$g_{y/N}(x/N)=\mathrm{e}^{-\theta(x/N)}H(x/N,y/N).$$

Then $g_{y/N}$ is smooth and

$$|g_{y/N}((x+1)/N)-g_{y/N}((x+1)/N)|\leqslant\frac{2}{N}\sup_x\left|\frac{\mathrm{d}}{\mathrm{d}x_1}H(x/N,y/N)\right|\leqslant\frac{1}{N}C\left(\frac{\mathrm{d}}{\mathrm{d}x_1}H\right).$$

Therefore, Lemma 5.1 implies

$$\tilde{\phi}_N(x,y)\leqslant\frac{C}{N}\mathrm{e}^{-\theta(x/N)}\mathrm{e}^{-\theta(y/N)}.$$

Now we deal with (16).

$$N\nabla\tilde{\phi}_N(.,y)(x)=N^2\Delta\phi_N(.,y)(x+1).$$

Using equality (19), we obtain

$$\begin{aligned}N^2\Delta\phi_N(.,y)(x)&=\mathrm{e}^{-\theta(y/N)}N^2\Delta(\mathrm{e}^{-\theta(./N)}H(./N,y/N))(x)K_N(x,y)\\ &\quad+\mathrm{e}^{-\theta(y/N)}\mathrm{e}^{-\theta(x/N)}H(x/N,y/N)N^2\Delta K_N(.,y)(x)\\ &\quad+\mathrm{e}^{-\theta(y/N)}N\nabla(\mathrm{e}^{-\theta(./N)}H(./N,y/N))(x)N\nabla K_N(.,y)(x)\\ &\quad+\mathrm{e}^{-\theta(y/N)}N\nabla(\mathrm{e}^{-\theta(./N)}H(./N,y/N))(x-1)N\nabla K_N(.,y)(x-1).\end{aligned}$$

A Taylor expansion gives

$$N^2\Delta(\mathrm{e}^{-\theta(./N)}H(./N,y/N))(x)\leqslant C(\theta,H)\mathrm{e}^{-\theta(x/N)}$$

and

$$N\nabla(\mathrm{e}^{-\theta(./N)}H(./N,y/N))(x)\leqslant C(\theta,H)\mathrm{e}^{-\theta(x/N)}$$

and Lemma 5.1 allows us to conclude. Inequality (17) can be shown in the same way. □

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