JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **93**, 142–194 (1983)

On the Three-Dimensional Lunar Problem and Other Perturbation Problems of the Kepler Problem

MARTIN KUMMER

Forschungsinstitut für Mathematik ETH Zürich
and
The University of Toledo, Toledo, Ohio 43606

Submitted by C. L. Dolph

INTRODUCTION

This work grew out of an attempt to carry over the methods of our study [1] of the restricted three body problem for high values of the Jacobian constant (also known as the lunar problem) from two to three dimensions. During this endeavor we found that the regularization of the Kepler problem by Kustaanheimo–Stiefel–Scheifele ([2, 3]) was the obvious tool to be applied. Although Stiefel–Scheifele give an excellent presentation of the so called KS transformation in their monograph [3], we find that a different notation employing Pauli matrices and complex variables (i.e., a notation which in its essential features was proposed by Jost [4])[1] is better suited for our purposes. This notation makes it obvious that the canonical version of the KS transformation is a canonical extension of a map $\pi: \mathbb{R}^4\backslash\{0\} = \mathbb{C}^4\backslash\{0\} \to \mathbb{R}^3\backslash\{0\}$ whose restriction to S^3 is the Hopf map. If $z, w \in \mathbb{C}^2$ are coordinates on the symplectic manifold $\{\mathbb{C}^4\backslash\{0\}, d\theta\}$, where $\theta = 2\,\mathrm{Im}\langle w, dz\rangle$ ($\langle\ ,\ \rangle$ being the standard inner product of $\mathbb{C}^2$) and the group $U(1)$ acts on $\mathbb{C}^4\backslash\{0\}$ via $e^{is}: (z, w) \to (e^{is}z, e^{is}w)$, then this action is exact symplectic (i.e., leaves θ invariant). Correspondingly, it is "generated" by the Hamiltonian or "moment": $2I = 2\,\mathrm{Re}\langle w, z\rangle$. (For the general notion of the "moment" of a Hamiltonian action of a Lie group on a symplectic manifold see, e.g., [5–11].) The relation $I = 0$ is nothing but the "bilinear relation" associated with the KS transformation. Thus the KS transformation explicitly reduces out the $U(1)$-action on the submanifold $(I^{-1}(0))' = I^{-1}(0) \cap \{z, w : z \neq 0\}$ of $\mathbb{C}^4\backslash\{0\}$. In other words it establishes an isomorphism between the symplectic spaces $(\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}^3$ and

[1] According to a personal communication by Waldvogel, the same notation was originally employed by Kustaanheimo.

0022-247X/83 $3.00

$(I^{-1}(0))'/U(1)$. This isomorphism then carries the Hamiltonian system of the Kepler problem (with fictitious time variable) into the one of four harmonic oscillators in resonance (compare [3, 32, 33].)

The final step in the regularization of the Kepler problem consists in lifting the restriction $z \neq 0$. In other words, the phase space of the regularized Kepler problem is $I^{-1}(0)/U(1)$, where $I^{-1}(0)$ is regarded as a submanifold of $\mathbb{C}^4\backslash\{0\}$.

At this point we draw the reader's attention to the fact that $I^{-1}(0)$ plays a fundamental role in a completely different context: Penrose [12] identifies the compactified manifold of null lines of Minkowski space with $I^{-1}(0)/\mathbb{C}^*$ (our notation), where $\mathbb{C}^*$ is the group of nonzero complex numbers acting on $\mathbb{C}^4$ in an obvious way. Indeed, like a compactified surface of constant negative energy of the Kepler problem, Penrose's manifold possesses the topological character $S^2 \times S^3$ (see [12, p. 354; 17, p. 629; 33]. Penrose calls the points of $I^{-1}(0)$ "null-twistors." Since the linear transformations leaving I invariant comprise the group $\tilde{U}(2, 2) = U(1) \times SU(2, 2)$, the group $SU(2, 2)$ plays a fundamental role in the theory of twistors (see also [13, pp. 58–73]). It follows that the same group should also play a fundamental role in the Kepler problem. That this is indeed the case has already been demonstrated on the level of Lie algebras by Baumgarte [14]. Baumgarte, adopting some ideas presented by Barut [15] in his study of the quantum mechanical Kepler problem to classical mechanics, exhibits the Kepler Hamiltonian, as well as the Hamiltonian of four harmonic oscillators in resonance, as members of a "generator-set" of the Lie algebra $so(4, 2) \cong su(2, 2)$ over their respective phase spaces and shows that the KS transformation carries the two representations of this Lie algebra into each other, thereby sending the oscillator Hamiltonian into the Kepler Hamiltonian. This feature of the KS transformations is now easily understood from the point of view of twistor theory. The group $U(2, 2)$ acts symplectically on $\mathbb{C}^4\backslash\{0\}$, thereby leaving the "bilinear relation" $I = 0$ invariant. Moreover, $SU(2, 2)$ acts transitively on the phase space $I^{-1}(0)/U(1)$ so that the latter can be realized as a certain orbit of $SU(2, 2)$ in the dual of the Lie algebra $su(2, 2)^*$ (equipped with the symplectic structure discovered for such orbits by Kirillov [7]. See also, e.g., [6, 8, 10, 11, 16].)

Another realization of our phase space is T^+S^3, that is to say, the cotangent bundle of the 3-sphere from which the zero section has been removed.

Sternberg and Guillemin [16, p. 174–178], using methods that differ considerably from ours, have previously exhibited T^+S^3 as a homogeneous symplectic space of the identity component $SO_0(4, 2)$ of $SO(4, 2)$ which is doubly covered by $SU(2, 2)$.

Since T^+S^3 is again six dimensional, we may ask about its relationship to the original six-dimensional phase space $(\mathbb{R}^3\setminus\{0\})\times\mathbb{R}^3$ of the Kepler problem. We find that the original phase space is symplectically embedded into T^+S^3 as an open submanifold. In fact, the explicit formulae that we obtain via the KS transformation for this embedding map agree up to an automorphism of T^+S^3 precisely with those given by Moser in [17]. Indeed, one verifies immediately that the embedding takes the Kepler flow on a surface of fixed negative energy into the geodesic flow on the unit (*co*-) tangent bundle of S^3.

If our only concern were with the Kepler problem, all that has been said so far would be of academic significance only. Since $I^{-1}(0)/U(1)$ is also the correct phase space for a wide class of perturbations of the Kepler problem, however, our constructions retain their value when such problems are under investigation. This class of perturbations consists of potentials that (besides being dependent on the positions vector) may also depend linearly on the momentum. The restricted three body problem for large values of the Jacobian constant belongs to this class. This problem is studied in some detail in Sections 5 and 6.

In Section 4 we discuss common features of perturbations of the class just defined. We prepare the perturbation by bringing the dominant terms into normal form. Approaches to finding the normal form of a (resonant) Hamiltonian are discussed at many places in the literature (see, e.g., [6, 18, 19]). We shortly discuss our own approach which in its final version was influenced by two sources: (1) Moser's presentation in [18] and his personal lectures and (2) some written correspondence about this topic with Churchill (see also [19]).

Terminating the normal form at a certain order and discarding the remainder leaves us with an approximation to the original Hamiltonian whose flow—according to KMA theory (see [18, 20, 21])—sheds some light on the flow of the original Hamiltonian. In the sequel this approximation will be referred to as "truncated Hamiltonian." Since the harmonic oscillator Hamiltonian (which replaces the Kepler Hamiltonian), here denoted by J, is invariant under the subgroup $U(2)\times U(2)$ of the group $U(2, 2)$, and since the prepared perturbation is in involution with J and hence is $U(1)\times U(1)$-invariant, it defines a Hamiltonian system on the reduced space $I^{-1}(0)\cap J^{-1}(1)/U(1)\times U(1) = [SU(2)/U(1)]\times[SU(2)/U(1)] = S^2\times S^2$ which obviously is also the orbit space of the Kepler problem. (The topological nature of this orbit space was first determined by Moser in [17]. See also [3].) We realize each sphere of the product $S^2\times S^2$ as a submanifold in $su(2)^*$ (identified as $\mathbb{R}^3$) and give an explicit description of the flow that the truncated Hamiltonian induces on $S^2\times S^2$. This construction is analogous to the one presented by the author in a simpler situation in [22–25]. (In this connection see also [17, 35].) In particular, critical points of the truncated Hamiltonian give rise to periodic solutions which in our approximation are still Kepler ellipses whose elements (for fixed energy), however, are functions of the perturbation parameter. Also, as the perturbation parameter increases, the frequency of the periodic solution will in general undergo a displacement away from the Kepler frequency. In short, the flow of the truncated Hamiltonian appears as a product of the quotient flow on $S^2 \times S^2$ with the (accelerated or decelerated) Kepler flow. In cases in which the truncated Hamiltonian is not only in involution with J but with, let us say, the third component of the angular momentum L_3 as well (as is the case in the problem studied in Sections 5 and 6), the truncated Hamiltonian becomes integrable and a further reduction is possible. The result is a Hamiltonian system defined on (part of) an orbit of $SU(1, 1)$ in $su(1, 1)^*$ which we realize as (part of) a hyperboloid (compare also [23, 24]). Critical points of this doubly reduced Hamiltonian correspond to families of quasiperiodic solutions with two frequencies that the original (truncated) Hamiltonian possesses on each energy surface, the family parameter being $v = L_3/J$. These quasiperiodic solutions are Kepler ellipses whose planes are slowly rotating about the 3-axis. Here, the elements as well as the two crucial frequencies are functions of v and the perturbation parameter.

In Section 3 we apply the methods explained in the first two sections to the three-dimensional restricted three body problem for high values of the Jacobian constant C. By an appropriate scaling it is indeed possible to view this problem as a particular perturbation problem of the Kepler problem, the perturbation parameter being the inverse square root of the Jacobian constant. As pointed out by Moser in a personal conversation, the corresponding normal form must be of integrable type. It is therefore possible to apply the full machinery of the first two sections to our problem. First we note that the truncated Hamiltonian of our problem possesses on each Jacobi surface the four well-known periodic solutions (compare [17]): the two circular ones which are already present in the two-dimensional problem and the two collision orbits that are perpendicular to the plane of the two primaries. For a thorough study of the latter ones for arbitrary values of the Jacobian constant but for small values of μ (= mass of one of the primaries), consult [26].

Besides these four periodic solutions we find on each Jacobi surface four families of quasiperiodic solutions with two frequencies of the general type discussed above. We denote these four families by the critical points $\mathbf{e}_{\pm}(v)$, $\mathbf{e}_0^{\pm}(v)$ that give rise to them. Here the parameter which labels the members of each family agrees asymptotically for $C \to \infty$ with the normalized third component of the angular momentum $v \approx \nu^{-1} C^{1/2} L_3$ and it is confined to vary in the interval $0 < |v| < 1$ (ν = mass of the primary in the neighborhood of which the massless body moves: $0 < \nu < 1$). The members of the family $\mathbf{e}_0^{+}(v)$ are ellipses in the plane of the primaries which for $v \to \pm 1$ tend toward the two circular solutions. This family is already present in the two-dimensional problem and in that problem it can be continued (in the sense of KAM theory) to the full Hamiltonian, thereby also guaranteeing the stability of the circular solutions ([1, 27]).

In the three-dimensional problem we can only assert that this family is stable in our integrable approximation. Although most members of the family can be continued to the full problem (see Section 6) their actual stability cannot even be investigated by the KAM theory.

The families $\mathbf{e}_{\pm}(v)$ exist and are stable only for $0 < |v| < v_0$, where $v_0 = \frac{1}{5}\sqrt{15}$. Both of these families close in on one of the two collision orbits for $v \to 0$, whereas for $|v| \to v_0$ their members become circular orbits on planes that enclose the angle $\arccos v_0 \approx 39.23°$ with the plane of the primaries. Finally, the fourth family $\mathbf{e}_0^-(v)$ consists of circular solutions and it is unstable for $0 < |v| < v_0$ and stable for $v_0 < |v| < 1$. Since for $v \to \pm 1$ these solutions tend to the circular solutions of the planar problem, we see that in our approximation, by means of joining the families $\mathbf{e}_{\pm}(v)$ and $\mathbf{e}_0^-(v)$ at $v_0(-v_0)$, it is possible to embed the direct (retrograde) circular solution of the planar problem and the two collision solutions into one continuous stable family of quasiperiodic solutions with two frequencies (see Fig. 4).[2]

In Section 6 we take up the subtle problem of continuation of the four families of quasiperiodic solutions with two frequencies as well as those with three frequencies which "surround" them. The quasiperiodic solutions supported by these 3-tori correspond to Kepler ellipses whose planes and Laplace vectors are not only rotating about the 3-axis but are also subjected to small oscillations. According to KAM theory, the majority of these tori (in the sense of measure theory) can be continued to the exact problem if only a certain determinant does not vanish. Since the three frequencies (in the rotating system) have different orders of magnitude in terms of the perturbation parameter, a straightforward application of this determinant condition is not possible. Nevertheless we are able to prove that certain 3-tori in the neighborhood of the four families of 2-tori for a "very short" interval of the perturbation parameter persist in the exact problem.

As far as the continuation of the 2-tori themselves is concerned, a suitable application of the theory explained in [28, 29] shows that only a "discrete" family of those tori survive the onslaught of the full perturbation. In particular, the perturbation parameter (and therefore the Jacobian constant) labeling a surviving torus must be known with absolute precision (see Theorem 5).

[2] After this manuscript had been prepared half way, the author learned about the publication of [31] in which some features of these solutions are already described. (See in particular pp. 576 and 578).

2. Review of the Spinor Regularization of the Kepler Problem

We start our exposition with a review of the spinor regularization of the Kepler problem by Kustaanheimo, Stiefel, and Scheifele ([2, 3]) using a more compact notation (compare [4]).

For this purpose, we consider the smooth manifolds $\mathbb{R}^4\backslash\{0\} = \mathbb{C}^2\backslash\{0\}$[3] and $\mathbb{R}^3\backslash\{0\}$ with global coordinates $x = (z_1, z_2)$ ($z_i (i = 1, 2)$ = complex numbers) and $\mathbf{x} = (x_1, x_2, x_3)$, respectively. Next, we define the submersion

$$\pi\colon \mathbb{C}^2\backslash\{0\} \to \mathbb{R}^3\backslash\{0\} \tag{2.1}$$

by means of the formula

$$\mathbf{x} \circ \pi(z) = \langle z, \boldsymbol{\sigma} z\rangle, \tag{2.2}$$

where

$$\boldsymbol{\sigma} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \tag{2.3}$$

is the vector of Pauli matrices. In (2.2) we think of $z \in \mathbb{C}^2$ as a column and the notation $\langle z, w\rangle$ ($z, w \in \mathbb{C}^2$) is used to denote the usual inner product of $\mathbb{C}^2$,

$$\langle z, w\rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2. \tag{2.4}$$

Obviously, (2.1) defines a principal $U(1)$-bundle with group action e^{is}: $(z_1, z_2) \to (e^{is} z_1, e^{is} z_2)$, π being essentially the Hopf map. If in the cotangent bundles $T^*(\mathbb{C}^2\backslash\{0\}) = (\mathbb{C}^2\backslash\{0\}) \times \mathbb{C}^2$ and $T^*(\mathbb{R}^3\backslash\{0\}) = (\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}^3$ coordinates (z, w) and $(\mathbf{x}, \mathbf{y})$ are introduced, then their canonical 1-forms may be written in the form

$$\theta = 2\,\mathrm{Im}\langle w, dz\rangle, \qquad \theta_0 = \mathbf{y} \cdot d\mathbf{x}. \tag{2.5}$$

Accordingly, the canonical symplectic structures of these cotangent bundles are defined by $d\theta$ and $d\theta_0$. Here θ is manifestly invariant under the lifted group action $e^{is}\colon (z, w) \to (e^{is}z, e^{is}w)$ and possesses the "moment" $2I$, where

$$I = \mathrm{Re}\langle w, z\rangle \tag{2.6}$$

(i.e., $2I$ is the Hamiltonian generating the $U(1)$-action on $T^*(\mathbb{C}^2\backslash\{0\})$.) The spinor regularization of Kustaanheimo–Stiefel is based on the fact that the

[3] In this paper $\mathbb{C}^n$ will always be regarded as a real manifold.

map π has a lift (denoted by the same symbol) to the cotangent bundle with the property

$$\pi^*\theta_0 = \theta|_{(I^{-1}(0))'}. \tag{2.7}$$

Here, $\theta|_{(I^{-1}(0))'}$ denotes the restriction of θ to the seven-dimensional submanifold $I^{-1}(0) \cap \{z, w : z \neq 0\}$ of $\mathbb{C}^4$. This lift of π is obtained by supplementing (2.2) by the formula

$$\mathbf{y} \circ \pi = \langle z, z\rangle^{-1} \operatorname{Im}\langle w, \boldsymbol{\sigma} z\rangle. \tag{2.8}$$

In order to prove (2.7) we first note that the following formula holds for arbitrary elements $u, w, z \in \mathbb{C}^2$:

$$\langle u, \boldsymbol{\sigma} z\rangle \cdot \boldsymbol{\sigma} w = 2z\langle u, w\rangle - \langle u, z\rangle w. \tag{2.9}$$

(As in (2.5) the dot denotes the usual inner product of $\mathbb{R}^3$.) Adding to (2.9) the relation obtained from (2.9) by interchanging u and z yields

$$\operatorname{Re}\langle z, \boldsymbol{\sigma} u\rangle \cdot \boldsymbol{\sigma} w = u\langle z, w\rangle + z\langle u, w\rangle - \operatorname{Re}\langle z, u\rangle\, w. \tag{2.10}$$

Multiplying both sides of (2.10) with $z^\dagger = (\bar{z}_1, \bar{z}_2)$ from the left and taking real parts we obtain

$$\operatorname{Re}\langle z, \boldsymbol{\sigma} u\rangle \cdot \operatorname{Re}\langle z, \boldsymbol{\sigma} w\rangle = \langle z, z\rangle \operatorname{Re}\langle u, w\rangle - \operatorname{Im}\langle z, u\rangle \operatorname{Im}\langle z, w\rangle. \tag{2.11}$$

Replacing in (2.11) u by dz and w by $i^{-1}w$ gives

$$\operatorname{Re}\langle z, \boldsymbol{\sigma} dz\rangle \cdot \operatorname{Im}\langle w, \boldsymbol{\sigma} z\rangle = \langle z, z\rangle \operatorname{Im}\langle w, dz\rangle - \operatorname{Im}\langle z, dz\rangle \operatorname{Re}\langle z, w\rangle.$$

Substituting in this expression the left sides of (2.2) and (2.8) finally yields for $(z, w) \in (I^{-1}(0))'$

$$(\mathbf{y} \circ \pi)\, d(\mathbf{x} \circ \pi) = 2 \operatorname{Im}\langle w, dz\rangle.$$

This, however, is precisely relation (2.7) which we set out to prove. Replacing u and w in (2.11) first by z then by $i^{-1}w$ we find (with $r = |\mathbf{x}|$) on $(I^{-1}(0))'$

$$r \circ \pi = \langle z, z\rangle, \tag{2.12a}$$

$$|\mathbf{y}|^2 \circ \pi = \langle z, z\rangle^{-1}\langle w, w\rangle. \tag{2.12b}$$

We are now in a position to give a concise description of the spinor regularization of the Kepler Hamiltonian

$$H_0 = \tfrac{1}{2}|\mathbf{y}|^2 - 1/r \tag{2.13}$$

on the energy surface $H_0 = -\tfrac{1}{2}$.

In the following a Hamiltonian system on a (even-dimensional) smooth manifold M will be a triple of objects, the first being the Hamiltonian, the second being the fundamental two-form defining the symplectic structure on M and thereby telling us how to associate a system of differential equations with the Hamiltonian, and the third being the time variable that enters these differential equations. For example, with $(H(\mathbf{x}, \mathbf{y}), d\theta_0, t)$ (θ_0 given in (2.5)) we associate the differential equations

$$\frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{y}}, \qquad \frac{d\mathbf{y}}{dt} = -\frac{\partial H}{\partial \mathbf{x}}. \tag{2.14}$$

Stated in a more coordinate-independent way, this means that the vector field X_H determined by the condition

$$X_H \lrcorner\, d\theta_0 = -dH \tag{2.15}$$

must be interpreted as the velocity field of the flow induced by H on M. In order to regularize the Hamiltonian system $(H_0, d\theta_0, t)$, where H_0 is the Kepler Hamiltonian (2.13), we first consider the auxiliary Hamiltonian system $(H = \frac{1}{2}|\mathbf{y}|^2 - K/r,\ d[\theta_0 - K ds], t)$ on the phase space of dimension eight that is obtained from $T^*(\mathbb{R}^3\backslash\{0\})$ by adjoining the variables K and s. The differential equations associated with this extended Hamiltonian system are Eqs. (2.14) supplemented by the following two equations:

$$\frac{ds}{dt} = -\frac{\partial H}{\partial K} = \frac{1}{r}, \qquad \frac{dK}{dt} = 0. \tag{2.16}$$

Now, by a well-known theorem (see, e.g., [20, p. 266]), the extended Hamiltonian system induces on the manifold $H = -\frac{1}{2}$ the Hamiltonian system $(K_0, d\theta_0, s)$, where K_0 is obtained from H by solving the equation $H = -\frac{1}{2}$ for K. We obtain

$$K_0 = \tfrac{1}{2}(|\mathbf{y}|^2 + 1)\, r. \tag{2.17}$$

It is clear (and also directly verifiable) that the vector field of H_0 on the surface $H_0 = -\frac{1}{2}$ agrees up to the scaling factor r^{-1} with the one induced by K_0 on the surface $K_0 = 1$. Note, however, that the "energy" surface $K_0 = 1$ is still noncompact[4] so that the regularization is not yet complete. The main step leading toward spinor regularization consists in the replacement of K_0 by $J = K_0 \circ \pi$. On account of relations (2.12) we find

$$J = \tfrac{1}{2}[\langle z, z\rangle + \langle w, w\rangle]. \tag{2.18}$$

[4] Actually, the topological character of this surface is $S^2 \times \mathbb{R}^3$ (see [32]).

Thus we replace the Hamiltonian system $(K_0, d\theta_0, s)$ on $T^*(\mathbb{R}^3\backslash\{0\})$ by the Hamiltonian system $(J, d\theta, s)$ (θ defined in (2.5)) which at first is regarded as being restricted to that portion $(I^{-1}(0))'$ of the surface $I^{-1}(0)$ on which $z \neq 0$. Reducing out the $U(1)$-action on the manifold $(I^{-1}(0))'$ we recover the first Hamiltonian system from the second. In fact, formulae (2.7) and (2.12) show that such a reduction is accomplished explicitly by the map π.

In order to complete the regularization of the Kepler problem we have to treat the point $z=0$ on the same footing as all the other points. This is achieved simply by extending the domain of definition of K_0 to all points $(z, w) \neq (0, 0)$. (The reason why we exclude the origin of $\mathbb{C}^4$ will become clear later.) The phase space of the regularized Kepler problem is therefore the quotient manifold $I^{-1}(0)/U(1)$, where $I^{-1}(0)$ is viewed as a submanifold of $\mathbb{C}^4\backslash\{0\}$.

In Section 3 we shall obtain several realizations of this phase space and we shall make contact with previous work of Moser [17], Baumgarte [14], and Guillemin and Sternberg [16]. Here we only remark that there is a close connection between Penrose's conformal completion of Minkowski space [12] and our description of the compactification of a negative energy surface of the Kepler problem. Indeed, selecting a fixed spacelike hyperplane $\mathbb{R}^3$ in Minkowski space, a null line is uniquely determined by the following data: (i) a directional vector which can be identified with a point of S^2, and (ii) the point of intersection with the hyperplane $\mathbb{R}^3$. Thus the manifold of null lines of Minkowski space is identified with $S^2 \times \mathbb{R}^3$. On the other hand, our map π associates with each point $(z, w) \in (I^{-1}(0))'$ the point $((\mathbf{x} \circ \pi) \cdot (r \circ \pi)^{-1}, \mathbf{y})$ of $S^2 \times \mathbb{R}^3$ in such a way that two points are mapped into the same point of $S^2 \times \mathbb{R}^3$ precisely if they lie on the same orbit of $\mathbb{C}^*$ (=group of nonzero complex numbers acting on 4-tuples of complex numbers in an obvious way). Thus, the manifold of null lines of Minkowski space appears as quotient manifold $(I^{-1}(0))'/\mathbb{C}^*$.

If Minkowski space is completed by adjoining a null cone at infinity, the new manifold of null lines is $I^{-1}(0)/\mathbb{C}^*$ (i.e., again the restriction $z \neq 0$ is lifted). Since obviously

$$I^{-1}(0)/\mathbb{C}^* = (I^{-1}(0) \cap J^{-1}(1))/U(1),$$

we see that the manifold of null lines of Minkowski space and a surface of negative energy of the Kepler problem are diffeomorphic (before and after compactification). In both cases the compactified manifold is $S^2 \times S^3$. In fact, lifting the restriction $z \neq 0$ amounts to replacing $\mathbb{R}^3$ in $S^2 \times \mathbb{R}^3$ by S^3. Here, $\mathbb{R}^3$ is first viewed as $S^3\backslash\{\text{one point}\}$ via a stereographic projection and then the missing point is filled in (compare Moser [17, p. 629] and Penrose [12, p. 354]). Here $I^{-1}(0)/U(1)$ is not only the appropriate phase space for the regularized Kepler Hamiltonian but also for a wide class of

Hamiltonians that are representable as suitable perturbations of the Kepler Hamiltonian. More specifically, we consider the class of Hamiltonians

$$H = \tfrac{1}{2}\,|\mathbf{y}|^2 - (K/r) + \varepsilon\mathfrak{B}(\mathbf{x}, \mathbf{y}, \varepsilon), \tag{2.19}$$

where $\mathfrak{B}$ is real analytic in $\mathbf{x}, \mathbf{y}, \varepsilon$ (ε sufficiently small) and at most linear in $\mathbf{y}$. Going through the same series of transformations as with the Kepler problem, we find that the vector field induced by H on the surface $H = -\frac{1}{2}$ differs from the one induced by

$$K = \tfrac{1}{2}(|\mathbf{y}|^2 + 1)\, r + \varepsilon r\mathfrak{B}(\mathbf{x}, \mathbf{y}, \varepsilon) \tag{2.20}$$

on $K = 1$ just by a scaling factor r^{-1}. Since $\mathfrak{B}$ is at most linear in $\mathbf{y}$ we may write

$$\mathfrak{B} = \mathfrak{B}_0(\mathbf{x}, \boldsymbol{\varepsilon}) + \boldsymbol{\mathfrak{B}}_1(\mathbf{x}, \varepsilon) \cdot \mathbf{y}$$

and we obtain

$$\begin{aligned} K \circ \pi = \tfrac{1}{2}[\langle z, z\rangle + \langle w, w\rangle] + \varepsilon r\mathfrak{B}_0(\mathbf{x}, \varepsilon) \\ + \varepsilon\boldsymbol{\mathfrak{B}}_1(\mathbf{x}, \varepsilon) \cdot \operatorname{Im}\langle w, \boldsymbol{\sigma} z\rangle, \end{aligned} \tag{2.21}$$

where $\mathbf{x} = \langle z, \boldsymbol{\sigma} z\rangle$ and $r = \langle z, z\rangle$. Obviously, this Hamiltonian is real analytic on $\mathbb{C}^4$ and it also defines such a Hamiltonian on $I^{-1}(0)/U(1)$.

Before closing this section we shall seek an expression for the angular momentum $\mathbf{L} \circ \pi$, where $\mathbf{L} = \mathbf{x} \times \mathbf{y}$. For this purpose we shall make use of the relation ($\times$ = cross product)

$$\langle w, v\rangle\langle u, \boldsymbol{\sigma} z\rangle + i\langle w, \boldsymbol{\sigma} v\rangle \times \langle u, \boldsymbol{\sigma} z\rangle = 2\langle u, \boldsymbol{\sigma} v\rangle\langle w, z\rangle - \langle w, \boldsymbol{\sigma} v\rangle\langle u, z\rangle \tag{2.22}$$

which holds for any elements $u, v, w, z \in \mathbb{C}^2$. This relation can be derived from (2.9) and the well-known relation

$$(\mathbf{a} \cdot \boldsymbol{\sigma})\, \boldsymbol{\sigma} = \mathbf{a}\sigma_0 + i(\boldsymbol{\sigma} \times \mathbf{a}) \tag{2.23}$$

which is valid for all 3-vectors $\mathbf{a}$. Here σ_0 is the 2×2 unit matrix. Indeed, if we apply both sides of (23) to $v \in \mathbb{C}^2$ and set $\mathbf{a} = \langle u, \boldsymbol{\sigma} z\rangle$, we obtain

$$(\langle u, \boldsymbol{\sigma} z\rangle \cdot \boldsymbol{\sigma})\, \boldsymbol{\sigma} v = \langle u, \boldsymbol{\sigma} z\rangle\, v + i(\boldsymbol{\sigma} v \times \langle u, \boldsymbol{\sigma} z\rangle). \tag{2.24}$$

Replacing w by $\boldsymbol{\sigma} v$ in (2.9), however, yields a relation which allows us to replace the left side of (2.24) by

$$2z\langle u, \boldsymbol{\sigma} v\rangle - \langle u, z\rangle\, \boldsymbol{\sigma} v.$$

If the resulting relation is multiplied from the left by $w^\dagger = (\bar{w}_1, \bar{w}_2)$, relation

(2.22) results. In order to obtain the desired expression for the angular momentum, we set $u = v = z$ in (2.22) and simultaneously replace w by iw. Thus, we obtain

$$\langle w, \boldsymbol{\sigma} z\rangle \times \langle z, \boldsymbol{\sigma} z\rangle = -i\langle z, \boldsymbol{\sigma} z\rangle\langle w, z\rangle + i\langle w, \boldsymbol{\sigma} z\rangle\langle z, z\rangle.$$

Equating imaginary parts on both sides yields

$$\langle z, \boldsymbol{\sigma} z\rangle \times \operatorname{Im}\langle w, \boldsymbol{\sigma} z\rangle = \langle z, \boldsymbol{\sigma} z\rangle \operatorname{Re}\langle w, z\rangle - \langle z, z\rangle \operatorname{Re}\langle w, \boldsymbol{\sigma} z\rangle. \tag{2.25}$$

Hence on account of (2.2), (2.8), and (2.25),

$$\mathbf{L} \circ \pi = -\operatorname{Re}\langle w, \boldsymbol{\sigma} z\rangle \qquad \text{on} \quad I^{-1}(0). \tag{2.26}$$

3. Realizations of the Space $I^{-1}(0)/U(1)$

In the last section we saw that a regularization of the Kepler problem is achieved by lifting the restriction $z \neq 0$ and regarding $I^{-1}(0)$ as a submanifold of $\mathbb{C}^4\backslash\{0\}$. This will be our convention for the remainder of this paper.

We also saw in the last section that $I^{-1}(0)/U(1)$ is not only the appropriate phase space for the Kepler problem but also for a wide class of problems that can be described in terms of a suitable perturbation of the Kepler problem. Therefore it may be important to have available alternate realizations of this symplectic manifold.

In what follows we shall show that $I^{-1}(0)/U(1)$ can be realized as an orbit of the Lie group $SU(2, 2)$ in the dual $su(2, 2)^*$ of its Lie algebra. This will give some geometrical insight into the purely algebraic constructions of Baumgarte [14] who (stimulated by Barut's quantum mechanical constructions [15]) exhibits the regularized Kepler Hamiltonian as member of the Lie algebra $so(4, 2) \cong su(2, 2)$ ($\cong$ means "isomorphic to").

There is another way of realizing the phase space $I^{-1}(0)/U(1)$, namely as T^+S^3; that is to say, as the cotangent bundle of the 3-sphere from which the zero-section has been removed. The fact that T^+S^3 is symplectomorphic to an orbit of the group $SO_0(4, 2)$ (remember $SO_0(4, 2)$ is the identity component of $SO(4, 2)$ which is doubly covered by $SU(2, 2)$) in $so(4, 2)^* \cong su(2, 2)^*$ was previously shown by Guillemin and Sternberg [16, pp. 174–178] in connection with their discussion of Moser's work [17]. In [17] Moser shows (among other things) that the flow of the Kepler Hamiltonian on a surface of fixed negative energy is diffeomorphic to the geodesic flow on the unit tangent bundle of the 3-sphere. We shall rederive both results starting from the phase space $I^{-1}(0)/U(1)$ of the "twistor"-regularization described previously. We feel that our approach will not only shed new light on the relationship between the KS regularization and Moser's regularization of the Kepler problem but we hope to convince the reader that it is particularly well suited for perturbations of the Kepler problem such as the three-dimensional restricted three body problem for high values of the Jacobian constant. This problem will be discussed in detail in Sections 5 and 6.

We start out by proving that $U(2,2)$ acts symplectically on $\{\mathbb{C}^4\backslash\{0\}, d\theta\}$, where θ was defined in (2.5). To this end we write θ in the form

$$\begin{aligned}\theta &= (1/i)(\bar{w}_1 dz_1 + \bar{w}_2 dz_2 - w_1 d\bar{z}_1 - w_2 d\bar{z}_2)\\ &= (1/i)[\langle w, dz\rangle + \langle z, dw\rangle].\end{aligned} \tag{3.1}$$

Here, we have identified two cohomologous 1-forms (i.e., two forms differing by an exact form). This convention will be in force throughout this paper since obviously the symplectic structure defined by $d\theta$ only depends on the cohomology class of θ. Next we subject (z, w) to the transformation

$$\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = \mathfrak{C}\begin{pmatrix} z\\ w\end{pmatrix} \qquad \text{with inverse} \qquad \begin{pmatrix} z\\ w\end{pmatrix} = \mathfrak{C}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \tag{3.2}$$

where $\mathfrak{C}$ is the matrix

$$\mathfrak{C} = 2^{-1/2}\begin{pmatrix}\sigma_0 & \sigma_0\\ \sigma_0 & -\sigma_0\end{pmatrix} \tag{3.3}$$

with σ_0 being the two-dimensional unit matrix. We obtain

$$\begin{aligned}\theta_{(\eta,\zeta)} &= (1/i)[\langle\eta, d\eta\rangle - \langle\zeta, d\zeta\rangle] = (1/i)(\eta^\dagger, \zeta^\dagger)\,\mathfrak{J}\begin{pmatrix} d\eta\\ d\zeta\end{pmatrix}\\ &= \operatorname{Im}[\langle\eta, d\eta\rangle - \langle\zeta, d\zeta\rangle].\end{aligned} \tag{3.4}$$

Here we think of η and ζ as columns and of $(\eta^\dagger, \zeta^\dagger)$ as row $(\bar{\eta}_1, \bar{\eta}_2, \bar{\zeta}_1, \bar{\zeta}_2)$. Moreover $\mathfrak{J}$ is the 4×4 matrix

$$\mathfrak{J} = \begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix}. \tag{3.5}$$

The right side of (3.4) is manifestly invariant under the obvious action $\binom{\eta}{\zeta} \to U\binom{\eta}{\zeta}$ of $U(2,2)$ on $\mathbb{C}^4$. In the following we identify the corresponding Lie algebra $u(2,2)$ with the one formed by all matrices $\mathfrak{A}$ having the property that $\mathfrak{J}\mathfrak{A}$ is a Hermitian matrix. The appropriate bracket for this Lie algebra is

$$[\mathfrak{A}, \mathfrak{B}] = -i(\mathfrak{A}\mathfrak{B} - \mathfrak{B}\mathfrak{A}). \tag{3.6}$$

In turn we identify the dual $u(2,2)^*$ with the same space of matrices which amounts to endowing this space with the inner product

$$\langle \mathfrak{A}, \mathfrak{B} \rangle = \mathrm{tr}(\mathfrak{J}\mathfrak{A}\mathfrak{J}\mathfrak{B}).$$

In the following we replace $U(2,2)$ by its four-fold covering group $\tilde{U}(2,2) = U(1) \times SU(2,2)$, possessing the same Lie algebra $u(2,2)$ as $U(2,2)$. Correspondingly $u(2,2)$ and $u(2,2)^*$ decompose into an orthogonal sum. For example,

$$u(2,2)^* = u(1)^* \oplus su(2,2)^*,$$

where $u(1)^*$ is spanned by the 4×4 unit matrix $\mathbb{1}$, and $su(2,2)^*$ is spanned by those elements of $u(2,2)^*$ having vanishing trace.

We also introduce the Poisson bracket corresponding to $id\theta$: Let f, g be two real-valued functions on $\mathbb{C}^4\backslash\{0\}$. We define

$$\{f, g\} = (\nabla f)^\dagger \mathfrak{J} \nabla g - (\nabla g)^\dagger \mathfrak{J} \nabla f = \mathrm{tr}\{[\bar{\nabla} f (\nabla g)^{\mathrm{T}} - (\bar{\nabla} g)(\nabla f)^{\mathrm{T}}]\, \mathfrak{J}\}, \quad (3.7)$$

where $(\nabla f)^{\mathrm{T}} = (\partial f/\partial \eta_1, \partial f/\partial \eta_2, \partial f/\partial \zeta_1, \partial f/\partial \zeta_2)$ and ∇f is the corresponding column vector. Similarily, $(\nabla f)^\dagger = (\bar{\nabla} f)^{\mathrm{T}} = (\partial f/\partial \bar{\eta}_1, \partial f/\partial \bar{\eta}_2, \partial f/\partial \bar{\zeta}_1, \partial f/\partial \bar{\zeta}_2)$ and $\bar{\nabla} f$ is the corresponding column vector.

The time rate of change of g under the flow induced on $\mathbb{C}^4\backslash\{0\}$ by f is given by

$$\dot{g} = i\{f, g\}. \quad (3.8)$$

It follows that the moment of the action of $U(2,2)$ on $\mathbb{C}^4\backslash\{0\}$ is

$$\psi(\eta, \zeta) = \mathfrak{J} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} (\eta^\dagger, \zeta^\dagger) = \begin{pmatrix} \eta\eta^\dagger & \eta\zeta^\dagger \\ -\zeta\eta^\dagger & -\zeta\zeta^\dagger \end{pmatrix}. \quad (3.9)$$

In more explicit language this means that, given an element $\mathfrak{A} \in su(2,2)$, the Hamiltonian

$$\psi_{\mathfrak{A}}(\eta, \zeta) = \langle \psi(\eta, \zeta), \mathfrak{A} \rangle = (\eta^\dagger, \zeta^\dagger)\, \mathfrak{J}\mathfrak{A} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \quad (3.10)$$

induces the flow $\binom{\eta}{\zeta} \to \exp(i\,\mathfrak{A}s)\binom{\eta}{\zeta}$ on $\mathbb{C}^4$. The equivariance of the moment finds its expression in the formula $(U \in U(2,2))$

$$\psi(\eta', \zeta') = (U^\dagger)^{-1}\, \psi(\eta, \zeta)\, U^\dagger \qquad \text{for} \quad \begin{pmatrix} \eta' \\ \zeta' \end{pmatrix} = U \begin{pmatrix} \eta \\ \zeta \end{pmatrix}. \quad (3.11)$$

Its infinitesimal counterpart is

$$\{\psi_{\mathfrak{A}}, \psi_{\mathfrak{B}}\} = -i\psi_{[\mathfrak{A}, \mathfrak{B}]}. \quad (3.12)$$

This formula is also verified directly by substituting (3.10) into the second expression for the Poisson bracket given in (3.7). The moment I of the $U(1)$-action that was already defined in (2.6) assumes the following form in the new variables:

$$I = \tfrac{1}{2}\psi_1(\eta, \zeta) = \tfrac{1}{2}[\langle \eta, \eta \rangle - \langle \zeta, \zeta \rangle]. \tag{3.13}$$

It is itself invariant under $U(2, 2)$. In fact $2I$ is the quadratic form of $\mathbb{C}(2, 2)$ so that if $g(\ ,\ )$ denotes the corresponding inner product, then $2I = g(a,\ )$, where we have set $a = (\eta, \zeta)$. Note that $I^{-1}(0)$ actually lies in the open submanifold $\{(\eta, \zeta): \eta \neq 0, \zeta \neq 0\}$ of $\mathbb{C}^4 \backslash \{0\}$.

We turn now to the proof of the fact that $I^{-1}(0)/U(1)$ is symplectomorphic to a certain orbit of $SU(2, 2)$ in $su(2, 2)^*$. For this purpose we first generalize the situation at hand and consider more generally a Hamiltonian $G = H \times K$-space $\{M, \omega\}$ with equivariant moment

$$\psi: M \to \mathfrak{g}^* = \mathfrak{g}^* \oplus \mathfrak{k}^*.$$

(Here, $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{k}$ are the Lie algebras associated with the groups G, H, and K; $\mathfrak{g}^*$, $\mathfrak{h}^*$, and $\mathfrak{k}^*$ are their duals; and ω is the 2-form defining the symplectic structure on M.) Let ψ_H, ψ_K be the components of ψ in $\mathfrak{h}^*$ and $\mathfrak{k}^*$, respectively. Observe that ψ_K is equivariant with respect to K and invariant with respect to H; an analogous statement is valid for ψ_H. In this situation the following theorem is valid (for background material see [6–11, 16]).

THEOREM 1. *Assume* (i) $\mu_0 \in \mathfrak{k}^*$ *is a regular value of* ψ_K *and that its isotropy subgroup* K_{μ_0} *in* K *acts freely and properly on* $\psi_K^{-1}(\mu_0)$ *so that* $\psi_K^{-1}(\mu_0)/K_{\mu_0}$ *is a symplectic manifold.* (ii) *The isotropy group of* μ_0 *in* G: $G_{\mu_0} = H \times K_{\mu_0}$ *acts transitively on* $\psi_K^{-1}(\mu_0)$. (iii) *For some* $a_0 \in \psi_K^{-1}(\mu_0)$, *let* $\lambda_0 = \psi_H(a_0)$. *Assume that* λ_0 *is a regular value of* ψ_H *and* K *acts transitively on* $\psi_H^{-1}(\lambda_0)$. *Then* ψ_H *defines a symplectomorphism between the spaces* $\psi_K^{-1}(\mu_0)/K_{\mu_0}$ *and* $\mathfrak{O}_H(\lambda_0)$ *(orbit of* H *through* λ_0 *in* $\mathfrak{h}^*$*) which intertwines the action of the group H. In particular, if* $X_{\mathfrak{g}^*}$, $Y_{\mathfrak{h}^*}$ *are the vector fields induced in* $\mathfrak{h}^*$ *by the one parameter groups* $\exp(Xt)$, $\exp(Yt)$, *where* $X, Y \in \mathfrak{h}$, *and if* $\omega_\lambda^{\mu_0}$ *is the canonical two form defining the symplectic structure on* $\mathfrak{O}_H(\lambda_0)$ *(discovered by Kirillov* [7]*), i.e.,*

$$\omega_\lambda^{\mu_0}(X_{\mathfrak{h}^*\lambda}, Y_{\mathfrak{h}^*\lambda}) = -\lambda([X, Y]) \tag{3.14}$$

for $\lambda \in \mathfrak{O}_H(\lambda_0)$, *then*

$$\psi_H^* \omega^{\mu_0} = \omega|_{\psi_K^{-1}(\mu_0)}. \tag{3.15}$$

We postpone the proof of this theorem to an appendix. Instead we apply it immediately to the situation at hand: Here $G = \tilde{U}(2, 2)$, $K = U(1)$, $H = SU(2, 2)$, $\psi_K = I$, $\mu_0 = 0$, and the application of Theorem 1 will give

THEOREM 2. *We have that $I^{-1}(0)/U(1)$ is symplectomorphic to the orbit of all elements $\mathfrak{A}$ of $su(2, 2)^*$ having the property that $J\mathfrak{A}$ is a one-dimensional orthogonal projection of the Hilbert space $\mathbb{C}^4$ (equipped with the norm $2J$, J given in* (2.18)).

Proof. Condition (i) of Theorem 1 is obviously satisfied. In order to check condition (ii) we have to show that $U(2, 2)$ acts transitively on $I^{-1}(0)$. This follows either by Witt's theorem or directly as follows: To $a = (\eta, \zeta)$ we associate $b = J^{-1}(\eta, -\zeta)$, where we think of J as being expressed in the variables η, ζ

$$J = \tfrac{1}{2}\psi_3(\eta, \zeta) = \tfrac{1}{2}[\langle \eta, \eta \rangle + \langle \zeta, \zeta \rangle]. \tag{3.16}$$

Notice that on $\mathbb{C}^4 \backslash \{0\}$, b is well defined. Setting $e = \frac{1}{2}(a + b)$, $f = \frac{1}{2}(a - b)$, we have on $I^{-1}(0)$, that $g(e, e) = -g(f, f) = 1$ and $g(e, f) = 0$. Choose vectors $e_0, f_0 \in \mathbb{C}(2, 2)$ so that (e, e_0, f, f_0) is an orthonormal basis in $\mathbb{C}(2, 2)$. Finally, let $U \in U(2, 2)$ be the matrix whose columns are the members of this basis. Then U carries the vector $e_1 = (1, 0, 1, 0)$ into a. Here our convention to regard $I^{-1}(0)$ as a submanifold of $\mathbb{C}^4 \backslash \{0\}$ rather than of $\mathbb{C}^4$ became essential.

Finally we check condition (iii) of Theorem 1 by setting $a_0 = e_1$ so that $\lambda_0 = \psi_H(e_1)$. Since ψ and ψ_H agree on $I^{-1}(0)$, ψ_H is given in (3.9) and one easily obtains $\psi_H^{-1}(\lambda_0) = (e^{i\alpha}, 0, e^{i\alpha}, 0)_{\alpha \in \mathbb{R}}$, so that all assumptions of Theorem 1 are satisfied. The nature of the orbit as described in Theorem 2 follows immediately from an inspection of formula (3.9).

Our next goal is to introduce suitable coordinates on this orbit. Combining the 3-vector of Pauli matrices (2.3) with σ_0 we obtain the 4-vector of Pauli matrices $\sigma = (\sigma_0, \boldsymbol{\sigma})$. With its help we construct an orthogonal basis of $u(2, 2)$ by setting

$$\mathfrak{M} = \tfrac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{N} = \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \quad \mathfrak{Q} = \tfrac{1}{2}\begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \mathfrak{P} = \frac{1}{2i}\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}. \tag{3.17}$$

The Hamiltonians inducing the corresponding one-parameter flows in $\mathbb{C}^4$ are denoted by the corresponding latin letter so that

$$M = \psi_{\mathfrak{M}}, \qquad N = \psi_{\mathfrak{N}}, \qquad Q = \psi_{\mathfrak{Q}}, \qquad P = \psi_{\mathfrak{P}}. \tag{3.18}$$

The following relations are immediate consequences of these definitions:

$$J = M_0 - N_0, \qquad I = M_0 + N_0, \tag{3.19}$$

$$\mathbf{M} = \tfrac{1}{2}\langle \eta, \boldsymbol{\sigma}\eta\rangle, \qquad \mathbf{N} = -\tfrac{1}{2}\langle \zeta, \boldsymbol{\sigma}\zeta\rangle, \tag{3.20}$$

$$\mathbf{Q} = \operatorname{Re}\langle \eta, \boldsymbol{\sigma}\zeta\rangle, \qquad \mathbf{P} = \operatorname{Im}\langle \eta, \boldsymbol{\sigma}\zeta\rangle, \tag{3.21}$$

$$Q_0 = \operatorname{Re}\langle \eta, \zeta\rangle, \qquad P_0 = \operatorname{Im}\langle \eta, \zeta\rangle. \tag{3.22}$$

There exist several relations between these functions. For instance, replacing z by η and $u = w$ first by ζ and then by $-i\zeta$ in relation (2.11) yields

$$\mathbf{Q}^2 + Q_4^2 = J^2 - I^2, \qquad \mathbf{P}^2 + P_4^2 = J^2 - I^2. \tag{3.23}$$

Here, for reasons that become clear shortly, we have set

$$Q_4 = -P_0, \qquad P_4 = Q_0. \tag{3.24}$$

Making the same substitutions also in (2.25) yields

$$\begin{aligned} \tfrac{1}{2}(I + J)\,\mathbf{Q} &= \mathbf{M}P_4 + (\mathbf{M} \times \mathbf{P}), \\ -\tfrac{1}{2}(I + J)\,\mathbf{P} &= \mathbf{M}Q_4 + (\mathbf{M} \times \mathbf{Q}). \end{aligned} \tag{3.25}$$

Using also that $\mathbf{M}^2 = (\frac{1}{2}(I + J))^2$ (see formula preceding (4.5)), we deduce from (3.25) that

$$\mathbf{P} \cdot \mathbf{Q} + P_4 Q_4 = 0 \tag{3.26}$$

whenever $I + J \neq 0$. In particular this is the case on the surface $I^{-1}(0)$.

Now let $(\eta, q) = (\eta_1, \eta_2, \mathbf{q}, q_4)$ denote global coordinates on $\mathbb{C}^2 \times \mathbb{R}^4$. By setting $\eta \circ F = \eta$, $q \circ F = J^{-1}\tilde{Q}$, where $\tilde{Q} = (\mathbf{Q}, Q_4)$, we define a smooth (i.e., real analytic) map $F: [\mathbb{C}^2 \backslash \{0\}] \times \mathbb{C}^2 \to [\mathbb{C}^2 \backslash \{0\}] \times \mathbb{R}^4$. The following Lemma gives a deeper insight into the nature of the manifold $I^{-1}(0)$.

LEMMA. *If the map F is restricted to $I^{-1}(0)$, it becomes a diffeomorphism onto $[\mathbb{C}^2 \backslash \{0\}] \times S^3$, where $S^3 = \{q\colon \|q\| = 1\}$, $\|q\| = (\mathbf{q}^2 + q_4^2)^{1/2}$.*

Proof. Let $G\colon [\mathbb{C}^2 \backslash \{0\}] \times \mathbb{R}^4 \to [\mathbb{C}^2 \backslash \{0\}] \times \mathbb{C}^2$ be the smooth map defined by $\eta \circ G = \eta$, $(\zeta \circ G)(\eta, q) = (\mathbf{q} \cdot \boldsymbol{\sigma})\,\eta - iq_4\eta$. Then $(\zeta \circ G \circ F)(\eta, \zeta) = J^{-1}[(\mathbf{Q} \cdot \boldsymbol{\sigma})\,\eta - iQ_4\eta]$. If we set $u = w = \eta$, $z = \zeta$ in relation (2.10) we obtain

$$\begin{aligned} \mathbf{Q} \cdot \boldsymbol{\sigma}\eta &= \operatorname{Re}\langle \zeta, \boldsymbol{\sigma}\eta\rangle\, \boldsymbol{\sigma}\eta = \eta\langle \zeta, \eta\rangle + \zeta\langle \eta, \eta\rangle - \operatorname{Re}\langle \zeta, \eta > \eta \\ &= \langle \eta, \eta > \zeta + i\operatorname{Im}\langle \zeta, \eta > \eta \end{aligned}$$

so that $(\zeta \circ G \circ F)(\eta, \zeta) = J^{-1}(I + J)\,\zeta$. The right side of the last relation reduces to ζ on the surface $I^{-1}(0)$. Furthermore, we find on account of

(2.23) that $q \circ F \circ G = (J \circ G)^{-1}\langle \eta, \eta \rangle q$, where $(J \circ G)(\eta, q) = \frac{1}{2}[1 + \mathbf{q}^2 + q_4^2]\langle \eta, \eta \rangle$ so that $q \circ F \circ G$ reduces to q on S^3.

Combining this result with the observation that F and G leave η unaffected completes the proof of the Lemma.

COROLLARY. *We have that* $I^{-1}(0)/U(1)$ *is diffeomorphic to* $(\mathbb{R}^3 \backslash \{0\}) \times S^3$.

Proof. We first note that if we let $U(1)$ act trivially on S^3, then F becomes an isomorphism of $U(1)$-spaces. Let π' be the map π introduced in (2.1) with the difference, however, that in the present context the coordinates of $\mathbb{C}^2\backslash\{0\}$ are η and those of $\mathbb{R}^3\backslash\{0\}$ will be denoted by $\mathbf{m}$ so that $(\mathbf{m} \circ \pi')(\eta) = 2\mathbf{M}$ ($\mathbf{M}$ defined in (3.20)). Since π' defines a principal $U(1)$-bundle, it follows that the map $\tilde{\pi}: I^{-1}(0) \to (\mathbb{R}^3\backslash\{0\}) \times S^3$ defined by the formula: $\tilde{\pi} = (\pi' \times \mathrm{id}) \circ F|_{I^{-1}(0)}$ (where id is the identity on S^3) induces a diffeomorphism between the spaces $I^{-1}(0)/U(1)$ and $(\mathbb{R}^3\backslash\{0\}) \times S^3$.

This proof also shows that $\mathbf{m} \circ \tilde{\pi} = 2\mathbf{M}$, $q \circ \tilde{\pi} = J^{-1}\tilde{Q}$ is a set of coordinates on $I^{-1}(0)/U(1)$ (and hence also on our $SU(2,2)$-orbit in $su(2,2)^*$). Now $(\mathbb{R}^3\backslash\{0\}) \times S^3$ can be viewed as T^+S^3, i.e., as a cotangent bundle of the 3-sphere from which the zero-section has been removed. Indeed, the map $j: (\mathbb{R}^3\backslash\{0\}) \times S^3 \to \mathbb{R}^8$ defined by $q \circ j = q$ and $(p \circ j)(\mathbf{m}, q) = (-q_4\mathbf{m} + (\mathbf{q} \times \mathbf{m}),\ \mathbf{q} \cdot \mathbf{m})$ yields a parametrization of $T^+S^3 = \{p, q \in \mathbb{R}^8$: $\mathbf{q}^2 + q_4^2 = 1,\ \mathbf{q} \cdot \mathbf{p} + p_4 q_4 = 0,\ p \neq 0\}$. Finally, setting $\hat{\pi} = j \circ \tilde{\pi}$ we find $(p \circ \hat{\pi})(\eta, \zeta) = J^{-1}(-Q_4\, 2\mathbf{M} + 2(\mathbf{Q} \times \mathbf{M}),\ 2\mathbf{Q} \cdot \mathbf{M}) = (\mathbf{P}, P_4)$, where the last equality holds on account of (3.25). Combining this result with the obvious relation $q \circ \hat{\pi} = J^{-1}\tilde{Q}$ we find in view of (3.26):

$$\begin{aligned}\hat{\pi}^*(\mathbf{p} \cdot d\mathbf{q} + p_4 dq_4) &= \mathbf{P} \cdot d(J^{-1}\mathbf{Q}) + P_4 d(J^{-1}Q_4) \\ &= J^{-1}(\mathbf{P} \cdot d\mathbf{Q} + P_4 dQ_4).\end{aligned}$$

Replacing u by $-i\zeta$, z by η, and w by $d\zeta$ in (2.11) yields

$$\mathrm{Im}\langle \eta, \boldsymbol{\sigma}\zeta \rangle \,\mathrm{Re}\langle \eta, \sigma d\zeta \rangle = -\langle \eta, \eta \rangle \,\mathrm{Im}\langle \zeta, d\zeta \rangle + \mathrm{Re}\,\langle \eta, \zeta \rangle \,\mathrm{Im}\langle \eta, d\zeta \rangle.$$

Since $\mathbf{P} \cdot d\mathbf{Q}$ is obtained from the left side of the last expression by antisymmetrization in η, ζ, we find

$$\mathbf{P} \cdot d\mathbf{Q} = J[\mathrm{Im}\langle \eta, d\eta \rangle - \mathrm{Im}\langle \zeta, d\zeta \rangle] + P_4[\mathrm{Im}\langle \eta, d\zeta \rangle + \mathrm{Im}\langle d\eta, \zeta \rangle]$$

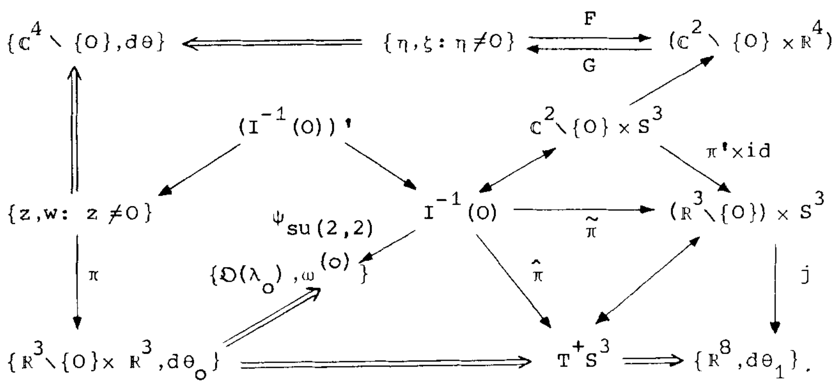
or finally

$$\hat{\pi}^*\theta_1|_{T^+S^3} = \theta|_{I^{-1}(0)}, \tag{3.27}$$

where θ_1 is the canonical 1-form of $\mathbb{R}^8$,

$$\theta_1 = (\mathbf{p} \cdot d\mathbf{q} + p_4 dq_4). \tag{3.28}$$

Combining this result with the fact that $\hat{\pi}$ by its very construction establishes a diffeomorphism between $I^{-1}(0)/U(1)$ and T^+S^3 shows that T^+S^3 with its canonical symplectic structure defined by the 2-form $d\theta_1|_{T^+S^3}$ provides us with a second realization of the symplectic space $I^{-1}(0)/U(1)$. Since also $\mathfrak{O}(\lambda_0)$ (as described in Theorem 2) constitutes such a realization, we essentially recover the result by Guillemin and Sternberg [16, pp. 174–178] who prove that T^+S^3 is symplectomorphic to a certain orbit of $SO_0(4, 2)$ in $so(4, 2)^*$. (Note that $SO_0(4, 2) \cong SU(2, 2)/(\mathbb{1}, -\mathbb{1})$ (see [13, p. 60)] and that the action (3.11) of $SU(2, 2)$ does not distinguish between U and $-U$.) We summarize our previous discussion in the following diagram:



All nonlabeled arrows represent imbeddings of submanifolds. A double arrow represents a symplectic imbedding (the symplectic structure being that of the larger space). Observe also that restrictions of maps are denoted by the same symbols as the unrestricted maps. We expect the left arrow at the bottom of the diagram to represent essentially Moser's transformation (see [17]). A detailed calulation that we omit yields

$$\begin{aligned} q &= [r(1+\mathbf{y}^2)]^{-1}(\mathbf{x}(1+\mathbf{y}^2) - 2(\mathbf{x}\cdot\mathbf{y})\,\mathbf{y}, -2(\mathbf{x}\cdot\mathbf{y})) \\ p &= (r\mathbf{y}, \tfrac{1}{2}(1-\mathbf{y}^2)\,r), \qquad \text{where} \quad r = |\mathbf{x}|. \end{aligned} \tag{3.29}$$

(See [32].)

The image in T^+S^3 is the complement of the manifold $S^3 \times \{\text{neg. } p_4\text{-axis}\}$. On it the map possesses an inverse which is described by the formulae

$$\mathbf{x} = (\|p\| + p_4)\,\mathbf{q} - q_4\mathbf{p}), \qquad \mathbf{y} = (\|p\| + p_4)^{-1}\mathbf{p}. \tag{3.30}$$

It is not difficult to verify that formulae (3.29) define a symplectic injection of $(\mathbb{R}^3\backslash\{0\})\times\mathbb{R}^3$ into T^+S^3. The same interpretation applies to Moser's transformation which in our notation takes the form

$$q=-(1+\mathbf{y}^2)^{-1}[2\mathbf{y}, 1-y^2], \quad p=(\tfrac{1}{2}(\mathbf{y}^2+1)\,x-(\mathbf{y}\cdot\mathbf{x})\,\mathbf{y}, -\mathbf{x}\cdot\mathbf{y}). \tag{3.31}$$

Both transformations carry Hamiltonian (2.17) into the Hamiltonian $\|p\|$ on the surface $\|p\|=1$ a flow whose orbits project onto the geodesic lines on S^3. Observe that if transformation (3.29) is followed by the symplectic S^3. Observe that if transformation (3.29) is followed by the symplectic automorphism of T^+S^3 that replaces q by $\|p\|^{-1}p$ and p by $-\|p\|\,q$, then transformation (3.31) results (compare [17]).

4. Preparations for the Study of the Perturbed Kepler Problem: Reducing out the Kepler Flow

In this section we first present a detailed study of the Kepler flow using our "twistor"-regularization and then proceed to construct the orbit manifold of the Kepler flow explicitly. It is well known that this manifold has topological character $S^2\times S^2$ (see, e.g., [3, 17]). By definition of the orbit manifold, the Keepler flow leaves this space pointwise fixed. Any perturbation of the Kepler problem, however, will via its normal form induce a flow on $S^2\times S^2$. We shall present a recipe for the explicit construction of this flow. It is intuitively obvious that the product of this reduced flow with the Kepler flow constitutes a first approximation to the perturbed Kepler problem under investigation. The precise nature of this approximation will not be discussed in general. For the case of the three-dimensional restricted three body problem in the limit of large values of the Jacobian constant, however, such a discussion is presented in Section 6.

We first note that the subgroup of $U(2, 2)$ leaving the regularized Kepler Hamiltonian (2.18) invariant is $U(2)\times U(2)$. Indeed, subjecting this Hamiltonian to the transformation of variables (3.2) will identify this Hamiltonian with the expression J as given in (3.16) and our contention becomes obvious. The action of $U(2)\times U(2)$ on $\mathbb{C}^4$ is described by the formulae

$$\eta'=U_1\eta, \qquad \zeta'=U_2\zeta, \qquad U_1, U_2\in U(2). \tag{4.1}$$

We note in passing that restricting (U_1, U_2) to $SU(2)\times SU(2)$ induces an element of $SO(4)$ on T^+S^3. Indeed, since $SU(2)$ is parametrised by S^3 by means of the association $a\in S^3\rightarrow a_4\sigma_0-i\mathbf{a}\cdot\boldsymbol{\sigma}\in SU(2)$, there is a linear correspondence $a'=Oa\in S^3$ such that

$$U_1(a_4\sigma_0-i\mathbf{a}\cdot\boldsymbol{\sigma})\,U_2=(a_4'\sigma_0-i\mathbf{a}'\cdot\boldsymbol{\sigma}). \tag{4.2}$$

Sandwiching both sides of (4.2) between η, ζ and taking imaginary and real parts we obtain, in view of definitions (3.21), (3.22), and (3.24)

$$\mathbf{Q}' \cdot \mathbf{a} + Q_4' a_4 = \mathbf{Q} \cdot \mathbf{a}' + Q_4 a_4', \qquad \mathbf{P}' \cdot \mathbf{a} + P_4' a_4 = \mathbf{P} \cdot \mathbf{a}' + P_4 a_4', \tag{4.3}$$

where the primed quantities on the left side are defined with the help of (η', ζ') (given in (4.1)) in the same way as the unprimed ones on the right side. Since O carries S^3 linearly into itself and since $SU(2) \times SU(2)$ is connected, it follows that $0 \in SO(4)$ and from (4.3) we conclude

$$\tilde{Q}' = O\tilde{Q}, \qquad \tilde{P}' = O\tilde{P}$$

($\tilde{Q} = (\mathbf{Q}, Q_4)$, etc.) (Compare also [30, p. 204; 33].)

Next we note that the "infinitesimal generators" of the $U(2) \times U(2)$ action are M and N given in (3.18). In particular, $\mathbf{M}, \mathbf{N}$ given in (3.20) generate the $SU(2) \times SU(2)$ action. We also introduce the vectors

$$\mathbf{L} = -\mathbf{M} - \mathbf{N} = \psi_{\mathfrak{L}}, \qquad \mathbf{A} = \mathbf{M} - \mathbf{N} = \psi_{\mathfrak{A}}, \tag{4.4}$$

where

$$\mathfrak{L} = \frac{1}{2}\begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \mathfrak{A} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \in su(2, 2).$$

(Here the notation is the same as in (3.18).) The vector $\mathbf{L}$ is none other than the angular momentum. This is easily checked by subjecting expression (2.26) to transformation (3.2). It will become clear below that $\mathbf{A}$ is the Laplace (= negative Lenz-) vector. (Of course, we may also transform back into the $\mathbf{x}$, $\mathbf{y}$ variables in which case we obtain $\mathbf{A} = [\mathbf{y} \times (\mathbf{y} \times \mathbf{x}) + \frac{1}{2}(\mathbf{y}^2 + 1)\,\mathbf{x}] \circ \pi$.)

Setting $z = u = \eta$ and $w = \zeta$ in (2.9) we find $\langle \eta, \boldsymbol{\sigma}\eta \rangle\, \boldsymbol{\sigma}\zeta = 2\eta\langle \eta, \zeta\rangle - \langle \eta, \eta\rangle\, \zeta$. Multiplying the last relation from the left by $-\frac{1}{4}\zeta^{\dagger}$ yields

$$\mathbf{M} \cdot \mathbf{N} = -\tfrac{1}{4}\langle \eta, \boldsymbol{\sigma}\eta\rangle\langle \zeta, \boldsymbol{\sigma}\zeta\rangle = \tfrac{1}{4}\langle \eta, \eta\rangle\langle \zeta, \zeta\rangle - \tfrac{1}{2}|\langle \eta, \zeta\rangle|^2.$$

Moreover, replacing z in (2.12a) by η and ζ, respectively, we find $\mathbf{M}^2 = \frac{1}{4}\langle \eta, \eta\rangle$, $\mathbf{N}^2 = \frac{1}{4}\langle \zeta, \zeta\rangle$. On $I^{-1}(0)$ these expressions simplify to

$$\mathbf{M} \cdot \mathbf{N} = \tfrac{1}{4}J^2 - \tfrac{1}{2}|\langle \eta, \zeta\rangle|^2, \qquad \mathbf{M}^2 = \mathbf{N}^2 = \tfrac{1}{4}J^2. \tag{4.5}$$

An immediate consequence of (4.5) are the relations

$$\mathbf{A}^2 = |\langle \eta, \zeta\rangle|^2, \qquad \mathbf{L}^2 + \mathbf{A}^2 = J^2, \qquad \mathbf{A} \cdot \mathbf{L} = 0. \tag{4.6}$$

which are all valid on $I^{-1}(0)$.

Expressing the position vector $\mathbf{x}$ and the momentum vector $\mathbf{y}$ (whose z-w expressions are given in (2.2) and (2.8)) in the ζ, η variables yields

$$\begin{aligned} \mathbf{x} &= \mathrm{A} + \mathbf{Q}, & r &= J + P_4, \\ \mathbf{y} &= (J + P_4)^{-1}\mathbf{P}, & |\mathbf{y}| &= (J + P_4)^{-1}(J - P_4). \end{aligned} \tag{4.7}$$

(By dropping the symbol π on $\mathbf{x}, \mathbf{y}$ we employ the usual abuse of notation according to which the same symbol can be used to denote an independent or a "dependent" variable, (i.e., a function) depending on the context.)

Observe that under the Kepler flow $\langle \zeta, \eta \rangle$ is multiplied by e^{is}. Thus, in terms of the variable

$$u = \arg\langle \zeta, \eta \rangle, \tag{4.8}$$

the Kepler flow is simply represented by the translation $u \to u + s$. Hence, varying u and keeping all other variables constant we expect the endpoint of $\mathbf{x}$ (given in (4.7)) to trace out an ellipse that lies in a plane perpendicular to $\mathbf{L}$, has one focus at the origin, and has center $\mathbf{A}$. In order to prove this contention we introduce the right-handed orthonormal basis

$$\mathbf{f}_1 = |\mathbf{A}|^{-1}\mathbf{A}, \qquad \mathbf{f}_2 = \mathbf{f}_3 \times \mathbf{f}_1, \qquad \mathbf{f}_3 = |\mathbf{L}|^{-1}\mathbf{L}. \tag{4.9}$$

Setting $w = v = \eta$, $u = z = \zeta$ in (2.22) and equating imaginary parts on both sides yields

$$-\tfrac{1}{2}\langle \eta, \boldsymbol{\sigma}\eta \rangle \times \langle \zeta, \boldsymbol{\sigma}\zeta \rangle = -\mathrm{Im}[\langle \zeta, \boldsymbol{\sigma}\eta \rangle \langle \eta, \zeta \rangle] = \mathbf{P}P_4 + \mathbf{Q}Q_4.$$

Setting

$$e = |\mathbf{A}|\, J^{-1}, \tag{4.10}$$

we find on account of (4.6), (4.8), (4.10), (3.22), and (3.24) that

$$P_4 = Je \cos u, \qquad Q_4 = Je \sin u, \tag{4.11}$$

and therefore

$$\mathbf{L} \times \mathbf{A} = Je(\mathbf{P} \cos u + \mathbf{Q} \sin u). \tag{4.12}$$

Furthermore, we obtain from (4.7)

$$r^2 = |\mathbf{x}|^2 = J^2 e^2 + 2\mathbf{A} \cdot \mathbf{Q} + \mathbf{Q}^2 = J^2 + 2JP_4 + P_4^2. \tag{4.13}$$

Combining (4.11) with relations (3.23) and (3.26) (where $I = 0$) we find

$$\begin{aligned} \boldsymbol{Q}^2 &= J^2 - Q_4^2 = J^2(1 - e^2 \sin^2 u), \\ \mathbf{P} \cdot \mathbf{Q} &= -P_4 Q_4 = -J^2 e^2 \cos u \sin u. \end{aligned} \tag{4.14}$$

Replacing $\mathbf{Q}^2$ and P_4^2 in (4.13) by their values, given in (4.14) and (4.11), respectively, yields

$$\mathbf{A} \cdot \mathbf{Q} = JP_4 = J^2 e \cos u. \tag{4.15}$$

Let

$$X_k = \mathbf{x} \cdot \mathbf{f}_k \qquad (k = 1, 2, 3) \tag{4.16}$$

be the components of $\mathbf{x}$ with respect to basis (4.9). Using (4.15) we find

$$X_1 = J(e + \cos u). \tag{4.17}$$

Since according to (4.9), (4.10), and (4.12)

$$\mathbf{f}_2 = |\mathbf{L}|^{-1}(\mathbf{P} \cos u + \mathbf{Q} \sin u),$$

we compute in view of (4.14)

$$X_2 = \mathbf{f}_2 \cdot \mathbf{x} = \mathbf{f}_2 \cdot (\mathbf{A} + \mathbf{Q}) = \mathbf{f}_2 \cdot \mathbf{Q} = |\mathbf{L}|^{-1} J^2 (1 - e^2) \sin u.$$

However, on account of (4.6) and (4.10),

$$\mathbf{L}^2 = J^2 - \mathbf{A}^2 = J^2(1 - e^2), \tag{4.18}$$

so that

$$X_2 = J(1 - e^2)^{1/2} \sin u. \tag{4.19}$$

Finally, a suitable application of (2.11) yields on $I^{-1}(0)$

$$X_3 = \mathbf{f}_3 \cdot \mathbf{x} = \mathbf{f}_3 \cdot (\mathbf{A} + \mathbf{Q}) = \mathbf{f}_3 \cdot \mathbf{Q} = -|\mathbf{L}|^{-1}[(\mathbf{M} + \mathbf{N}) \cdot \mathbf{Q}] = 0.$$

These formulae confirm our conjecture that by keeping all quantities except u constant, $\mathbf{x}$ traces out an ellipse in a plane perpendicular to $\mathbf{L}$ with one focus at 0 and center $\mathbf{A}$, u being the so-called "eccentric anomaly" (see Fig. 1).

In order to motivate our explicit construction of the orbit manifold of the Kepler problem, we first return to the perturbation problem described by Hamiltonian (2.21). We shall show that a careful preparation of this Hamiltonian will naturally lead us to a construction of the orbit manifold of the Kepler problem. The view of normal form theory that will be presented in the sequel was strongly influenced by Moser's lectures and by some correspondence with R. Churchill (see also [6, 18, 19, 35]).

Expressing the perturbation terms also in the new variables, Hamiltonian (2.21) becomes

$$K = J + \varepsilon \mathbf{P} \cdot \mathfrak{B}_1(\mathbf{A} + \mathbf{Q}, \varepsilon) + \varepsilon(J + P_4)\, \mathfrak{B}_0(\mathbf{A} + \mathbf{Q}, \varepsilon). \tag{4.20}$$

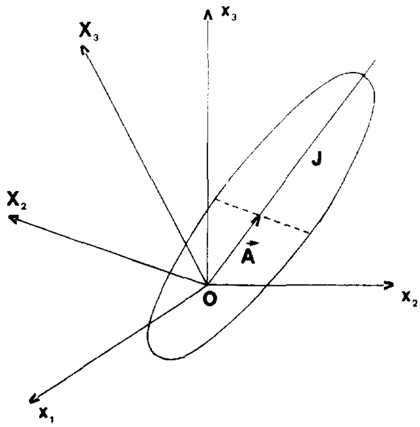


FIGURE 1

In the following we shall denote the perturbation term simply by $\varepsilon\mathfrak{B}_1(\varepsilon)$, keeping only in mind that $\mathfrak{B}_1(\varepsilon)$ can be expanded into a formal power series in ε whose coefficients lie in some Lie algebra $\mathfrak{K}$ so that K can be written in the form

$$K = J + \varepsilon\mathfrak{B}_1^{(0)} + \varepsilon^2\mathfrak{B}_1^{(1)} + \varepsilon^3\mathfrak{B}_1^{(2)} + \cdots . \tag{4.21}$$

In the case of Hamiltonian (4.20), $\mathfrak{K}$ can be taken to be the Lie algebra of real analytic functions in the "generators" of the Lie algebra $su(2, 2)$, equipped with Poisson bracket (3.7). In the following, however, $\mathfrak{K}$ is any Lie algebra and temporarily we write the Lie product like an ordinary one. (Of course, since the product is antisymmetric we must be careful about the order of the terms.) Here J is viewed as a linear operator over $\mathfrak{K}$ sending an element $S \in \mathfrak{K}$ into $JS = -SJ$ (this operator is usually denoted by ad J). The basic assumption underlying normal form theory is that $\mathfrak{K}$ splits under J,

$$\mathfrak{K} = \operatorname{Ker} J \oplus \operatorname{Ran} J. \tag{4.22}$$

In order to see that (4.22) is satisfied in the present context, we write all functions as power series in Q, P with coefficients depending on J, $\mathbf{M}$, $\mathbf{N}$, where relations (3.23) and (3.26) are used to simplify as much as possible. Then $\operatorname{Ker} J$ consists of "constant terms" only, whereas $\operatorname{Ran} J$ consists of power series in Q and P having no constant terms.

Returning to the general case, we denote the unique solution $S \in \operatorname{Ran} J$ of the equation $JS = \mathfrak{B}$ ($\mathfrak{B} \in \operatorname{Ran} J$) by $S = \Gamma\mathfrak{B}$ and caret and tilde are used to denote the projections onto $\operatorname{Ker} J$ and $\operatorname{Ran} J$ that correspond to the splitting in (4.22). Starting from (4.21), we first multiply from the left with $\exp \varepsilon S_1$, where $S_1 = \Gamma\tilde{\mathfrak{V}}_1^{(0)}$ and obtain

$$J + \varepsilon\mathfrak{W}^{(1)} + \varepsilon^2\mathfrak{V}_2^{(0)} + \mathcal{O}(\varepsilon^3), \tag{4.23}$$

where

$$\mathfrak{W}^{(1)} = \hat{\mathfrak{V}}_1^{(0)}, \qquad \mathfrak{V}_2^{(0)} = \tfrac{1}{2}S_1^2 J + S_1\mathfrak{V}_1^{(0)} + \mathfrak{V}_1^{(1)} = \tfrac{1}{2}S_1\tilde{\mathfrak{V}}_1^{(0)} + S_1\mathfrak{W}_1 + \mathfrak{V}_1^{(1)}. \tag{4.24}$$

Transforming (4.23) by multiplying from the left by $\exp(\varepsilon S_2)$ ($S_2 = \Gamma\tilde{\mathfrak{V}}_2^{(0)}$) yields

$$J + \varepsilon\mathfrak{W}^{(1)} + \varepsilon^2\mathfrak{W}^{(2)} + \mathcal{O}(\varepsilon^3),$$

where

$$\mathfrak{W}^{(2)} = \tfrac{1}{2}(S_1\tilde{\mathfrak{V}}_1^{(0)})\hat{} + \hat{\mathfrak{V}}_1^{(1)}. \tag{4.25}$$

Proceeding in this way, higher and higher order terms are "squeezed" into Ker J. Note, however, that since Ker J is itself a Lie algebra, we may let our series of transformations follow by an arbitrary transformation $\exp(\varepsilon T(\varepsilon))$, where $T(\varepsilon)$ is a polynomial in ε with coefficients in Ker J, without destroying what we have achieved in the first place, namely, that all terms of the transformed Hamiltonian up to a certain order lie in Ker J. For example, such an additional transformation will replace $\mathfrak{W}^{(2)}$ by the expression $\mathfrak{W}^{(2)} + T^{(0)}\mathfrak{W}^{(1)}$, which by an appropriate choice of $T^{(0)}$ may be simpler than $\mathfrak{W}^{(2)}$. This remark was communicated to the author by Moser who, after studying the author's work [1], pointed out that the Hamiltonian of [1, p. 1344, (37)] could be simplified by an appropriate choice of $T^{(0)}$.

In any case, what we learn from the previous discussion is that apparently there exists a canonical transformation which brings our Hamiltonian into the form

$$K = J + \varepsilon\mathfrak{W}(J, \mathbf{M}, \mathbf{N}, \varepsilon) + \mathcal{O}(\varepsilon^{n+1}). \tag{4.26}$$

Here, $\mathfrak{W}$ is a polynomial of degree $n-1$ in ε whose coefficients are real analytic functions of the seven variables J, $\mathbf{M}, \mathbf{N}$. Notice, however, that relations (4.5) exist between these variables. (The fact that we stay within the set of analytic functions by applying transformations of the type $\exp \varepsilon S$ can be seen for instance by realizing that $\exp \varepsilon S$ is nothing but the local flow of the Hamiltonian system $(S, d\theta, \varepsilon)$. This flow, however, is analytic in all variables for sufficiently small ε if S has this property.) Dropping the term $\mathcal{O}(\varepsilon^{n+1})$ in (4.26), we are left with a Hamiltonian, in the following called truncated Hamiltonian, whose flow can essentially be represented as the product of the Kepler flow with a suitable flow that $\mathfrak{W}$ induces on the orbit manifold $S^2 \times S^2$ of the Kepler flow. A precise description of the latter flow requires, however, that this orbit manifold be constructed explicitly. Obviously, it is obtained by reducing out the action of $U(1) \times U(1)$ on the submanifold $S^3 \times S^3 = \{(M_0, N_0) = (J/2, -J/2)\}$ of $\mathbb{C}^2 \times \mathbb{C}^2$. (Set $I = 0$ in (3.19).) Now, the symplectic space $\{C^4, d\theta\}$ is the product of the symplectic spaces $\{\mathbb{C}^2, d\theta_+\}$ $(\theta_+ = \text{Im}\langle \eta, d\eta \rangle)$ and the symplectic space $\{\mathbb{C}^2, -d\theta_-\}$ $(\theta_- = \text{Im}\langle \zeta, d\zeta \rangle)$ so that it is sufficient to construct the symplectic manifold $S^3/U(1)$, where $S^3 = \{\eta | M_0 = (J/2)\}$. In this situation Theorem 1 becomes applicable with

$$K = U(1), \qquad H = SU(2), \qquad \mu_0 = J/2. \tag{4.27}$$

Identifying $su(2)$ and $su(2)^*$ with $\mathbb{R}^3$ (the pairing being the usual dot product) we find that the moment ψ_H is simply $\mathbf{M} = \frac{1}{2}\langle \eta, \boldsymbol{\sigma}\eta \rangle$, i.e., again the Hopf map. Since $SU(2)$ acts on $\mathbb{R}^3$ via rotations, the reduced space becomes a sphere of radius $J/2$. This is also seen from relation (4.5).

According to Theorem 1, the symplectic form on S^2 is

$$\omega_{\mathbf{M}}(\mathbf{X}, \mathbf{Y}) = \mathbf{M} \cdot (\mathbf{X} \times \mathbf{Y}) \cdot (4/J^2). \tag{4.28}$$

Summarizing, we see that the orbit space of the Kepler flow can be realized as the product of two spheres of radius $J/2$ which we think imbedded in $\mathbf{M} - \mathbf{N}$ space, the latter being equipped with the product structure $\omega_{\mathbf{M}} + \omega_{\mathbf{N}}$, where $\omega_{\mathbf{M}}$ is defined in (4.28). Accordingly, truncated Hamiltonian (4.26) induces on $S^2 \times S^2$ a flow that is governed by the differential equations

$$\dot{\mathbf{M}} = \mathbf{M} \times \nabla_{\mathbf{M}}\mathfrak{W}, \qquad \dot{\mathbf{N}} = \mathbf{N} \times \nabla_{\mathbf{N}}\mathfrak{W}. \tag{4.29}$$

A more direct way of obtaining these differential equations starts from the "commutation" relations

$$\{M_1, M_2\} = -iM_3, \qquad (\{N_1, N_2\} = -iN_3) \tag{4.30}$$

and cyclic which are proved by setting $\mathfrak{A} = \mathfrak{M}_1, \mathfrak{B} = \mathfrak{M}_2$ $(\mathfrak{A} = \mathfrak{N}_1, \mathfrak{B} = \mathfrak{N}_2)$ in relations (3.12) and taking the commutation relations for the Pauli matrices

$$[\sigma_1, \sigma_2] = 2\sigma_3 \tag{4.31}$$

into account. (In turn (4.31) follows from (2.23) and the definition of the bracket given in (3.6).) Now, on account of (3.8) and (4.30), we compute

$$\begin{aligned} \dot{M}_1 = i\{K, M_1\} = i\nabla_{\mathbf{M}}K \cdot \{\mathbf{M}, M_1\} &= i\nabla_{M_2}K\{M_2, M_1\} + i\nabla_{M_3}K\{M_3, M_1\} \\ &= \nabla_{M_3}KM_2 - \nabla_{M_2}KM_3 = (\mathbf{M} \times \nabla_{\mathbf{M}}K)_1. \end{aligned}$$

This relation together with those obtained from it by a cyclic permutation of the subscripts make up the first of vector equations (4.29). The second is proved similarily.

In particular, periodic orbits of the truncated Hamiltonian correspond to critical points of $\mathfrak{W}$ on $S^2 \times S^2$, i.e., to points $(\mathbf{M}, \mathbf{N})$ at which the equations

$$\nabla_{\mathbf{M}} \mathfrak{W} = \lambda_1 \mathbf{M}, \qquad \nabla_{\mathbf{N}} \mathfrak{W} = \lambda_2 \mathbf{N} \tag{4.32}$$

are satisfied for a pair of "multipliers" $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. Suppose $(\mathbf{M}, \mathbf{N})$ is such a critical point. The corresponding periodic orbit must be a Kepler ellipse in a plane perpendicular to $\mathbf{L} = -\mathbf{M} - \mathbf{N}$ with center at $\mathbf{A} = \mathbf{M} - \mathbf{N}$ (see Fig. 1). The major axis of this ellipse is J which for $\varepsilon = 0$ equals one (since $K = 1$). For nonzero values of ε, however, this is no longer the case. Indeed, since $\mathfrak{W}$ in (4.32) does not only depend on $\mathbf{M}, \mathbf{N}$, but also on J, ε, the vectors $\mathbf{M}, \mathbf{N}$ (as well as the multipliers λ_1, λ_2) satisfying (4.32) will in general be functions of J, ε. Substituting these functions into $\mathfrak{W}(J, \mathbf{M}, \mathbf{N}, \varepsilon)$ will produce a function $\tilde{\mathfrak{W}}(J, \varepsilon)$ of J, ε only. The major axis of our ellipse as a function of ε is then obtained by solving the equation

$$J + \varepsilon \tilde{\mathfrak{W}}(J, \varepsilon) = 1$$

for J. By substituting this function in place of J back into $\mathbf{M}$ and $\mathbf{N}$, we finally obtain all elements of our Kepler ellipse as functions of ε.

We note in passing that points $(\mathbf{M}, \mathbf{M})$ correspond to circular orbits and points $(\mathbf{M}, -\mathbf{M})$ to collision orbits.

The "quotient" flow governed by Eqs. (4.29) determines the flow of the truncated Hamiltonian almost completely. It is possible to introduce a symplectic chart on $I^{-1}(0)/U(1)$ which is adapted to this situation. In order to exhibit this chart we first introduce the polar representation for these variables,

$$\eta_k = (Y_k)^{1/2} e^{i\alpha_k}, \qquad \zeta_k = (Z_k)^{1/2}\, e^{-i\beta_k} \qquad (k = 1, 2). \tag{4.33}$$

Substituting these expressions into (3.4), we find

$$\theta|_{I^{-1}(0)} = J d\chi + Y d\alpha + Z d\beta, \tag{4.34}$$

where we have set

$$\begin{gathered} Y_2 = Y, \qquad Z_2 = Z, \qquad \chi = \alpha_1 + \beta_1, \\ \alpha = \alpha_2 - \alpha_1, \qquad \beta = \beta_2 - \beta_1, \end{gathered} \tag{4.35}$$

so that we also have

$$Y_1 = J - Y, \qquad Z_1 = J - Z. \tag{4.36}$$

Notice that all these functions, and therefore also θ as given in (4.34), are defined on that portion of $I^{-1}(0)/U(1)$ for which Y_1, Y_2, Z_1, and Z_2 are all nonzero. Correspondingly, we have to put the following restrictions on Y and Z:

$$0 < Y < J, \qquad 0 < Z < J, \qquad J > 0. \tag{4.37}$$

It turns out that (Y, α) parametrizes the first and (Z, β) the second sphere of the orbit space $S^2 \times S^2$ everywhere except for the poles. In order to see this, we express **M**, **N** in the new coordinates

$$\begin{aligned}\mathbf{M} &= (\mathrm{Re}(\bar{\eta}_1 \eta_2), \mathrm{Im}(\bar{\eta}_1 \eta_2), \tfrac{1}{2}(Y_1 - Y_2)) \\ &= ((J - Y)^{1/2} Y^{1/2} \cos\alpha, (J - Y)^{1/2} Y^{1/2} \sin\alpha, J/2 - Y)\end{aligned} \tag{4.38}$$

$$\begin{aligned}\mathbf{N} &= (-\mathrm{Re}(\bar{\zeta}_1 \zeta_2), -\mathrm{Im}(\bar{\zeta}_1 \zeta_2), \tfrac{1}{2}(Z_2 - Z_1)) \\ &= (-(J - Z)^{1/2} Z^{1/2} \cos\beta, (J - Z)^{1/2} Z^{1/2} \sin\beta, Z - (J/2)).\end{aligned} \tag{4.39}$$

We can interpret (4.38) as a map which assigns to each point P': $(Y^{1/2}\cos\alpha,\ Y^{1/2}\sin\alpha, 0)$ a point P: (M_1, M_2, M_3) on the sphere. Geometrically this relationship can be described as follows: Let P'' be the point with the same M_1-M_2 coordinates as P' but lying on a sphere of radius $J^{1/2}$ whose center is the south pole S: $(0, 0, -J/2)$ of the first sphere. Then the ray SP'' intersects the first sphere in the point P (see Fig. 2).

Formula (4.39) allows a similar interpretation: The point P' in this case has coordinates $(-Z^{1/2}\cos\beta, Z^{1/2}\sin\beta, 0)$ and the south pole in the previous construction is replaced by the north pole.

Since J is the regularized Kepler Hamiltonian, the angle χ like u is just

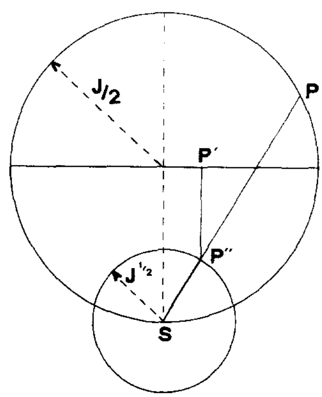


FIGURE 2

translated by s under the Kepler flow. In order to find a relation between the two angle-like variables, we substitute the right sides of (4.33) into (4.8) and obtain in view of (4.35) and (4.36)

$$u = \chi + \Xi, \tag{4.40a}$$

where

$$\Xi = \arg[(J - Y)^{1/2}(J - Z)^{1/2} + Y^{1/2}Z^{1/2}e^{i(\alpha+\beta)}]. \tag{4.40b}$$

Since the change in time of Ξ is completely determined by the flow on the orbit space $S^2 \times S^2$ and since furthermore on account of (4.34), (4.38), and (4.39),

$$\dot{\chi} = 1 + \varepsilon \left\{ \frac{\partial \mathfrak{W}}{\partial J} + \frac{1}{2}\left(\nabla_{\mathbf{M}} \mathfrak{W} \cdot \mathbf{M} + \nabla_{M_3} \mathfrak{W} \frac{J}{2} \right) \times \left(\frac{J}{2} + M_3 \right)^{-1} + \frac{1}{2}\left(\nabla_{\mathbf{N}} \mathfrak{W} \cdot \mathbf{N} - \nabla_{N_3} \mathfrak{W} \frac{J}{2} \right)\left(\frac{J}{2} - N_3 \right)^{-1} \right\}, \tag{4.41}$$

we see that Eqs. (4.29) and (4.41) determine the flow of truncated Hamiltonian (4.26) completely. In particular, for a periodic solution for which Ξ is constant, χ may be replaced by the eccentric anomaly u and in view of (4.32) we obtain

$$\dot{u} = 1 + \varepsilon[\partial \mathfrak{W}/\partial J + (\lambda_1 + \lambda_2)(J/4)]. \tag{4.42}$$

Formula (4.42) is easily seen to be valid also when the critical point happens to have M_3 coordinate $-J/2$ or N_3 coordinate $J/2$ although the right side of (4.41) is undefined at such points.

Let $(S^2)_0 \times ((S^2)^0)$ be a sphere from which the south pole (north pole) has been removed. By using rectangular coordinates for P' instead of polar coordinates, i.e., by setting

$$\xi_1 = (Y)^{1/2} e^{i\alpha}, \qquad \bar{\xi}_2 = Z^{1/2} e^{i\beta}, \tag{4.43}$$

we extend our previous parametrization to $(S^2)_0 \times (S^2)^0$. Fundamental 1-form (4.34) in the chart $(J, \chi, \xi = (\xi_1, \xi_2))$ takes the form

$$\theta|_{I=0} = J d\chi + \operatorname{Im}\langle \xi, \sigma_3 d\xi \rangle. \tag{4.44}$$

Here, σ_3 is the third Pauli matrix and $\xi \in \mathbb{C}^2$ is thought of as a column. According to (4.37), the range of the new chart is

$$|\xi_1|^2 < J, \qquad |\xi_2|^2 < J, \qquad J > 0. \tag{4.45}$$

Moreover, the third component of the angular momentum becomes

$$L_3 = -M_3 - N_3 = Y - Z = \langle \xi, \sigma_3 \xi \rangle. \tag{4.46}$$

It follows that L_3 induces on $S^2 \times S^2$ a flow in which both spheres rotate counterclockwise with unit angular speed. Restricted to the surface $L_3 = V$ this flow is made up of a one-parameter family of products of circles: $M_3 = M_3^0$, $N_3 = -V - M_3^0$, where

$$-J/2 < M_3^0 \leqslant -V + (J/2) \qquad \text{for} \quad 0 \leqslant V < J$$

and

$$-V - (J/2) < M_3^0 \leqslant J/2 \qquad \text{for} \quad -J < V \leqslant 0$$

(see Fig. 3).

Ignoring for the moment restrictions (4.45), we recognize from (4.44) that the group $U(1, 1)$ acts symplectically on $\mathbb{C}^2$ via the action $\xi \to U\xi$ ($U \in U(1, 1)$). Theorem 1 becomes applicable with $K = U(1)$ and $H = SU(1, 1)$. The moment ψ_K is L_3 whereas an identification of $su(1, 1)$ and $su(1, 1)^*$ with $\mathbb{R}^3$ allows us to view the moment of the $SU(1, 1)$ action as a map from $\mathbb{C}^2$ to $\mathbb{R}^3$ which takes the point ξ into the point

$$\mathbf{V} = (\mathrm{Re}(\xi_1 \bar{\xi}_2), \mathrm{Im}(\xi_1 \bar{\xi}_2), \tfrac{1}{2}(|\xi_1|^2 + |\xi_2|^2)). \tag{4.47}$$

For $V \neq 0$ this map takes the surface $L_3 = |\xi_1|^2 - |\xi_2|^2 = V$ into the upper sheet of the hyperboloid

$$V_3^2 - V_1^2 - V_2^2 = V^2/4, \qquad V_3 > 0. \tag{4.48}$$

(In our identification of $su(1, 1)^*$ with $\mathbb{R}^3$, the coadjoint action of $SU(1, 1)$ is the obvious action of the orthochronous Lorentz group $SO_0(2, 1)$. Accor-

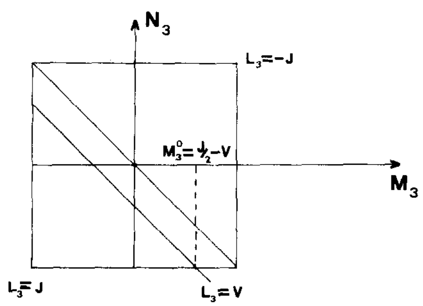


FIGURE 3

dingly, orbits of this action are hyperboloids of type (4.48).) The (Kirillov-) symplectic structure on hyperboloid (4.45) is defined by the two-form

$$\omega_{\mathbf{V}}(\mathbf{X}, \mathbf{Y}) = [(\mathbf{V} \times \mathbf{X}) \cdot \mathbf{Y}](2/V)^2. \tag{4.49}$$

Taking restrictions (4.45) into account also, we obtain

THEOREM 3. *Reducing out the flow of L_3 on a hypersurface $L_3 = V$ $(0 < |V| < J)$ of $S^2 \times S^2$ produces that portion of hyperboloid* (4.48) *which is defined by the inequalities*

$$\tfrac{1}{2}|V| \leqslant V_3 < J - \tfrac{1}{2}|V|. \tag{4.50}$$

The symplectic structure is given in (4.49).

This result attains significance whenever the perturbation term $\mathfrak{W}(J, \mathbf{M}, \mathbf{N}, \varepsilon)$ "commutes" with L_3 thereby defining a function $\mathfrak{W}(J, L_3, \mathbf{V}, \varepsilon)$ on the domain described by (4.48) and (4.50). Under this assumption the flow of truncated Hamiltonian (4.26) is governed by the differential equations

$$\dot{\mathbf{V}} = \nabla_{\mathbf{V}} \mathfrak{W} \times \hat{\mathbf{V}}, \tag{4.51}$$

where $\hat{\mathbf{V}} = (V_1, V_2, -V_3)$. Equations (4.51) can also be derived from the Poisson bracket relations of the components of $\mathbf{V}$ which with respect to the symplectic structure $i d\theta$ (θ given in (4.44)) take on the following form:

$$\{V_1, V_2\} = -iV_3, \qquad \{V_2, V_3\} = iV_1, \qquad \{V_3, V_1\} = iV_2.$$

Using (3.8), we find

$$\begin{aligned} \dot{V}_1 &= i\nabla_{\mathbf{V}} \mathfrak{W} \cdot \{\mathbf{V}, V_1\} = i\nabla_{V_2} \mathfrak{W} \{V_2, V_1\} + i\nabla_{V_3} \mathfrak{W} \{V_3, V_1\} \\ &= -V_2 \nabla_{V_3} \mathfrak{W} - V_3 \nabla_{V_2} \mathfrak{W} = (\nabla_{\mathbf{V}} \mathfrak{W} \times \hat{\mathbf{V}})_1, \end{aligned}$$

and similarily for the other components. Instead of working with the variables $\mathbf{V}$ and V, it is often preferable to work with the normalized variables

$$\mathbf{v} = J^{-1}\mathbf{V}, \qquad v = J^{-1}V = J^{-1}L_3. \tag{4.52}$$

In these normalized variables the region described by (4.48) and (4.50) is defined by the relations

$$v_3^2 - v_1^2 - v_2^2 = v^2/4, \quad \tfrac{1}{2}|v| \leqslant v_3 < 1 - (|v|/2) \quad (0 < |v| < 1) \tag{4.53}$$

and Eqs. (4.51) become

$$J\dot{\mathbf{v}} = (\nabla_{\mathbf{v}} \tilde{\mathfrak{W}} \times \hat{\mathbf{v}}), \tag{4.54}$$

where $\tilde{\mathfrak{W}}(J, v, \mathbf{v}, \varepsilon) = \mathfrak{W}(J, Jv, J\mathbf{v}, \varepsilon)$.

It is clear that to any critical point of the function $\tilde{\mathfrak{W}}(\mathbf{v})$ (we suppress the other variables momentarily) in region (4.53), truncated Hamiltonian (4.26) possesses a quasiperiodic solution with two frequencies. In order to investigate the nature of these quasiperiodic solutions we express the vectors **M** and **N** in terms of the variables **v** and v. Using the two auxiliary functions

$$C_{\pm} = (1 - v_3 \mp (v/2))^{1/2}(v_3 \mp (v/2))^{1/2}, \tag{4.55}$$

we find

$$\begin{aligned} M_1 + iM_2 &= JC_+ e^{i\alpha}, & M_3 &= J([(1-v)/2] - v_3), \\ -N_1 - iN_2 &= JC_- e^{-i\varphi} e^{i\alpha}, & -N_3 &= J([(1+v)/2] - v_3), \end{aligned} \tag{2(4.56)}$$

where $\varphi = \alpha + \beta$ is the phase of $v_1 + iv_2$ and α (defined in (4.35)) is arbitrary. These formulae show that to a critical point of $\tilde{\mathfrak{W}}$ in region (4.53) we obtain a one-parameter family of Kepler ellipses, the family parameter being α. The planes of these ellipses are perpendicular to the vectors

$$J^{-1}(L_1 + iL_2, L_3) = [e^{i\alpha}(C_- e^{-i\varphi} - C_+), v] \tag{4.57}$$

and the centers of the ellipses are located at

$$(A_1 + iA_2, A_3) = J[e^{i\alpha}(C_- e^{-i\varphi} + C_+), 1 - 2v_3]. \tag{4.58}$$

From (4.55), (4.57), and (4.6) we find

$$\begin{aligned} J^{-}\mathbf{L}^2 &= 2v_3 - 2v_3^2 - 2[(1-v_3)^2 - (v^2/4)]^{1/2} v_1 + (v^2/2) \\ J^{-2}\mathbf{A}^2 &= 1 - J^{-2}\mathbf{L}^2. \end{aligned} \tag{4.59}$$

The condition that **e** is a critical point of $\tilde{\mathfrak{W}}$ can be written in the form

$$(\nabla_{\mathbf{v}} \tilde{\mathfrak{W}})(\mathbf{e}) = -\lambda \hat{\mathbf{e}}, \qquad (\lambda = \text{multiplier of critical point } \mathbf{e} \text{ of } \tilde{\mathfrak{W}}), \tag{4.60}$$

where $\mathbf{e} = (e_1, e_2, -e_3)$. It shows that in general v_3 and φ in (4.57)–(4.59) will be functions of v, J, and ε. Ultimately, they may be viewed as functions of v and ε only since J can be eliminated from the condition

$$J + \varepsilon\tilde{\mathfrak{W}}(J, v, \mathbf{e}, \varepsilon) = 1. \tag{4.61}$$

In order to obtain a complete description of the flow of the Hamiltonian $J + \varepsilon\tilde{\mathfrak{W}}$ we have to augment differential equations (4.51) by two equations involving angle variables. For this purpose we use the angles $\varphi = \alpha + \beta$ and α. On account of (4.34) and (4.46) we find

$$\theta|_{I=0} = Jd\chi + Zd\varphi + Vd\alpha. \tag{4.62}$$

In order to write down the Hamiltonian equations for χ and α we express $\mathbf{V}$ in terms of V, Z, φ:

$$\mathbf{V} = ([Z(V+Z)]^{1/2} \cos\varphi, [Z(V+Z)]^{1/2} \sin\varphi, Z + (V/2)). \tag{4.63}$$

The two equations which in conjunction with Eqs. (4.51) describe the flow of our Hamiltonian $J + \varepsilon\tilde{\mathfrak{W}}$ are

$$\begin{aligned} \dot{\chi} &= 1 + \varepsilon \left[\frac{\partial\tilde{\mathfrak{W}}}{\partial J} - J^{-1}\left(v\frac{\partial\tilde{\mathfrak{W}}}{\partial v} + \mathbf{v}\cdot\frac{\partial\tilde{\mathfrak{W}}}{\partial \mathbf{v}}\right)\right], \\ \dot{\alpha} &= J^{-1}\frac{\varepsilon}{2}\left[2\frac{\partial\tilde{\mathfrak{W}}}{\partial v} + (v_3 + \tfrac{1}{2}v)^{-1}\left(\frac{\partial\tilde{\mathfrak{W}}}{\partial v_1}v_1 + \frac{\partial\tilde{\mathfrak{W}}}{\partial v_2}v_2\right) + \frac{\partial\tilde{\mathfrak{W}}}{\partial v_3}\right]. \end{aligned} \tag{4.64}$$

If $\mathbf{v} = \mathbf{e}$ is a critical point, then Ξ (defined in (4.40)) is again constant so that χ may be replaced by the eccentric anomaly u. On account of (4.60) we obtain

$$\dot{u} = 1 + \varepsilon\left[\frac{\partial\tilde{\mathfrak{W}}}{\partial J} - J^{-1}\left(v\frac{\partial\tilde{\mathfrak{W}}}{\partial v} + \lambda\frac{v^2}{4}\right)\right], \quad \dot{\alpha} = \varepsilon J^{-1}\left[\frac{\partial\tilde{\mathfrak{W}}}{\partial v} + \lambda\frac{v}{4}\right]. \tag{4.65}$$

These differential equations are all formulated in terms of the artifical time s. In order to obtain the corresponding expressions in terms of the physical time t we first observe that as a consequence of (2.16), (4.7), and (4.11) we have

$$\dot{t} = J + P_4 = J(1 + e\cos u). \tag{4.66}$$

Hence, introducing the mean anomaly $l = u + e\sin u$ we find

$$dl/dt = J^{-1}\dot{u}, \qquad d\alpha/dt = J^{-1}\dot{\alpha}(1 + e\cos u)^{-1}. \tag{4.67}$$

If the expressions given in (4.65) are substituted into (4.67) for $\dot{u}$ and $\dot{\alpha}$, the two frequencies of the quasiperiodic solution corresponding to our critical point $\mathbf{e}$ are expressed in terms of the physical time t.

5. The Three-Dimensional Restricted Three Body Problem for High Values of the Jacobian Constant (Lunar Problem)

In order to apply the theory developed in Sections 1–3 to the Hamiltonian of the three-dimensional restricted three body problem for high values of the Jacobian constant, we first have to prepare the corresponding Hamiltonian,

$$H = \tfrac{1}{2}|\mathbf{y}|^2 - (x_1 y_2 - x_2 y_1) - \tilde{F}(\mathbf{x}), \qquad \tilde{F}(\mathbf{x}) = \nu\,|\mathbf{x} - \mu\mathbf{i}|^{-1} + \mu\,|\mathbf{x} + \nu\mathbf{i}|^{-1}. \tag{5.1}$$

($\mathbf{i}$ = unit vector along x_1-axis; ν, μ = masses of the primaries: $\nu + \mu = 1$) in such a manner that it assumes form (2.19). Assuming that the massless body is confined to move in Hill's region of the mass ν, we first shift this mass into the origin by means of the substitution $\mathbf{x} - \mu\mathbf{i} \to \mathbf{x}$. We obtain

$$H = \tfrac{1}{2}|\mathbf{y}|^2 - (x_1 y_2 - x_2 y_1) - \mu y_2 - F(\mathbf{x}), \tag{5.2}$$

$$F(\mathbf{x}) = \nu r^{-1} + \mu\,|\mathbf{x} + \mathbf{i}|^{-1}. \tag{5.3}$$

Hamiltonian (5.2) and the Hamiltonian

$$H = \tfrac{1}{2}|\mathbf{y}|^2 - (x_1 y_2 - x_2 y_1) + \mu(1 - x_1) - F(\mathbf{x}) \tag{5.4}$$

induce flows in the phase space $(\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}^3$ whose projections onto $\mathbb{R}^3 \backslash \{0\}$ ($\mathbf{x}$ space) agree. This is checked by eliminating $\mathbf{y}$ from the corresponding Hamiltonian equations. Remembering that r stays small, we may expand the term $|\mathbf{x} + \mathbf{i}|^{-1}$ with respect to $\mathbf{x}$

$$|\mathbf{x} + \mathbf{i}|^{-1} = 1 - x_1 + \tfrac{1}{2}G(\mathbf{x}) + \mathcal{O}(r^3),$$

where $G(\mathbf{x}) = 3x_1^2 - r^2$. Substituting this expression into (5.3) yields

$$H = \tfrac{1}{2}|\mathbf{y}|^2 - (\nu/r) - (x_1 y_2 - x_2 y_1) - (\mu/2)\,G(\mathbf{x}) + \mathcal{O}(r^3). \tag{5.5}$$

Our goal is to study the flow of this Hamiltonian on the (Jacobi) surface

$$H = -\tfrac{1}{2}\varepsilon^{-2}, \qquad \text{where} \quad \varepsilon \ll 1. \tag{5.6}$$

In order to bring Hamiltonian (5.4) into the form of (2.19) we stretch variables according to the recipe

$$x = \nu\varepsilon^2 \hat{x}, \qquad y = \varepsilon^{-1}\hat{y}, \qquad H = \varepsilon^{-2}\hat{H}, \qquad t = \kappa\hat{t}, \tag{5.7}$$

where the reciprocal of the quantity

$$\kappa = \nu\varepsilon^3 \tag{5.8}$$

is the Kepler frequency (mean motion).

Dropping the carets in (5.6) again, we obtain the Hamiltonian

$$H = \tfrac{1}{2}|\mathbf{y}|^2 - (1/|\mathbf{x}| - \kappa(x_1 y_2 - x_2 y_1) - (\mu/2)\kappa^2 G(\mathbf{x}) + \mathcal{O}(\varepsilon^8 \nu^3). \tag{5.9}$$

Thus we have succeeded in bringing the Hamiltonian of the three-dimensional restricted three body problem into the form of (2.19) to which the theory explained previously is applicable. It requires that Hamiltonian (5.9) be brought into normal form. To that end we first switch to the regularized form K of the Hamiltonian and express it in terms of the generators of $SU(2, 2)$ with the result

$$K = J - \kappa L_3(J + Q_0) - (\mu/2)\kappa^2(J + Q_0)[3(A_1 + Q_1)^2 - (J + Q_0)^2] + \mathcal{O}(\varepsilon^8). \tag{5.10}$$

Comparring this form of the Hamiltonian with the general form (4.21) shows that $\kappa = \nu\varepsilon^3$ takes over the role of ε. (The fact that the "third-order" term is of order ε^8 rather than of order κ^3 is irrelevant.) Apparently, the following identifications must be made:

$$\mathfrak{V}_1^{(0)} = -L_3(J + Q_0), \qquad \mathfrak{V}_1^{(1)} = -(\mu/2)(J + Q_0)[3(A_1 + Q_1)^2 - (J + Q_0)^2].$$

Using the notation introduced in Section 4 in connection with (4.23)–(4.25) we find

$$\mathfrak{W}^{(1)} = \hat{\mathfrak{V}}_1^{(0)} = -JL_3, \qquad \tilde{\mathfrak{V}}_1^{(0)} = -Q_0 L_3, \qquad S_1 = \Gamma\tilde{\mathfrak{V}}_1^{(0)} = iP_0 L_3.$$

The last equality follows from $\{J, P_0\} = iQ_0$, which in turn is a consequence of (3.12) with $\mathfrak{A} = \frac{1}{2}\mathfrak{J}$, $\mathfrak{B} = \mathfrak{P}_0$. Indeed, we find $\{J, iP_0 L_3\} = iL_3\{J, P_0\} = -Q_0 L_3$. Moreover, since $\{P_0, Q_0\} = -iJ$ as a consequence of (3.12) and $[\mathfrak{P}_0, \mathfrak{Q}_0] = \frac{1}{2}\mathfrak{J}$ and since $\mathfrak{L}_3$ commutes with $\mathfrak{P}_0, \mathfrak{Q}_0$, we find

$$\begin{aligned}\tfrac{1}{2}\{S_1, \tilde{\mathfrak{V}}_1^{(0)}\} &= -\tfrac{1}{2}i\{P_0 L_3, Q_0 L_3\} = -(i/2)L_3^2\{P_0, Q_0\} \\ &= -\tfrac{1}{2}JL_3^2 \in \operatorname{Ker} J.\end{aligned}$$

It remains to calculate $\mathfrak{V}_1^{(1)}$, where

$$\begin{aligned}\mathfrak{V}_1^{(1)} = \frac{\mu}{2}\,[&J^3 + 3J^2 Q_0 + 3JQ_0^2 + Q_0^3 - 3JA_1^2 - 3JQ_1^2 \\ &- 6JA_1 Q_1 - 3Q_0 A_1^2 - 3Q_0 Q_1^2 - 6Q_0 A_1 Q_1].\end{aligned}$$

We only give the result,

$$\hat{\mathfrak{V}}_1^{(1)} = (\mu J/2)[\tfrac{5}{2}J^2 - 3(L_2^2 + L_3^2) - \tfrac{15}{2}A_1^2 - \tfrac{3}{2}L_1^2].$$

This expression can be simplified by transforming with $\exp(\kappa T)$, where

$$T = i\mu(-\tfrac{3}{8}L_1 L_2 + \tfrac{15}{8}A_1 A_2).$$

According to the remark following (4.25), the modified term becomes

$$\hat{\mathfrak{V}}_1^{(1)} + \{T, \mathfrak{W}^{(1)}\} = (\mu/2)J[-\tfrac{5}{4}J^2 + \tfrac{3}{2}\mathbf{L}^2 - \tfrac{3}{4}L_3^2 + \tfrac{15}{4}A_3^2].$$

Summarizing, we see that the regularized Hamiltonian of the three-dimensional three body problem for values of the Jacobian constants ε^{-2} ($\varepsilon \ll 1$) can be transformed into the form

$$K = J + \kappa\mathfrak{W} + \mathcal{O}(\varepsilon^8), \tag{5.11}$$

where

$$\mathfrak{W} = -JL_3 - (\kappa/8)J[5\mu J^2 + (4 + 3\mu)L_3^2 - 6\mu\mathbf{L}^2 - 15\mu A_3^2]. \tag{5.12}$$

Now K is in form (4.26) with $\mathfrak{W}$ given in (5.12). The critical points of $\mathfrak{W}$ on $S^2 \times S^2$ are those of its dominating term $-JL_3$, namely, $(J/2)(\pm\mathbf{k}, \pm\mathbf{k})$ (where $\mathbf{k}$ is the unit vector in the 3-direction). The points $\pm(J/2)(\mathbf{k}, \mathbf{k})$ represent circular orbits in the plane of the primaries whereas $\pm(J/2(\mathbf{k}, -\mathbf{k})$ represent collision orbits perpendicular to this plane (compare [17, p. 632]).

For later use we also compute the corresponding sum of the multipliers. We find

$$\lambda_1 + \lambda_2 = \pm 4 - \kappa \cdot (4 - 3\mu)J \tag{5.13}$$

for the circular orbits and

$$\lambda_1 + \lambda_2 = \kappa 15\mu J \tag{5.14}$$

for the collision orbits.

Since $\mathfrak{W}$ is in involution not only with J but also with L_3, we see that

$$E = 6(L_1^2 + L_2^2) + 15A_3^2$$

is also an integral of truncated Hamiltonian (5.11). Furthermore, the three integrals J, L_3, and E are mutually in involution so that the truncated Hamiltonian represents an integrable approximation to the Hamiltonian of our problem. Since on account of (4.6)

$$L_1^2 + L_2^2 + \mathbf{A}^2 = J^2 - L_3^2 \tag{5.15}$$

is also an integral and since a circular solution in the plane of the primaries is characterized by the fact that all quantities on the left of the last equation are zero, these quantities stay small on neighbouring solutions, proving that in our approximation the circular solutions are stable. A similar argument that starts from the observation that

$$\tfrac{9}{4}L_3^2 + \tfrac{15}{4}(L_1^2 + L_2^2 + \mathbf{A}^2) - \tfrac{1}{4}E = \tfrac{9}{4}\mathbf{L}^2 + \tfrac{15}{4}(A_1^2 + A_2^2)$$

is also an integral of our integrable approximation shows that in this approximation also the two collision orbits are stable.

Notice that the condition $J + \kappa\mathfrak{W} = 1$ subjects the three integrals to a relation, which when solved for J assumes the form

$$J = 1 + \kappa L_3 + (\kappa^2/8)[5\mu + 3(4-\mu)L_3^2 - \mu E] + \mathcal{O}(\kappa^3). \tag{5.16}$$

In particular, on a circular solution this relation becomes

$$J = 1 \mp \kappa + (\kappa^2/4)(\mu + 10) + \mathcal{O}(\kappa^3) \tag{5.17}$$

and on a collision solution

$$J = 1 - \tfrac{5}{4}\mu\kappa^2 + \mathcal{O}(\kappa^3). \tag{5.18}$$

The first solution corresponds to a rotation of the massless body along a circle of radius $\nu\varepsilon^2 J$ (J given in (5.17)) in the plane of the primaries whereas the second corresponds to an "oscillation" perpendicular to that plane with amplitude $2\nu\varepsilon^2 J$ (J given in (5.18)). Here, the factor $\nu\varepsilon^2$ is due to the stretching of variables given in (5.7). We also find

$$\frac{\partial\mathfrak{W}}{\partial J} = \pm J - \frac{15}{8}\mu\kappa J^2 - \frac{\kappa}{8}(4-3\mu)J^2$$

on a circular orbit and $\partial\mathfrak{W}/\partial J = 0$ on a collision orbit. Substituting these values together with (5.13) and (5.14) into (4.42) yields

$$\dot{u} = 1 \pm 2\kappa J - \tfrac{3}{4}\mu\kappa^2 J^2 - \tfrac{3}{2}\kappa^2 J^2 \tag{5.19}$$

on a circular orbit and

$$\dot{u} = 1 + \tfrac{15}{4}\mu\kappa^2 J^2 \tag{5.20}$$

on a collision orbit. Substituting the right sides of (5.19) and (5.20) into the first of Eqs. (4.67) and simultaneously replacing J by its value as given in (5.17) and (5.18) yields

$$du/dt = \kappa^{-1}(1 \pm 3\kappa - 3\kappa^2 - \mu\kappa^2) + \mathcal{O}(\kappa^2)$$

for the frequency of the circular solutions (in agreement with [1, Eq. (44)]) and

$$dl/dt = \kappa^{-1}(1 + 5\mu\kappa^2), \qquad l = u + \sin u,$$

for the one of the collision orbits.

We turn now to the determination of the quasiperiodic solutions with two frequencies. For this purpose we express $\mathfrak{W}$ in terms of the variables $\mathbf{v}$ and v by making use of relations (4.57)–(4.59). We obtain

$$\tilde{\mathfrak{W}} = -J^2 v + \kappa J^3[-(v^2/2) + \tfrac{5}{4}\mu + 3\mu\tilde{F}(v_1, v_3, v)], \tag{5.21}$$

where

$$\tilde{F}(v_1, v_3, v) = -2v_3 + 2v_3^2 - \tfrac{1}{2}[(1 - v_3)^2 - (v^2/4)]^{1/2}\, v_1. \tag{5.22}$$

We look now for critical points of $\tilde{F}$ in region (4.53). For simplicity we assume $v > 0$. This assumption will be in force for the remainder of this paper. In fact, it is not difficult to see that the case $v < 0$ can always be reduced to the case $v > 0$ by a time reversal which is implemented by the replacement $\zeta \leftrightarrow \bar{\eta}$, $\varepsilon \to -\varepsilon$ (see (4.46)[5]). Writing down condition (4.60) with $(\tilde{\mathfrak{W}}, \lambda)$ replaced by $(\tilde{F}, (2\lambda)^{-1})$ (we find it more convenient to work with $(2\lambda)^{-1}$ than with λ) we obtain the relations

$$\begin{aligned} v_1 &= \lambda[(1 - v_3)^2 - (v^2/4)]^{1/2}, \qquad v_2 = 0, \\ v_3 &= \lambda(\lambda - 4)[\lambda^2 - 8\lambda + 1]^{-1}. \end{aligned} \tag{5.23}$$

Since we also have $v_3^2 - v_1^2 = \frac{1}{4}v^2$, we can eliminate v_1 and v_3 from the last two equations. We find that λ must either take the values ± 1 or it must be a solution of the following quadratic equation

$$\lambda^2 - 2(4 - \sqrt{15}\, v^{-1})\lambda + 1 = 0. \tag{5.24}$$

Actually, at first we find that λ could also be a solution of an equation that differs from (24) by a replacement $-\sqrt{15} \to +\sqrt{15}$. Since the corresponding critical point does not lie in region (4.53), however, it can be discarded. The solutions of (5.24) are found to be

$$\lambda_\pm = -2\sqrt{15}\, v^{-1}\delta_\pm, \qquad \delta_\pm = -(2/\sqrt{15})\, v + \tfrac{1}{2}(1 \pm \rho(v)), \tag{5.25}$$

where

$$\rho(v) = [1 - (8/\sqrt{15})\, v + v^2]^{1/2} = [(v_0 - v)(v_0^{-1} - v)]^{1/2} \tag{5.26}$$

[5] In the original variables this transformation takes the form $t \to -t$; $x_1 \to x_1$, $x_2 \to -x_2$, $x_3 \to x_3$; $y_1 \to -y_1$, $y_2 \to y_2$, $y_3 \to -y_3$.

and

$$v_0 = \sqrt{3/5} = \tfrac{1}{5}\sqrt{15} = 0.77459667.... \tag{5.27}$$

Since $\rho(v)$ must be a real number, the interval in which v is allowed to vary is reduced from the unit interval to $(0, v_0)$. Using (5.23) we compute the critical points corresponding to $\lambda_\pm$ and find

$$\mathbf{e}_\pm(v) = (-(\delta_\pm)^{1/2}, 0, \tfrac{1}{2}(1 \pm \rho(v))). \tag{5.28}$$

One checks that for $0 < v < v_0$, $\delta_\pm$ is positive and that $\mathbf{e}_\pm(v)$ lies indeed in region (4.53). A simple computation using (5.23) also yields the critical points

$$\mathbf{e}_0^\pm(v) = \tfrac{1}{2}(\pm(1 - v^2)^{1/2}, 0, 1)$$

corresponding to $\lambda = \pm 1$. They exist for all values of v in (0, 1). Using (4.57) and (4.58), we compute the elements e, $\mathbf{A}$, $\mathbf{L}$ of the Kepler ellipses that correspond to these critical points. Setting first arbitrarily $\alpha = 0$ we find

$$e = (1 - (v/v_0))^{1/2}, \qquad \mathbf{L} = J(-((v/v_0) - v^2)^{1/2}, 0, v);$$
$$\mathbf{A} = \mp J((vv_0 - v^2)^{1/2}, 0, \rho(\mathrm{v}))$$

for $\mathbf{e}_\pm(v)$,

$$e = (1 - v^2)^{1/2}, \qquad \mathbf{L} = J(0, 0, v); \qquad \mathbf{A} = J((1 - v^2)^{1/2}, 0, 0) \tag{5.29}$$

for $\mathbf{e}_0^+(v)$, and finally

$$e = 0, \qquad \mathbf{L} = J(-(1 - v^2)^{1/2}, 0, v), \qquad \mathbf{A} = 0$$

for $\mathbf{e}_0^-(v)$.

In these formulae, J has to be determined as the solution of Eq. (4.61) with $\tilde{\mathfrak{W}}$ given in (5.21). We find for a critical point $\mathbf{v} \equiv \mathbf{e} = (e_1, 0, e_3)$ of $\tilde{F}$,

$$J = 1 + v\kappa + \tfrac{5}{2}v^2\kappa^2 - \kappa^2[\tfrac{5}{4}\mu + 3\mu\, f(v)] + \mathcal{O}(\kappa^3), \tag{5.30}$$

where

$$f(v) = \tilde{F}(e_1, e_3, v) = -2e_3 + 2e_3^2 - (e_1^2/2\lambda). \tag{5.31}$$

Setting $\mathbf{e} = \mathbf{e}_\pm(v)$ and $\mathbf{e} = \mathbf{e}_0^+(v)$, respectively, we find (in obvious notation)

$$f_\pm(v) = \tfrac{1}{2}v^2 - \tfrac{1}{4}\sqrt{15}\, v, \qquad f_0^\pm(v) = \mp \tfrac{1}{8}(1 - v^2) - \tfrac{1}{2}. \tag{5.32}$$

In writing down the expressions for the elements $\mathbf{A}$, $\mathbf{L}$ of the Kepler ellipses we have arbitrarily put $\alpha = 0$ (see (4.57) and (4.58)). Actually, these vectors revolve about the 3-axis with angular velocity $d\alpha/dt$.

We now turn to the calculation of this angular velocity which represents one of the frequencies of the quasiperiodic solution under investigation. At the same time we shall obtain the second frequency, namely, dl/dt, where l is the mean anomaly. For this purpose we first compute $\dot{u}$ and $\dot{\alpha}$ according to recipe (4.65) remembering also that we have to replace λ by $\frac{3}{2}\kappa J^3 \mu \lambda^{-1}$ as a consequence of our earlier replacement $(\tilde{\mathfrak{W}}, \lambda) \to (\tilde{F}, (2\lambda)^{-1})$ (see (5.21) and the remark preceding (5.23))

$$\dot{u} = 1 - Jv\kappa + J^2\kappa^2\{-\tfrac{1}{2}v^2 + \tfrac{15}{4}\mu + 3\mu[3f(v) - (v^2/8)(\lambda^{-1} + \lambda)]\},$$
$$\dot{\alpha} = -\kappa J + J^2\kappa^2[\tfrac{3}{8}\mu(\lambda^{-1} + \lambda) - 1]\, v.$$

Substituting these expressions together with the value of J given in (5.30) into (4.67) yields:

$$\frac{dl}{dt} = \frac{1}{\kappa} - 2v + \kappa\left\{-2v^2 + 5\mu + 3\mu\left[4f(v) - \frac{v^2}{8}(\lambda^{-1} + \lambda)\right]\right\} + \mathcal{O}(\kappa^2), \tag{5.33}$$

$$\frac{d\alpha}{dt}(1 + e\cos u) = -1 - \kappa\left[1 - \frac{3\mu}{8}(\lambda^{-1} + \lambda)\right] v + \mathcal{O}(\kappa^2). \tag{5.34}$$

In these formulae also, the stretching of the time variable given in (5.7) has been taken into account. In order to get rid of the term $(1 + e\cos u)$ on the left side of (5.34), we average over t and in this manner we find for the average rate of rotation of the orbital plane about the 3-axis

$$\overline{d\alpha}/dt = -1 - \kappa[1 - \tfrac{3}{8}\mu(\lambda^{-1} + \lambda)]\, v + \mathcal{O}(\kappa^2). \tag{5.34'}$$

Using the values of $f(v)$ given in (5.32), we specialize these formulae to the case of the four families of quasiperiodic solutions with two frequencies. The result for $\mathbf{e}_{\pm}(v)$ is

$$\begin{aligned} dl/dt &= (1/\kappa) - 2v + \kappa[-2v^2 + 5\mu + 3\mu(v^2 - \tfrac{15}{4}v_0 v)] + \mathcal{O}(\kappa^2), \\ \overline{d\alpha}/dt &= -1 - \kappa[(1 - 3\mu)\, v + \tfrac{15}{4}\mu v_0] + \mathcal{O}(\kappa^2). \end{aligned} \tag{5.35}$$

For $\mathbf{e}_0^+(v)$ we have

$$\begin{aligned} dl/dt &= (1/\kappa) - 2v + \kappa[-2v^2 - \tfrac{5}{2}\mu + \tfrac{3}{4}\mu v^2] + \mathcal{O}(\kappa^2), \\ \overline{d\alpha}/dt &= -1 - \kappa(1 - \tfrac{3}{4}\mu)\, v + \mathcal{O}(\kappa^2). \end{aligned} \tag{5.36}$$

Finally, for $\mathbf{e}_0^-(v)$,

$$\begin{aligned} du/dt &= (1/\kappa) - 2v + \kappa[-2v^2 + (\mu/2) - \tfrac{3}{4}\mu v^2] + \mathcal{O}(\kappa^2), \\ \overline{d\alpha}/dt &= -1 - \kappa(1 + \tfrac{3}{4}\mu)\, v + \mathcal{O}(\kappa^2). \end{aligned} \tag{5.37}$$

(Since the last family consists of circular solutions, we can replace the mean anomaly l by the eccentric anomaly u and apply (5.34) instead of (5.34′).)

For a discussion of the four families $\mathbf{e}_0^\pm(v), \mathbf{e}_\pm(v)$ of quasiperiodic solutions with two frequencies we refer the reader back to the latter part of Section 1. All statements made there with the exception of those pertaining to the stability of our solutions (which will be dealt with shortly) can be deduced from relations (5.29) (see also Fig. 4). Here we only add some comments on expressions (5.35)–(5.37) for the two frequencies. The term κ^{-1} in the formula for dl/dt is the Kepler frequency and the second term gives the correction due to rotation of the coordinate system (Coriolis term).

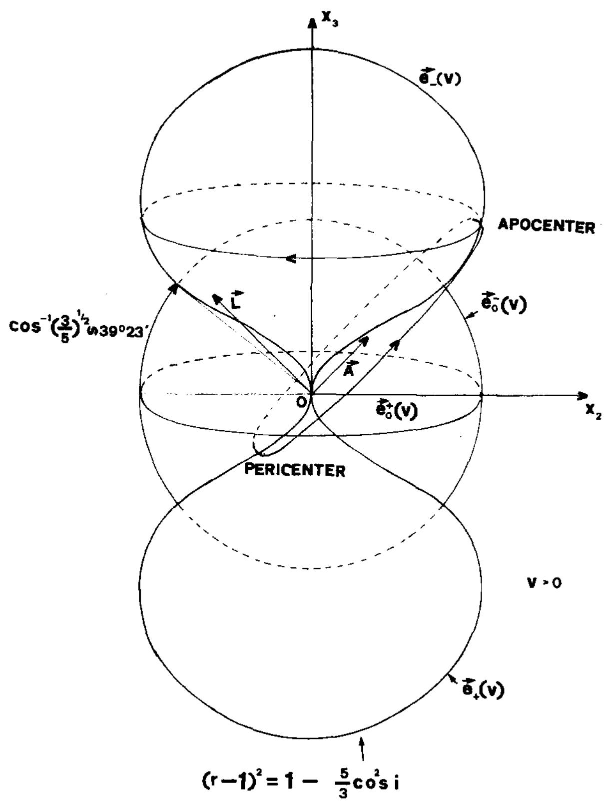


FIGURE 4

Only the third term gives the correction due to the specific potential of the restricted three body problem. The second term in the formulae for $\overline{d\alpha/dt}$ gives a first approximation to the average rate of rotation of the orbital plane about the 3-axis.

We finally turn to a discussion of the stability of our quasiperiodic solutions with two frequencies. Let $\mathbf{e} = (e_1, 0, e_3)$ $(e_3 > 0)$ be one of the critical points of function (5.22) in region (4.53). By means of the canonical transformation of variables

$$\xi_1 = Ch(\gamma/2)\,\hat{\xi}_1 + Sh(\gamma/2)\,\hat{\xi}_2, \qquad \xi_2 = Sh(\gamma/2)\,\hat{\xi}_1 + Ch(\gamma/2)\,\hat{\xi}_2, \tag{5.38}$$

where $\mathrm{Tanh}\,\gamma = e_1(e_3)^{-1}$ and J, χ are kept unchanged (see (4.44)), we can succeed in moving the critical point into the position $(v/2)\,\mathbf{k}$. Indeed we easily check using formula (4.47) that the transformation of variables induces in $\mathbf{v}$-space the following transformation:

$$v_1 = (2/v)(\hat{v}_1 e_3 + \hat{v}_3 e_1), \qquad v_3 = (2/v)(\hat{v}_1 e_1 + \hat{v}_3 e_3). \tag{5.39}$$

In particular, the point $\hat{\mathbf{v}} = (v/2)\,\mathbf{k}$ corresponds to the point $\mathbf{e} = (e_1, 0, e_3)$. By inserting the right sides of (5.39) in place of v_1 and v_3 in (5.22) and dropping the carets again, we obtain the function

$$\begin{aligned} F(v_1, v_3, v) = &-(4/v)(v_1 e_1 + v_3 e_3) + (8/v^2)(v_1 e_1 + v_3 e_3)^2 \\ &-(1/v)[(1-(2/v)(v_1 e_1 + v_3 e_3))^2 - (v^2/4)]^{1/2} \cdot (v_1 e_3 + v_3 e_1). \end{aligned} \tag{5.40}$$

Clearly, if $\mathbf{e} = (e_1, 0, e_3)$ is a critical point of $\tilde{F}$ with multiplier $(2\lambda)^{-1}$, then $(v/2)\,\mathbf{k}$ is a critical point of F with the same multiplier. In order to study this function near the critical point we introduce the coordinates

$$x = \sqrt{2Z}\cos\varphi, \qquad y = \sqrt{2Z}\sin\varphi. \tag{5.41}$$

Using the variables (x, y) in place of (Z, φ) in formula (4.63) for $\mathbf{V}$, we obtain

$$\mathbf{V} = \tfrac{1}{2}[(2V + x^2 + y^2)^{1/2}x,\ (2V + x^2 + y^2)^{1/2}y,\ (V + x^2 + y^2)] \tag{5.42}$$

and the normalized vector $\mathbf{v}$ can simply be obtained from (5.42) by replacing V by v and (x, y) by $(J^{-1/2}x, J^{-1/2}y)$. The fundamental 1-form (4.62) has the following expression in the new coordinates:

$$\theta|_{I=0} = Jd\chi + xdy + Vd\alpha, \tag{5.43}$$

so that the coordinate patch defined by $(J, \chi, x, y, V, \alpha)$ is symplectic.

Substituting the expressions for $v_1 = J^{-1}V_1$, $v_3 = J^{-1}V_3$ that follow from (5.42) into (5.40), we obtain a function

$$f(v) + G(J^{-1/2}x, J^{-1/2}y, v) \tag{5.44}$$

where $f(v)$ was defined in (5.31). The critical point $\mathbf{e} = (e_1, 0, e_3)$ of F appears now as a critical point of G at the origin of the $x - y$ plane. Accordingly, G possesses an expansion about $(0, 0)$ of the type

$$G(x, y, v) = \tfrac{1}{2}(Ax^2 + By^2) + \sum_{k+l=3} B_{kl}x^k y^l + \sum_{k+l=4} C_{kl}x^k y^l + \mathcal{O}_5, \tag{5.45}$$

where $\mathcal{O}_5$ represents a power series in (x, y) starting with a term of order five. Defining

$$\Delta(v) = \tfrac{1}{16}((1/\lambda^2) + \lambda^2)\, v^2 - 4f(v) - \tfrac{15}{2}\, e_3(1 - e_3) \tag{5.46}$$

(where $(2\lambda)^{-1}$ is the multiplier corresponding to the critical point $\mathbf{e} = (e_1, 0, e_3)$ of $\tilde{F}$; see the remark preceding (5.23)), we find

$$B = (v/4\lambda), \qquad A = (4\lambda/v)\, \Delta(v). \tag{5.47}$$

The remaining coefficients entering (5.44) are presently not needed. They only play a role in Section 6 when the question of continuation of our solutions to the full Hamiltonian will be taken up. From (5.45) and (5.47) we recognize that $\Delta(\lambda)$ is precisely the Hessian of the critical point $\mathbf{e} = (e_1, 0, e_3)$. Accordingly, the corresponding quasiperiodic solution is stable if $\Delta(v) > 0$ and unstable if $\Delta(v) < 0$. Specializing (5.45) to the four critical points $\mathbf{e}_\pm(v)$ and $\mathbf{e}_0^\pm(v)$, we obtain (in obvious notation)

$$\begin{aligned} \Delta_\pm(v) &= \tfrac{15}{4}\, (v_0 - v)(v_0^{-1} - v) > 0 && \text{for} \quad 0 < v < v_0, \\ \Delta_0^+(v) &= \tfrac{1}{8}\, (5 - 3v^2) \qquad > 0 && \text{for} \quad 0 < v < 1, \\ \Delta_0^-(v) &= \tfrac{1}{8}\, (5v^2 - 3) \qquad > 0 && \text{for} \quad v_0 < v < 1, \end{aligned} \tag{5.48}$$

but $\Delta_0^-(v)$ is negative (!) for $0 < v < v_0$.

From these formulae the reader can deduce all the statements that were made in the introduction with respect to the stability of our quasi periodic solutions with two frequencies. For an overview of the zeroth approximation of our four families $\mathbf{e}_0^\pm(v)$, $\mathbf{e}_\pm(v)$ of quasiperiodic solutions with two frequencies for $v > 0$ we refer the reader to Fig. 4. A corresponding figure could be drawn for $v < 0$ with all arrows and the vector $\mathbf{L}$ reversed.

The locus of the apocenters and pericenters of the rotating Kepler ellipses from which our stable solutions emerge is a surface of revolution about the x_3-axis. In Fig. 4 we have drawn its intersection C with the $x_2 - x_3$ plane. If the absolute value of the inclination i of the Kepler orbit is less than arc cos

v_0(= critical inclination: compare [31]) then C is part of a circle. We have drawn the continuation of this circle beyond the critical inclination using an interrupted line in order to indicate that the family $\mathbf{e}_0^-(v)$ becomes unstable there. In fact, for these values of the inclination the families $\mathbf{e}_\pm(v)$ take over an one easily deduces from formula (5.29) that the curve C for $|i| \geqslant \arc\cos v_0$ has the equation

$$(r-1)^2 = 1 - \tfrac{5}{3}\cos^2 i.$$

(r, i polar coordinates in $x_2 - x_3$ plane.)

Since a stable critical point is a center (for the quotient flow), we see that the corresponding quasiperiodic solution with two frequencies is "surrounded" by quasiperiodic solutions with three frequencies. Since for these solutions the angle φ is subjected to small variations, it follows from (4.57) and (4.58) that they represent solutions in which the angular momentum as well as the Laplace vector, in addition to rotating about the 3-axis, are also subjected to small oscillations. In Section 6 we intend to show that these solutions can be continued to the full Hamiltonian in the sense of KAM-theory by checking the corresponding determinant condition. In order to prepare ourselves for this endeavour we have to bring $G(x, y, v)$ into normal form with respect to its quadratic term under the assumption $\Delta(v) > 0$. Applying techniques (4.21)–(4.25) to the Hamiltonian G (the Lie bracket being the one canonically associated with the form $dx \wedge dy$) we find for the normal form up to fourth order terms

$$\tilde{G}(x, y, v) = (\Delta(v))^{1/2} Z + k(v) Z^2 + \mathcal{O}_3, \tag{5.49}$$

where $Z = \frac{1}{2}(x^2 + y^2)$ (see (5.41)) and $k(v)$ is related to the coefficients of G (given in (5.45)) by the formula

$$2k\Delta = \Big\{3C_{40}B^2 + C_{22}\Delta + 3C_{04}A^2 - \frac{3}{2}B\Delta\left[4\left(\frac{B_{30}}{A}\right)^2 + \left(\frac{B_{30}}{A} + \frac{B_{12}}{B}\right)^2\right] - \frac{3}{2}A\Delta\left[4\left(\frac{B_{03}}{B}\right)^2 + \left(\frac{B_{03}}{B} + \frac{B_{21}}{A}\right)^2\right]\Big\}. \tag{5.50}$$

(Compare also [23, 24]). In the specific case under consideration we find $B_{21} = B_{03} = 0$ so that the second line of (5.50) does not give any contribution to k. Still the calculations are long and tedious and therefore are omitted here. We just give the final result,

$$2k_\pm \Delta_\pm = 210\imath^3 + 4\imath^2 - \frac{47}{4}\imath + 1, \qquad \imath = \frac{1}{10}\frac{v}{v_0},$$
$$2k_0^+\Delta_0^+ = \frac{3}{64}\left(v^2 - \frac{35}{3}\right), \qquad 2k_0^-\Delta_0^- = \frac{3}{64}\left(1 - \frac{35}{3}v^2\right). \tag{5.51}$$

Notice that in (5.49) we have actually introduced angle and action variables up to order five. It is clear from the method of the generating function that this can be done to any order as long as $0 < v < 1$ and $Z < z_0$, z_0 sufficiently small. Rather than denoting these variables by new symbols we assume in the following that Z and φ are action angle variables for the Hamiltonian G and the new Hamiltonian will be denoted by $\tilde{G}(v, Z)$. Thus (5.49) should be viewed as an expansion of $\tilde{G}$ in the variable Z about $Z = 0$.

6. Continuation of Quasiperiodic Solutions

In this final section we address ourselves to the question of continuation of the quasiperiodic solutions that we found in the last section. Expressing our Hamiltonian in the symplectic chart $(J, \chi, x, y, V, \alpha)$ near the quasiperiodic solution with two frequencies that corresponds to the critical point $\mathbf{e} = (e_1, 0, e_3)$, we find from (5.11), (5.21), and (5.44)

$$K = J - \kappa JV + \kappa^2[-\tfrac{1}{2}JV^2 + \tfrac{5}{4}\mu J^3 + 3\mu J^3 f(J^{-1}V) + 3\mu J^3 G(J^{-1/2}x, J^{-1/2}y, J^{-1}V)] + \mathcal{O}(\varepsilon^8). \tag{6.1}$$

On the surface $K = 1$, K defines a Hamiltonian system $(\Lambda, d\hat{\theta}, \chi)$, where $\hat{\theta}$ is given by the expression

$$\hat{\theta} = Zd\varphi + Vd\alpha = xdy + Vd\alpha, \tag{6.2}$$

and the Hamiltonian

$$\Lambda = -\kappa V - \tfrac{3}{2}\kappa^2 V^2 + 3\mu\kappa^2 f(V) + 3\mu\kappa^2 \tilde{G}(V, Z) + \mathcal{O}(\varepsilon^8) \tag{6.3}$$

is obtained from K by solving the equation $K = 1$ for $-J$ and dropping some irrelevant constant terms. Furthermore, the function $G(x, y, V)$ has been replaced by its normal form $\tilde{G}$ that was introduced at the end of the last section.

We stretch variables according to the recipe

$$V = v + \varepsilon\hat{V}, \qquad Z = z + \varepsilon\hat{Z}, \qquad \Lambda = \varepsilon\hat{\Lambda}, \tag{6.4}$$

where v and z are fixed. (The variable z introduced here has nothing to do with the variable of Section 1 denoted by the same symbol.) Subjecting our Hamiltonian (6.3) to transformation (6.4) and dropping the carets again as well as some constant terms, we obtain

$$\Lambda = 3\kappa^2[-(1/3\kappa) + g_1(v, z)]\, V + 3\kappa^2 g_2(v, z)\, Z + \mathcal{O}(\varepsilon^7), \tag{6.5}$$

where

$$
\begin{aligned}
g_1(v,z) &= -v + \mu f'(v) + \mu \tilde{G}_v(v,z), \\
g_2(v,z) &= \mu \tilde{G}_z(v,z).
\end{aligned}
\tag{6.6}
$$

Integrating the corresponding Hamiltonian equations over χ from $\chi = 0$ to $\chi = 2\pi$ yields a symplectic map of the surface $\chi = 0$ onto itself. In our variables this map assumes the form

$$
\begin{aligned}
\alpha_1 &= \alpha + 6\pi\kappa^2[-(1/3\kappa) + g_1(v,z)] + \mathcal{O}(\varepsilon^7), \qquad & V_1 &= V + \mathcal{O}(\varepsilon^7), \\
\varphi_1 &= \varphi + 6\pi\kappa^2 g_2(v,z) + \mathcal{O}(\varepsilon^7), & Z_1 &= Z + \mathcal{O}(\varepsilon^7).
\end{aligned}
\tag{6.7}
$$

Here the terms $\mathcal{O}(\varepsilon^7)$ represent terms that are real analytic in $\{v, z, \varphi, \alpha, Z, V, \varepsilon$: $0 < v < 1$, $0 < z < z_0$; Z, V, ε sufficiently small, z_0 defined at the end of Section 5$\}$ starting with a term of order ε^7. Now KMA theory (see, e.g., [18]) implies that the map

$$
\begin{aligned}
\alpha_1 &= \alpha + 6\pi\kappa^2\omega_1 + \mathcal{O}(\varepsilon^7), \qquad & V_1 &= V + \mathcal{O}(\varepsilon^7), \\
\varphi_1 &= \varphi + 6\pi\kappa^2\omega_2 + \mathcal{O}(\varepsilon^7), & Z_1 &= Z + \mathcal{O}(\varepsilon^7),
\end{aligned}
\tag{6.8}
$$

possesses a 3-parametric family of tori

$$
V = V_\omega(\alpha, \varphi, z, v, \varepsilon), \qquad Z = Z_\omega(\alpha, \varphi, z, v, \varepsilon) \tag{6.9}
$$

for every frequency vector $\omega = (\omega_1, \omega_2)$ that satisfies the irrationality condition

$$
|\omega_1 p + \omega_2 q| \geqslant \gamma(|p| + |q| + 1)^{-\tau} \tag{6.10}
$$

for all $(p, q) \in \mathbb{Z}^2 \backslash \{0\}$ with some constants (γ, τ): $0 < \gamma \leqslant 1$, $\tau > 0$. The functions V_ω, Z_ω are defined and real analytic for all (v, z) in a rectangle $0 < v < 1$, $0 \leqslant z < z_0$ provided only that $\varepsilon < \varepsilon_1(\gamma, \tau)$.

In the following we shall show that for all four families of quasiperiodic solutions with two frequencies the determinant condition

$$
\frac{\partial(g_1, g_2)}{\partial(v, z)}(v, z) \neq 0 \tag{6.11}
$$

is satisfied for $z = 0$ and therefore also in some interval $0 \leqslant z < z_0$. We think of z_0 being chosen in such a way that (i) $\tilde{G}(z, v)$ exists as a real analytic function for $0 \leqslant z < z_0$, $0 < v < 1$. (ii) Inequality (6.11) holds true there.

Let the image of the open square $\{(v, z): 0 < v < 1, 0 < z < z_0\}$ under the map

$$
\omega_1 = g_1(v, z), \qquad \omega_2 = g_2(v, z) \tag{6.12}
$$

be denoted by $\mathfrak{R}$. Our choice of z_0 ensures that $\mathfrak{R}$ is an open set in the ω_1-ω_2 plane. For any real number, let $\mathfrak{R}_r$ be the r-translate of $\mathfrak{R}$ in the ω_1-direction. Thus $(\omega_1, \omega_2) \in \mathfrak{R}_r$ iff $(\omega_1 - r, \omega_2) \in \mathfrak{R}$. Let

$$c(\varepsilon) = \tfrac{1}{3}\kappa^{-1} = (3\nu)^{-1}\varepsilon^{-3} \tag{6.13}$$

and let $\Sigma(\gamma, \tau)$ be the set of all ω-vectors satisfying (6.10) and lying in one of the sets $\mathfrak{R}_{-r}, r > c(\varepsilon_1(\gamma, \tau))$.

Since the ω vectors satisfying (6.10) with some γ, τ $(0 < \gamma \leqslant 1, \tau > 0)$ are dense in the ω plane, the set $\Sigma = \bigcup_{\gamma,\tau} \Sigma(\gamma, \tau)$ is certainly nonempty. In the following we shall show that for every $\omega \in \Sigma$ there exists a small ε-interval I_ω such that for $\varepsilon \in I_\omega$, map (6.7) possesses an invariant torus with frequency vector ω. Indeed, if $\omega \in \Sigma(\gamma, \tau)$, then $(\omega_1 + r, \omega_2) \in \mathfrak{R}$ for $r \in \tilde{I}_\omega \subset (c(\varepsilon_1(\gamma, \tau)), \infty)$, $\tilde{I}_\omega$ being a suitable open interval. Hence $(\omega_1 + c(\varepsilon), \omega_2) \in \mathfrak{R}$ for $\varepsilon \in I'_\omega =_{\text{def}} c^{-1}(\tilde{I}_\omega) \subset (0, \varepsilon_1(\gamma, \tau))$. Since the map defined by (6.12) is a local diffeomorphism, there exist functions $v_\omega(\varepsilon), z_\omega(\varepsilon)$ defined in some open interval $I_\omega \subset I'_\omega$ such that the point $(v_\omega(\varepsilon), z_\omega(\varepsilon))$ is mapped into $(\omega_1 + c(\varepsilon), \omega_2)$ by (6.12). Replacing (v, z) by $(v_\omega(\varepsilon), z_\omega(\varepsilon))$ in (6.7) converts map (6.7) into map (6.8) which is known to possess the invariant tori (6.9). Thus, carrying through the same replacement also in (6.9) we obtain the predicted family of tori.

It remains to check condition (6.11) (with $z = 0$) for the four families of quasiperiodic solutions. We find

$$\frac{\partial(g_1, g_2)}{\partial(v, z)}(v, 0) = \frac{\mu}{\Delta} D, \quad D = (-1 + \mu f''(v))\, 2\Delta(v)\, k(v) - \frac{\mu}{4}(\Delta'(v))^2,$$

i.e., in obvious notation

$$4D_\pm = -(1-\mu)(840\iota^3 + 16\iota^2 - 47\iota + 4) - 15\mu(15\iota - 2)^2 < 0$$

for $0 < \iota < \frac{1}{10}(\iota = \frac{1}{10}(v/v_0))$ and

$$256 D_0^+ = 140 - 35\mu - 12v^2 - 33\mu v^2 \geqslant 60 \qquad \text{for} \quad 0 < v < 1,$$

$$256 D_0^- = -12 - 3\mu + 140v^2 - 65\mu v^2 \geqslant 30 \qquad \text{for} \quad v_0 \leqslant v < 1.$$

Finally we turn to the question of the continuation of the stable quasiperiodic solutions with two frequencies themselves. In order to treat this question we first remark that the following theorem is an immediate consequence of the results contained in [28, 29]:

THEOREM 4. *Let $\mathfrak{C}(\gamma, \tau)$ be the class of $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ satisfying the inequalities*

$$|\omega_1 j_1 + \omega_2 j_2 - k| \geqslant \gamma(|j_1| + |j_2| + 1)^{-\tau} \tag{6.14}$$

for all $(j_1, j_2) \in \mathbb{Z}^2 \backslash \{0\}$ *and* $k = -2, -1, 0, 1, 2$ *with some constants* γ, $\tau: 0 < \gamma \leqslant 1$, $\tau > 1$. *Let* $\mathcal{O}(\varepsilon)$ *represent a convergent power series in the variables* $(x, y, J, v, \varepsilon)$ *which starts out with a term of order* ε *and whose coefficients are real analytic functions of the angle-like variables* χ, α *(with period* 2π*). Then there exist two real analytic functions* $\mu_{1_\omega}(\varepsilon)$, $\mu_{2_\omega}(\varepsilon)$ *such that the Hamiltonian system* (Λ, θ, s)

$$\Lambda = (\omega_1 + \varepsilon\mu_{1\omega}(\varepsilon)) J + (\omega_2 + \varepsilon\mu_{2\omega}(\varepsilon)) V + \tfrac{1}{2}(x^2 + y^2) + \mathcal{O}(\varepsilon), \quad (6.15)$$

$$\theta = J d\chi + x dy + V d\alpha \quad (6.16)$$

possesses a family of quasiperiodic solutions with two frequencies and frequency ratio ω_1/ω_2, *the family parameter* ε *varying in some interval* $(0, \varepsilon_0(\gamma, \tau))$.

Remark. In the following it will be important to note that the functions $\mu_{i\omega}(\varepsilon)$ $(\omega \in \mathfrak{C}(\gamma, \tau))$ $(i = \text{fixed} = 1$ or $2)$ possesses a common interval of definition with nonempty interior and that they are bounded there by a common bound. Since this situation will play a fundamental role in the sequel we shall call a family $\{f_\omega\}_{\omega \in \mathbb{R}^2}$ of real-valued functions (defined and real analytic on a open subset of some $\mathbb{R}^r$) *normal* if for every pair (γ, τ) $0 < \gamma \leqslant 1$, $\tau > 1$ the following holds: (i) the functions f_ω, $\omega \in \mathfrak{C}(\gamma, \tau)$ have a common domain of definition with nonempty interior, and (ii) they possess a common bound there.

Sketch of a proof of Theorem 4. From [29, p. 173] we deduce the existence of three normal families of functions $\lambda_{1\omega}(\varepsilon, a)$, $\lambda_{2\omega}(\varepsilon, a)$, $\sigma_\omega(\varepsilon, a)$ such that for ε and a sufficiently close to zero the Hamiltonian

$$\Lambda = (\omega_1 + \varepsilon\lambda_{1\omega}(\varepsilon, a)) J + (\omega_2 + \varepsilon\lambda_{2\omega}(\varepsilon, a)) V$$
$$+ \tfrac{1}{2}(1 + \varepsilon\sigma_\omega(\varepsilon, a))(x^2 + y^2) + (1 + a)\mathcal{O}(\varepsilon)$$

has a family of quasiperiodic solutions with frequency vector $\omega = (\omega_1, \omega_2) \in \mathfrak{C}(\gamma, \tau)$. By a stretching of the time variable $s \to (1 + \varepsilon\sigma_\omega(\varepsilon, a)) s$ the associated transformation of the Hamiltonian $\Lambda \to (1 + \varepsilon\sigma_\omega(\varepsilon, a))^{-1}\Lambda$ brings it into form (6.15), provided only that the equations

$$a = \varepsilon\sigma_\omega(\varepsilon, a), \qquad \omega \in \mathbb{R}^2,$$

possess a normal family $\{a_\omega(\varepsilon)\}_{\omega \in \mathbb{R}^2}$ of solutions. This follows, however, from the fact that $\{\partial\sigma_\omega(\varepsilon, a)/\partial a\}_{\omega \in \mathbb{R}^2}$ is a normal family of functions by constructing $a_\omega(\varepsilon)$ in the usual manner (i.e., by means of the contraction principle) as a fixed point of the map $a \to \varepsilon\sigma_\omega(\varepsilon, a)$.

In order for Theorem 4 to be applicable to the problem at hand we set $z=0$ in (6.5) and we obtain (see also (6.6) and (5.49))

$$\hat{\Lambda} = J + (-\kappa - 3\kappa^2 v + 3\mu\kappa^2 f'(v))\, V + 3\mu\kappa^2 (\Delta(v))^{1/2} \times \tfrac{1}{2}(x^2+y^2) + \mathcal{O}(\varepsilon^7). \tag{6.17}$$

Here, $\hat{\Lambda}$ differs from Λ by the additional term J. Clearly, the Hamiltonian system ($\hat{\Lambda}$, θ given in (6.16), s) is equivalent to ($\Lambda, \hat{\theta}, \chi$) as given in (6.2) and (6.3) in the sense that both systems induce the same flow in V, α, x, y-space. (Note, however, that J is only an auxiliary variable which should not be confused with the variable denoted by the same symbol in the previous sections). The symbol $\mathcal{O}(\varepsilon^7)$ in (6.17) represents a convergent power series in x, y, V, ε starting with a term of order seven in ε whose coefficients are real analytic functions in v, α, χ.

Stretching the time variable according to the recipe $s \to 3\mu\kappa^2(\Delta(v))^{1/2} s$ brings our Hamiltonian into the form

$$\Lambda = \kappa^{-2} a(v)\, J + (b(v) - \kappa^{-1} a(v))\, V + \tfrac{1}{2}(x^2+y^2) + \mathcal{O}(\varepsilon), \tag{6.18}$$

where

$$a(v) = (3\mu)^{-1}(\Delta(v))^{-1/2},$$
$$b(v) = (3\mu)^{-1}(\Delta(v))^{-1/2}(-3v + 3\mu f'(v)).$$

(Remember that we are concerned with the continuation of the stable quasiperiodic solutions so that $\Delta(v) > 0$.) Form (6.18) of our Hamiltonian invites comparison with the corresponding Hamiltonian (6.15) of Theorem 4. Indeed, $\mathcal{O}(\varepsilon)$ now represents a function of the type described in Theorem 4 except that it also depends on v but not on J. Accordingly, the functions $\mu_{1\omega}, \mu_{2\omega}$ in (6.15) not only depend on ε but also on v.

Agreement between (6.15) and (6.18) is achieved if we succeed in satisfying the equations

$$\kappa^{-2} a(v) = \tilde{\omega}_1(\varepsilon, v), \qquad b(v) - \kappa^{-1} a(v) = \tilde{\omega}_2(\varepsilon, v) \tag{6.19}$$

with some appropriate choice of v and ε. Here we have set $\tilde{\omega}_i(\varepsilon, v) = \omega_i + \varepsilon\mu_{i\omega}(\varepsilon, v)$ $(i = 1, 2)$, where again $\omega = (\omega_1, \omega_2)$ belongs to some class $\mathfrak{C}(\gamma, \tau)$.

Eliminating κ from the second equation in (6.19), we see that it may be replaced by the equation

$$b(v) - a(v)^{1/2}\tilde{\omega}_1^{1/2} = \tilde{\omega}_2. \tag{6.20}$$

We first show that given any v^* corresponding to a stable quasiperiodic solution of the truncated Hamiltonian we can find arbitrarily close to it a value v_ω which solves Eq. (6.20) in the case $\varepsilon = 0$ for an appropriate choice of ω in $\mathfrak{C}(\gamma, \tau)$. To this end we first consider the equation

$$b(v)\,\delta - a(v)^{1/2} + a(v^*)^{1/2} = q \tag{6.21}$$

and note that since $a(v)' \neq 0$ for δ, q near zero it possesses a solution $v = \hat{v}(\delta, q)$ such that $\hat{v}(0,0) = v^*$. Now (6.20), for $\varepsilon = 0$, can be written in the form

$$b(v)\,\omega_1^{-1/2} - a(v)^{1/2} + a(v^*)^{1/2} = q(\omega), \tag{6.22}$$

where $q(\omega) = \omega_2 \omega_1^{-1/2} + a(v^*)^{1/2}$.

Now let $D(\xi)$ be the unit disk in the $\omega_1 - \omega_2$ plane centered at $(\xi, b(v^*) - a(v^*)^{1/2}\xi^{1/2})$ and set

$$\Sigma(\gamma, \tau, v^*) = \left\{ \omega \in \bigcup_{\xi > 0} D(\xi), \omega \in \mathfrak{C}(\gamma, \tau) \right\}.$$

Here γ is chosen so small that each unit disk contains at least one point of $\mathfrak{C}(\gamma, \tau)$.

Clearly, $\Sigma(\gamma, \tau, v^*)$ contains ω vectors with arbitrarily large first components and we have

$$\lim_{\substack{\omega_1 \to \infty \\ \omega \in \Sigma(\gamma, \tau, v^*)}} q(\omega) = \lim_{\xi \to \infty} q(\xi, b(v^*) - a(v^*)^{1/2}\xi^{1/2}) = 0.$$

It follows therefore that for $\omega \in \Sigma(\gamma, \tau, v^*)$ and ω_1 sufficiently large

$$v_\omega = \hat{v}(\omega_1^{-1/2}, q(\omega)) \tag{6.23}$$

solves (6.20) in the case $\varepsilon = 0$. In the case $\varepsilon \neq 0$, the same argument goes through with ω replaced by $\tilde{\omega} = \tilde{\omega}(\varepsilon, v)$ so that in this case (6.23) becomes an equation for v:

$$v = \hat{v}(\tilde{\omega}_1^{-1/2}, q(\tilde{\omega})). \tag{6.24}$$

Notice, however, that since we have (in obvious notation)

$$\frac{\partial}{\partial v}\hat{v}(\tilde{\omega}_1^{-1/2}, q(\tilde{\omega})) = -\varepsilon(D_1\hat{v} + \tilde{\omega}_2 D_2\hat{v})\frac{1}{2}\tilde{\omega}_1^{3/2}\frac{\partial \mu_{1\omega}}{\partial v}$$

$$+ \varepsilon D_2\hat{v}\tilde{\omega}_1^{-1/2}\frac{\partial \mu_{2\omega}}{\partial v} \to 0$$

for $\omega_1 \to \infty$, there exists a solution $v = v_\omega(\varepsilon)$ of (6.24) such that $\{v_\omega(\varepsilon)\}_{\omega \in \mathbb{R}^2}$ is a normal family of functions (compare with argument at the end of the proof of Theorem 4). Since furthermore

$$\kappa_\omega(\varepsilon) = \tilde{\omega}_1(\varepsilon, v_\omega(\varepsilon))^{-1} a(v_\omega(\varepsilon))^{1/2}$$

is also a normal family of functions with the property that $\kappa_\omega(\varepsilon) \to 0$ uniformly in $\bigcap_{\omega \in \mathfrak{C}(\gamma, \tau)} \mathrm{dom}(\kappa_\omega)$ for $\omega_1 \to \infty$, the equation $\kappa_\omega(\varepsilon) = \nu\varepsilon^3$ has a unique solution ε_ω for ω_1 sufficiently large. Summarizing we see that Eqs. (6.19) have indeed solutions $\varepsilon_\omega, v_\omega(\varepsilon_\omega)$ if only ω_1 is sufficiently large in $\Sigma(\gamma, \tau, v^*)$, where γ in turn is chosen sufficiently small. The situation is most vividly described by the following theorem which is an immediate consequence of the foregoing considerations:

THEOREM 5. *Given any v^* corresponding to a family of stable two-tori of the truncated Hamiltonian, then there exist sequences $\varepsilon_n \to 0$, $v_n \to v^*$ such that the two-torus corresponding to v_n and lying on the Jacobi surface $H = -\frac{1}{2}\varepsilon_n^{-2}$ persists in the full problem (in a slightly displaced and deformed manner but with the same frequency ratio). The two-torus is densely filled with an orbit on which the third component of the angular momentum $L_3^{(n)}$ is fixed and the sequence $\{L_3^{(n)}\}$ (in the original unstretched variables) has the properties* $\lim_{n\to\infty} L_3^{(n)} = 0$ *and* $\lim_{n\to\infty} L_3^{(n)}/\nu\varepsilon_n = v^*$.

APPENDIX TO SECTION 3

We present here a sketch of a proof of Theorem 1 (see [6–11], in particular [10] for background material).

Since ψ_H is equivvariant and since by (ii) G_{μ_0} acts transitively on $\psi_H^{-1}(\mu_0)$, it is clear that

$$\psi_H(\psi_K^{-1}(\mu_0)) = \mathfrak{O}_H(\lambda_0).$$

In order to show that ψ_H induces a diffeomorphism between the spaces $\psi_K^{-1}(\mu_0)/K_{\mu_0}$ and $\mathfrak{O}_H(\lambda_0)$ it suffices to show that

$$\psi_H : \psi_K^{-1}(\mu_0) \to \mathfrak{O}_H(\lambda_0)$$

is a principal fiber bundle with structure group K_{μ_0}. Clearly, for any $\lambda = \mathrm{Ad}_h^{\#}\lambda_0 \in \mathfrak{O}_H(\lambda_0)$ ($\mathrm{Ad}^{\#}$ = coadjoint action of H), the submanifold

$$\psi_H^{-1}(\lambda) \cap \psi_K^{-1}(\mu_0)$$

is made up of K_{μ_0}-orbits. In order to show that it is made up of a single such orbit we invoke assumption (iii). We leave the details to the reader and instead turn now to a proof of formula (3.15). This will complete our sketch of the proof of Theorem 1 since formula (3.15) implies that the diffeomorphism induced by ψ_H between the spaces $\psi^{-1}(\mu_0)/K_{\mu_0}$ and $\mathfrak{O}_H(\lambda_0)$ is actually symplectic.

In order to prove (3.15) we first note that on account of the equivariance of ψ_H the vector fields X_M and $X_{\mathfrak{h}^*}$ induced by the H-action on M and $\mathfrak{h}^*$, respectively, are ψ_H-related, i.e.,

$$d\psi_H X_{M\alpha} = X_{\mathfrak{h}^* \psi(\alpha)}$$

for all $\alpha \in M$. (For notational convenience we have dropped the subscript H on the right side. This convention will be in force for the remainder of this proof). Another well-known consequence of the equivariance of ψ is the formula (see, e.g., [10, p. 147])

$$\{\hat{\psi}(X), \hat{\psi}(Y)\} = -\hat{\psi}([X, Y]),$$

where we have set $\hat{\psi}(X)(\alpha) = \psi(\alpha)(X)$ and where $\{\ ,\ \}$ is the Poisson bracket associated with ω. With the help of these formulae, relation (3.15) is easily proved:

$$\begin{aligned}(\psi^*\omega^{\mu_0})_\alpha(X_{M\alpha}, Y_{M\alpha}) &= \omega^{\mu_0}_{\psi(\alpha)}(X_{\mathfrak{h}^*\psi(\alpha)}, Y_{\mathfrak{h}^*\psi(\alpha)}) \\ &= -\langle \psi(\alpha), [X, Y]\rangle = -\hat{\psi}_\alpha([X, Y]) \\ &= \{\hat{\psi}(X), \hat{\psi}(Y)\}(\alpha) = \omega_\alpha(X_{M\alpha}, Y_{M\alpha}).\end{aligned}$$

Restricting α to $\psi_K^{-1}(\mu_0)$ yields (3.15).

Acknowledgments

In its initial stage this work was supported by a summer faculty fellowship of the University of Toledo (Summer 1980). It was completed when the author was the recipient of a stipend of the "Mathematisches Forschungsinstitut der ETH, Zuerich." Some discussions related to certain aspects of this work with Professors Ruessman, J. Moser, and R. Jost and with Dr. Eliasson are gratefully acknowledged. Last but not least, the author would like to thank Ms. Boller for her patient and superb typing of the manuscript.

References

1. M. Kummer, On the stability of Hill's solutions of the plane restricted three body problem, *Amer. J. Math.* **101** (1979), 1333–1354.
2. P. Kustaanheimo and E. Stiefel, Perturbation theory of Kepler motion based on spinor regularization, *J. Reine Angew. Math.* **218** (1965), 204–219.
3. E. L. Stiefel and G. Scheifele, "Linear and Regular Celestial Mechanics," Springer-Verlag, Berlin/Heidelberg/New York, 1971.
4. R. Jost, "Das H-Atom nach Kustaanheimo–Stiefel–Scheifele," unpublished manuscript.

5. J. M. SOURIAU, "Structure de Systèmes Dynamiques," Dunod, Paris, 1970.
6. R. ABRAHAM AND J. E. MARSDEN, "Foundations of Mechanics," Cummings, Reading, Mass., 1978.
7. A. A. KIRILLOV, "Elements of the Theory of Representations," Springer-Verlag, Berlin/New York, 1976.
8. J. MOSER, Various aspects of integrable Hamiltonian Systems, *in* "Progress in Mathematics, Vol. 8: Dynamical Systems," Birkhäuser, Basel/Boston, 1980.
9. J. MARSDEN AND A. WEINSTEIN, Reduction of symplectic manifolds with symmetries, *Rep. Math. Phys.* **5** (1974), 121–130.
10. J. MARSDEN, "Applications to Global Analysis in Mathematical Physics," Publish or Perish, 1974.
11. J. MARSDEN, Geometric methods in mathematical physics, *in* "Lectures at the NSF–CBMS Conference," Lowell, Mass., 1979.
12. R. PENROSE, Twistor algebra, *J. Math. Phys.* **8** (1967), 345–366.
13. S. STERNBERG, On the role of field theories in our physical conception of geometry, *in* "Differential Geometric Methods in Mathematical Physics II," Lecture Notes in Mathematics No. 676, pp. 1–80, Springer-Verlag, Berlin, 1977.
14. J. BAUMGARTE, Das Oszillator-Kepler Problem und die Lie-Algebra, *J. Reine Angew. Math.* **301** (1978), 59–76.
15. A. O. BARUT, "Dynamical Groups and Generalized Symmetries in Quantum Theory with Applications in Atomic and Particle Physics," Christchurch, 1972.
16. V. GUILLEMIN AND S. STERNBERG, "Geometric Asymptotics," Mathematical Surveys 14, p. 175–178, Amer. Math. Soc., Providence, R.I., 1977.
17. J. K. MOSER, Regularization of Kepler's problem and the averaging method on a manifold, *Comm. Pure Appl. Math.* **23** (1970), 609–636.
18. J. K. MOSER, Lectures on Hamiltonian systems, *Mem. Amer. Math. Soc.* **81** (1968), 1–60.
19. R. C. CHURCHILL, "Non-linear Oscillators at Low Energies," preprint.
20. C. L. SIEGEL AND J. K. MOSER, "Lectures on Celestial Mechanics," Springer-Verlag, Berlin/Heidelberg/New York, 1971.
21. V. I. ARNOLD AND A. AVEZ, "Ergodic Problems of Classical Mechanics," Benjamin, New York, 1968.
22. M. KUMMER, On resonant non-linearly coupled oscillators with two equal frequencies, *Comm. Math. Phys.* **48** (1976), 53–79.
23. M. KUMMER, On resonant classical Hamiltonians with two equal frequencies, *Comm. Math. Phys.* **58** (1978), 85–112.
24. M. KUMMER, On resonant Hamiltonians with two degrees of freedom near an equilibrium point, Lecture Notes in Physics No. 93, pp. 57–75, Springer-Verlag, Berlin/New York, 1977.
25. M. KUMMER, On the existence of invariant analytic two-manifolds in a resonant system with two degrees of freedom, *in* Lecture Notes in Pure and Applied Mathematics No. 70, pp. 91–107, Marcel Dekker, New York, 1981.
26. E. BELBRUNO, "A New Regularization of the Restricted Three Body Problem with Applications," Doctoral Thesis, New York University, New York, 1979.
27. C. CONLEY, On some new long periodic solutions of the plane restricted three body problem, *Comm. Pure Appl. Math.* **16** (1963), 449–467.
28. J. MOSER, On the theory of quasiperiodic motions, *SIAM Rev.* **8** (1966), 145–172.
29. J. MOSER, Convergent series expansions for quasi-periodic motions, *Math. Ann.* **169** (1967), 136–176.
30. S. HELGASON, "Differential Geometry and Symmetric Spaces," Academic Press, New York, 1962.

31. W. H. JEFFERYS AND J. MOSER, Quasiperiodic solutions for the three-body problem, *Astronom. J.* **71** (1966), 568–578. (See also Y. KOZAI, *Astronom. J.* **67** (1962), 591.)
32. M. KUMMER, On the regularization of the Kepler problem, *Comm. Math. Phys.* **84** (1982), 133–152.
33. T. IWAI, On a "conformal" Kepler problem and its reduction, *J. Math. Phys.* **22** (1981), 1633–1639.
34. M. KUMMER, On the construction of the reduced phase space of a Hamiltonian system with symmetry, *Indiana Univ. Math. J.* **30** (1981), 281–291.
35. R. CHURCHILL, M. KUMMER AND D. ROD, "On Averaging, Reduction, and Symmetry in Hamiltonian Systems," *Diff. Equations*, in press.