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Baire Category in Spaces of Measures

George Koumoullis

Department of Mathematics, University of Athens, Panepistemiopolis, 157 84 Athens, Greece

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1. INTRODUCTION

Let X be a topological space and let $M(X)$ denote the space of non-negative finite *Borel measures* on X (i.e., measures defined on the σ-algebra $\mathscr{B}(X)$ of Borel sets in X). The *weak topology* on $M(X)$ is the smallest

tinuous. It is well known [13] that if X is completely regular, then the relative weak topology on the space $M_t(X)$ of tight (or Radon) measures on X coincides with the weak topology induced by the space of bounded real-valued continuous functions on X. Recall that a measure in $M(X)$ is called tight if it is inner regular with respect to compact sets.

Throughout all topological statements in $M(X)$ will be with respect to the weak topology. We are concerned with the Baire category in $M(X)$ and in finite products of X. For the properties of sets involving the Baire category (sets of the first or second category, sets with the Baire property, etc.) we refer the reader to [6] and [10]. The main object of this paper is to prove in Section 2 (in a more general form) the following theorem.

THEOREM A. *Let X be a Hausdorff space and R a subset of $X \times X$ of the first category. Then $(\mu \times \mu)^*(R) = 0$ for all $\mu \in M(X)$ except for a set of measures of the first category in $M(X)$.*

We note that $\mu \times \mu$ denotes the simple product measure defined on the product σ-algebra $\mathscr{B}(X) \otimes \mathscr{B}(X)$ and that $(\mu \times \mu)^*$ denotes the outer measure induced by $\mu \times \mu$ and defined on all subsets of $X \times X$.

Theorem A is useful in proving existence theorems using the Baire category method in $M(X)$. For instance, if $\mathscr{E}$ is a family of subsets of X such that the set $R = \bigcup \{E \times E : E \in \mathscr{E}\}$ is of the first category in $X \times X$, then, by Theorem A, in the sense of category almost all measures μ in $M(X)$ vanish on $\mathscr{E}$ (i.e., $\mu^*(E) = 0$ for all $E \in \mathscr{E}$). A special case of this situation is the following corollary which is essentially Lemma 3.5 in [4]—a result that in fact led us to prove Theorem A.

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COROLLARY. *Let $f: X \to Y$ be a continuous function between Hausdorff spaces such that f is not constant at any nonempty open subset of X. Then, in the sense of category, for almost all $\mu \in M(X)$ we have $\mu(f^{-1}(\{y\})) = 0$ for every $y \in Y$.*

Indeed, the set $R = \bigcup \{f^{-1}(\{y\}) \times f^{-1}(\{y\}) : y \in Y\} = \{(x, x') \in X \times X : f(x) = f(x')\}$ is closed and nowhere dense in $X \times X$.

The above corollary when f is the identity yields the following: if X is a Hausdorff space without isolated points then in the sense of category almost all measures in $M(X)$ vanish on singletons (cf. Theorem 6.1 in [11] and Theorem 1 in [3]). Other applications of Theorem A are presented in [5].

Section 3 contains some consequences of Theorem A, among which is a partial converse of Theorem A. Namely, we assume that $M(X)$ is of the second category in itself (note that Theorem A is trivial if $M(X)$ is of the first category in itself) and prove that the conclusion of Theorem A is also a sufficient condition in order that a subset R of $X \times X$ with the Baire property is of the first category. The second category subsets of $X \times X$ are characterized similarly, replacing $(\mu \times \mu)^* (R) = 0$ by $(\mu \times \mu)^* (R) > 0$.

It follows from the above characterizations that if R is a subset of $X \times X$ with the Baire property, then either the set $\{\mu \in M(X) : (\mu \times \mu)^* (R) > 0\}$ or its complement $\{\mu \in M(X) : (\mu \times \mu)^* (R) = 0\}$ is of the first category in $M(X)$. In order to explain this phenomenon we introduce in Section 4 the class of invariant subsets of $M(X)$, which contains the above sets, and prove the following category analogue of the zero–one law: if E is an invariant subset of $M(X)$ with the Baire property then either E or $M(X) \setminus E$ is of the first category in $M(X)$.

Finally we make two general remarks. It is clear that the focus of the results described above is centered upon spaces X such that $M(X)$ is of the second category in itself (or, equivalently, $M(X)$ is a Baire space (see Section 4)). A class of such spaces is examined in [5]. Here we mention that every Čech-complete space (in particular, every compact Hausdorff space) has this property. Indeed, the space $M_t(X)$ of tight measures on a Čech-complete space X is also Čech-complete (cf. Theorem 17, Part II in [14]) and is of course dense in $M(X)$. Therefore $M(X)$ is of the second category.

In this paper we are concerned with the Baire category primarily in $M(X)$. However, the results continue to hold if we replace $M(X)$ by any of the usual spaces of measures M encountered in topological measure theory, e.g., $M = M_t(X)$, or $M = M_\tau(X)$, the space of τ-additive measures. Recall that a measure μ in $M(X)$ is called τ-additive if μ is inner regular with respect to closed sets and $\mu(G) = \sup_\alpha \mu(G_\alpha)$ for every net $\{G_\alpha\}$ of open sets filtering up to G. To cover all these cases we will state our results for a dense subset M of $M(X)$.

2. THE MAIN THEOREM

In this section we prove Theorem A of the Introduction in the more general form of subsets of X^n, where n is a positive integer; namely,

THEOREM 2.1. *Let X be a Hausdorff space, M a dense subset of $M(X)$, $n \in \mathbb{N}$, and R a subset of X^n of the first category. Then $(\mu \times \cdots \times \mu)^* (R) = 0$ for all $\mu \in M$ except for a set of measures of the first category in M.*

In proving the theorem it will be helpful to use the following concept of independence and Lemma 2.2, the second part of which is the lemma in Section 3 of [7].

Let X be a topological space, $n \in \mathbb{N}$, and R a subset of X^n. A subset A of X is said to be *R-independent* if for every $(x_1, ..., x_n) \in A^n$ with distinct coordinates (i.e., $x_i \neq x_j$ for $i \neq j$), we have $(x_1, ..., x_n) \notin R$. More generally, a family $(A_i)_{i \in I}$ of subsets of X is said to be *R-independent* if for every $(i_1, ..., i_n) \in I^n$ with distinct coordinates and for every $x_{i_j} \in A_{i_j}$, $j = 1, ..., n$, we have $(x_{i_1}, ..., x_{i_n}) \notin R$.

LEMMA 2.2. *Let X be a topological space, $n \in \mathbb{N}$, and R a subset of X^n.*

(a) *If R is closed in X^n, then for every finite R-independent subset $\{x_1, ..., x_m\}$ of X, with $x_i \neq x_j$ for $i \neq j$, there exists an R-independent family $(V_i)_{i=1, ..., m}$ of open subsets of X such that $x_i \in V_i$ for $i = 1, ..., n$.*

(b) *If R is nowhere dense in X^n, then for every finite family $(U_i)_{i=1, ..., m}$ of open nonempty subsets of X, there exists an R-independent family $(V_i)_{i=1, ..., m}$ of open nonempty subsets of X such that $V_i \subset U_i$ for $i = 1, ..., m$.*

Proof. We assume that $m \geqslant n$, the case $m < n$ being trivial.

(a) First observe that, since R is closed in X^n, we have the following: for every $(y_1, ..., y_n) \in X^n \backslash R$ and every family $(G_i)_{i=1, ..., n}$ of open subsets of X such that $y_i \in G_i$ for $i = 1, ..., n$, there exists a family $(W_i)_{i=1, ..., n}$ of open subsets of X such that $y_i \in W_i \subset G_i$ and $(\prod_{i=1}^n W_i) \cap R = \emptyset$.

Now let $\{\tau_1, ..., \tau_k\}$ be the set of all $\tau \in \{1, ..., m\}^n$ with $\tau(i) \neq \tau(j)$ for $i \neq j$, $i, j \in \{1, ..., n\}$. We shall construct inductively families $(V_i^j)_{i=1, ..., m}$, for $j = 0, 1, ..., k$, of open subsets of X such that

$$V_i^0 = X \quad \text{and} \quad x_i \in V_i^j \subset V_i^{j-1} \qquad \text{for every} \quad i = 1, ..., m \text{ and } j = 1, ..., k$$

and

$$\left(\prod_{i=1}^n V_{\tau_j(i)}^j \right) \cap R = \emptyset \qquad \text{for every} \quad j = 1, ..., k.$$

Assume that $(V_i^j)_{i=1, ..., m}$ has been constructed for some j, $0 \leqslant j < k$. Since $\{x_1, ..., x_m\}$ is R-independent we have $(x_{\tau_{j+1}(1)}, ..., x_{\tau_{j+1}(n)}) \notin R$. Thus, by the above observation, we find a family $(V^{j+1}_{\tau_{j+1}(i)})_{i=1, ..., n}$ of open subsets of X such that $x_{\tau_{j+1}(i)} \in V^{j+1}_{\tau_{j+1}(i)} \subset V^{j}_{\tau_{j+1}(i)}$ and $(\prod_{i=1}^n V^{j+1}_{\tau_{j+1}(i)}) \cap R = \varnothing$. If $i \in \{1, ..., m\} \backslash \{\tau_{j+1}(1), ..., \tau_{j+1}(n)\}$, we set $V_i^{j+1} = V_i^j$.

Finally, we set $V_i = V_i^k$ for $i = 1, ..., m$ and it is clear that $(V_i)_{i=1, ..., m}$ is the required family.

(b) As has already been mentioned, (b) is given in [7]. A proof, similar to that of (a), can be given beginning with the observation that, since R is nowhere dense in X^n, for every family $(G_i)_{i=1, ..., n}$ of nonempty open subsets of X there exists a family $(W_i)_{i=1, ..., n}$ of nonempty open subsets of X such that $W_i \subset G_i$ and $(\prod_{i=1}^n W_i) \cap R = \varnothing$. Then, as in the proof of (a), we construct inductively suitable families $(V_i^j)_{i=1, ..., m}$ for $j = 0, 1, ..., k$ so that $(V_i^k)_{i=1, ..., m}$ is the required family.

We shall use several times in the sequel (not only in the proof of Theorem 2.1) the following two simple facts about the weak topology of $M(X)$. If A is a dense subset of X, the set of measures in $M(X)$ that are carried by a finite subset of A is dense in $M(X)$ (cf. Theorem 10, Part II in [14]). These measures are expressed in the form $\sum_{i=1}^n t_i \delta_{x_i}$, where $t_i \geqslant 0$, $x_i \in A$, and δ_{x_i} denotes the Dirac measure at x_i. It is also easy to see that if G is a nonempty open subset of X, the set $\{\mu \in M(X) : \mu(G) > 0\}$ is (open) dense in $M(X)$.

Proof of Theorem 2.1. We can assume without loss of generality that $M = M(X)$. Indeed, if the set $\{\mu \in M(X) : (\mu \times \cdots \times \mu)^* (R) > 0\}$ is of the first category in $M(X)$, then the intersection of this set with M is of the first category in M because M is dense in $M(X)$. We can also assume that R is closed nowhere dense in X^n because R is included in a countable union of closed nowhere dense sets.

First we consider the simple case $n = 1$, which also follows from the arguments in [1]. We set

$$C = \{\mu \in M(X) : \mu(R) > 0\} \qquad \text{and} \qquad C_\varepsilon = \{\mu \in M(X) : \mu(R) \geqslant \varepsilon\}$$

for every $\varepsilon > 0$. Since $C = \bigcup_{n=1}^\infty C_{1/n}$, it suffices to show that C_ε is nowhere dense in $M(X)$ for every $\varepsilon > 0$. Let D be the set of measures in $M(X)$ that are carried by a finite subset of $X \backslash R$. Note that D is dense in $M(X)$ because $X \backslash R$ is dense in X. Let μ be a measure in D, i.e., $\mu = \sum_{i=1}^m t_i \delta_{x_i}$, where $t_i \geqslant 0$ and $x_i \in X \backslash R$. For every $i = 1, ..., m$, we choose an open neighborhood V_i of x_i such that $V_i \cap R = \varnothing$ and set $W = \{\nu \in M(X)$: $\nu(X \backslash \bigcup_{i=1}^m V_i) < \varepsilon\}$. Then W is an open neighborhood of μ in $M(X)$ and $W \cap C_\varepsilon = \varnothing$. This shows that $D \cap cl_{M(X)} C_\varepsilon = \varnothing$ and so C_ε is nowhere dense in $M(X)$, completing the proof for $n = 1$.

We now assume that $n \geqslant 2$ and set

$$G_0 = \{x \in X : x \text{ is isolated in } X\} \qquad \text{and} \qquad G_1 = X \setminus \bar{G}_0.$$

Since there are no isolated points in R, $R \cap G_0^n = \emptyset$. Thus

$$R = \bigcup_{\sigma \in \{0,1\}^n} R \cap S(\sigma(1), ..., \sigma(n)),$$

where

$$S(\sigma(1), ..., \sigma(n)) = \prod_{i=1}^{n} G_{\sigma(i)}$$

for $\sigma \in \{0, 1\}^n \setminus \{(0, 0, ..., 0\}$ and

$$S(0, 0, ..., 0) = \bigcup_{i=1}^{n} \{(x_1, ..., x_n) \in X^n : x_i \in \bar{G}_0 \setminus G_0\}.$$

It suffices to prove the theorem when R is replaced by the set $R \cap S(\sigma(1), ..., \sigma(n))$ for some $\sigma \in \{0, 1\}^n$, that is, when R is a relatively closed subset of $S(\sigma(1), ..., \sigma(n))$ and is of course nowhere dense in X^n.

If R is included in $S(0, 0, ..., 0)$, then

$$\{\mu \in M(X) : (\mu \times \cdots \times \mu)^* (R) > 0\} \subset \{\mu \in M(X) : \mu(\bar{G}_0 \setminus G_0) > 0\},$$

where the set of measures on the right is of the first category in $M(X)$ because $\bar{G}_0 \setminus G_0$ is closed nowhere dense in X and the theorem holds for $n = 1$. Thus, we can assume that R is a nonempty relatively closed subset of $S(\sigma(1), ..., \sigma(n))$ where $\sigma \in \{0, 1\}^n \setminus \{(0, 0, ..., 0)\}$.

We set

$$C_\varepsilon = \{\mu \in M(X) : (\mu \times \cdots \times \mu)^* (R) \geqslant \varepsilon\}$$

for every $\varepsilon > 0$. As in the case $n = 1$, it suffices to prove that C_ε is nowhere dense in $M(X)$ for every $\varepsilon > 0$. Fix an $\varepsilon > 0$ throughout the proof.

We also set

$$I_0 = \{i \in \{1, ..., n\} : \sigma(i) = 0\}, \qquad I_1 = \{i \in \{1, ..., n\} : \sigma(i) = 1\}$$

and let n_i be the number of elements of I_i for $i = 0, 1$. Thus, $n_0 + n_1 = n$ and $n_1 > 0$. If $n_0 > 0$ we identify $S(\sigma(1), ..., \sigma(n))$ with $G_0^{I_0} \times G_1^{I_1}$ and for every $z = (z(i))_{i \in I_0} \in G_0^{I_0}$ we set

$$R_z = \{w = (w(j))_{j \in I_1} : (z, w) \in R\}.$$

Claim I. For every $z \in G_0^{I_0}$, R_z is closed and nowhere dense in $G_1^{I_1}$.

Indeed, R_z is closed in $G_1^{I_1}$ because R is closed in $S(\sigma(1), ..., \sigma(n)) = G_0^{I_0} \times G_1^{I_1}$. Also, if $V = \operatorname{int}_{G_1^{I_1}}(R_z)$ then $\{z\} \times V$ is open in $X^n = X^{I_0} \times X^{I_1}$ and is included in R. Since R is nowhere dense in X^n, $V = \emptyset$ and so R_z is nowhere dense.

Next we observe that every measure $\mu \in M(X)$ carried by a finite subset of $G_0 \cup G_1$ with $\mu(G_1) > 0$ has the following expression:

$$\mu = \sum_{i=1}^{m_0} t_i \delta_{x_i} + \sum_{j=1}^{m_1} s_j \delta_{y_j}, \qquad (*)$$

where $m_0 \geqslant 0$; $m_1 \geqslant 1$; $x_1, ..., x_{m_0}$ and $y_1, ..., y_{m_1}$ are distinct points of G_0 and G_1, respectively; and $t_i, s_j \in \mathbb{R}$, $t_i, s_j > 0$. Since $G_0 \cup G_1$ is dense in X and $\{\mu \in M(X) : \mu(G_1) > 0\}$ is open dense in $M(X)$, it follows that the set of measures of this form is dense in $M(X)$. For every $r > 0$ we denote by D_r the set of measures given by $(*)$ so that the following additional properties are satisfied:

(i) $m_1 \geqslant n_1$ and $m_1^{n_1} - m_1(m_1-1)\cdots(m_1-n_1+1) < (r/s_j)^{n_1}$ for every $j = 1, ..., m_1$; and

(ii) $\{y_1, ..., y_{m_1}\}$ is R-independent and R_z-independent for every $z \in \{x_1, ..., x_{m_0}\}^{I_0}$.

Claim II. For every $r > 0$, D_r is dense in $M(X)$.

Let μ be a measure given by $(*)$. Since the set of these measures is dense in $M(X)$, Claim II follows if we show that $\mu \in cl_{M(X)} D_r$.

We set

$$\tilde{R} = \begin{cases} \bigcup \{R_z : z \in \{x_1, ..., x_{m_0}\}^{I_0}\}, & \text{if } m_0 > 0 \text{ and } n_0 > 0 \\ R, & \text{otherwise.} \end{cases}$$

By Claim I, $\tilde{R}$ is closed nowhere dense in $G_1^{I_1}$ if either $m_0 > 0$ or $n_0 = 0$.

For every $j = 1, ..., m_1$, let $U_j \subset G_1$ be an open neighborhood of y_j. Since G_1 has no isolated points, for every $j = 1, ..., m_1$ there are pairwise disjoint nonempty open sets $U_{j,k} \subset U_j$, $k = 1, ..., p$, where p is sufficiently large (to be specified later). Now we choose nonempty open sets $V_{j,k} \subset U_{j,k}$ for $j = 1, ..., m_1$ and $k = 1, ..., p$, such that the family $(V_{j,k})_{j,k}$ is $\tilde{R}$-independent. We can do this by Lemma 2.2(b) when either $m_0 > 0$ or $n_0 = 0$ because then $\tilde{R}$ is nowhere dense in $G_1^{I_1}$. The case where $m_0 = 0$ and $n_0 > 0$ is trivial (take $V_{j,k} = U_{j,k}$). Finally, we choose $y_{j,k} \in V_{j,k}$ and set

$$\mu_{(U_1, ..., U_{m_1})} = \sum_{i=1}^{m_0} t_i \delta_{x_i} + \sum_{j=1}^{m_1} \sum_{k=1}^{p} \frac{s_j}{p} \delta_{y_{j,k}}$$

and

$$\mu_{U_j} = \sum_{k=1}^{p} \frac{s_j}{p} \delta_{y_{j,k}}.$$

It is easy to see that for every $j = 1, ..., m_1$, the net (μ_{U_j}), where the family of open neighborhoods U_j of y_j is directed in the obvious way, converges to $s_j \delta_{y_j}$. It follows that the net $(\mu_{(U_1, ..., U_{m_1})})$ converges to μ. Also, the above expression of $\mu_{(U_1, ..., U_{m1})}$ is as in $(*)$ and satisfies property (ii) of D_r. If, moreover, p is chosen so that $pm_1 \geqslant n_1$ and for every $j = 1, ..., m_1$

$$m_1^{n_1} - m_1 \left(m_1 - \frac{1}{p}\right) \cdots \left(m_1 - \frac{n_1 - 1}{p}\right) < \frac{r^{n_1}}{s_j^{n_1}},$$

that is,

$$(m_1 p)^{n_1} - m_1 p(m_1 p - 1) \cdots (m_1 p - n_1 + 1) < r^{n_1} \Big/ \left(\frac{s_j}{p}\right)^{n_1},$$

then $\mu_{(U_1, ..., U_{m_1})}$ satisfies property (i) of D_r as well. Therefore, $\mu_{(U_1, ..., U_{m_1})} \in D_r$ and we have shown that $\mu \in cl_{M(X)} D_r$, completing the proof of Claim II.

Claim III. If $\mu \in D_r$, $0 < r < 1$, and we assume that either (a) $n_0 > 0$, $r^{n_1} \mu(G_0)^{n_0} < \varepsilon/3$, and $\mu(G_1) > r^{1/2}$ or (b) $n_0 = 0$ and $r^n < \varepsilon$, then there exists an open neighborhood W of μ such that $W \cap C_\varepsilon = \emptyset$.

We have that μ is given by $(*)$ so that properties (i) and (ii) are satisfied.

Case (a). Condition (a) is satisfied.

If $m_0 = 0$ (i.e., $\mu(G_0) = 0$) we set

$$W = \{ v \in M(X) : v(\bar{G}_0) < (\varepsilon/(\mu(G_1) + 1)^{n_1})^{1/n_0} \text{ and } v(\bar{G}_1) < \mu(G_1) + 1 \}.$$

Then W is an open neighborhood of μ and for every $v \in W$,

$$\begin{aligned}(v \times \cdots \times v)^* (R) &\leqslant (v \times \cdots \times v)(S(\sigma(1), ..., \sigma(n))) \\ &= v(G_0)^{n_0} v(G_1)^{n_1} < \varepsilon,\end{aligned}$$

so $W \cap C_\varepsilon = \emptyset$. Thus we can assume that $m_0 > 0$.

We set

$$\tilde{R} = \bigcup \{ R_z : z \in \{ x_1, ..., x_{m_0} \}^{I_0} \}.$$

It is clear that $\tilde{R}$ is closed in $G_1^{I_1}$ and, by property (ii) of D_r, $\{ y_1, ..., y_{m_1} \}$ is $\tilde{R}$-independent. By Lemma 2.2(a), we can find pairwise disjoint open sets $V_j \subset G_1$, $j = 1, ..., m_1$, such that $y_j \in V_j$ and $(V_j)_{j=1, ..., m_1}$ is $\tilde{R}$-independent. Let W be the set of all $v \in M(X)$ satisfying the following conditions:

$$\nu\left(\bar{G}_1 \Big\backslash \bigcup_{j=1}^{m_1} V_j\right) < \alpha, \quad \nu(V_j) > s_j - \beta, \qquad j = 1, ..., m_1,$$

$$\nu(\bar{G}_1) < \mu(G_1) + \beta,$$

$$\nu(\bar{G}_0 \backslash \{x_1, ..., x_{m_0}\}) < \alpha, \quad \nu(\{x_i\}) > t_i - \beta, \qquad i = 1, ..., m_0,$$

$$r^{n_1} \nu(\bar{G}_0)^{n_0} < \varepsilon/3 \qquad \text{and} \qquad \nu(G_1) > r^{1/2},$$

where $0 < \alpha < \beta < 1$ are sufficiently small (to be specified later). Clearly W is an open neighborhood of μ.

It will be convenient to set

$$Z = \{x_1, ..., x_{m_0}\}^{I_0}, \qquad T = \{1, ..., m_1\}^{I_1},$$

and

$$T_1 = \{\tau \in T : \tau(i) \neq \tau(j) \text{ if } i, j \in I_1, i \neq j\}.$$

Notice that T_1 has $m_1(m_1 - 1) \cdots (m_1 - n_1 + 1)$ elements. Finally we set

$$V = \bigcup_{z \in Z} \bigcup_{\tau \in T_1} \left(\prod_{i \in I_0} \{z(i)\} \times \prod_{j \in I_1} V_{\tau(j)} \right).$$

Thus V is a finite union of disjoint open rectangles, $V \subset S(\sigma(1), ..., \sigma(n))$, and $R \subset S(\sigma(1), ..., \sigma(n)) \backslash V$ by the choice of V_j, $j = 1, ..., m_1$. It remains to prove that α and β can be chosen so that for every $\nu \in W$

$$(\nu \times \cdots \times \nu)(V) > (\nu \times \cdots \times \nu)(S(\sigma(1), ..., \sigma(n))) - \varepsilon. \tag{1}$$

Indeed, then we should have $(\nu \times \cdots \times \nu)^* (R) < \varepsilon$ for every $\nu \in W$, i.e., $W \cap C_\varepsilon = \varnothing$.

First, let us note that for every $\nu \in M(X)$,

$$\begin{aligned}
(\nu \times \cdots \times \nu)(V) &= \sum_{z \in Z} \sum_{\tau \in T_1} (\nu \times \cdots \times \nu) \left(\prod_{i \in I_0} \{z(i)\} \times \prod_{j \in I_1} V_{\tau(j)} \right) \\
&= \sum_{z \in Z} \sum_{\tau \in T_1} \left(\prod_{i \in I_0} \nu(\{z(i)\}) \cdot \prod_{j \in I_1} \nu(V_{\tau(j)}) \right) \\
&= \left(\sum_{z \in Z} \prod_{i \in I_0} \nu(\{z(i)\}) \right) \cdot \left(\sum_{\tau \in T_1} \prod_{j \in I_1} \nu(V_{\tau(j)}) \right) \\
&= \left(\sum_{i=1}^{m_0} \nu(\{x_i\}) \right)^{n_0} \\
&\quad \cdot \left[\left(\sum_{j=1}^{m_1} \nu(V_j) \right)^{n_1} - \sum_{\tau \in T \backslash T_1} \prod_{j \in I_1} \nu(V_{\tau(j)}) \right].
\end{aligned} \tag{2}$$

We shall need some bounds of $\nu(\bar{G}_i)$, $i=0, 1$, and $\nu(V_j)$, $j=1, ..., m_1$, when $\nu \in W$. For this purpose we assume that

$$\beta < \frac{\mu(G_i)}{m_i+1} \qquad \text{for} \quad i=0, 1. \tag{3}$$

(We shall impose other restrictions on β later.) Then, by the definition of W, we have

$$\nu(\bar{G}_0) \geqslant \sum_{i=1}^{m_0} \nu(\{x_i\}) > \sum_{i=1}^{m_0} (t_i - \beta) = \mu(G_0) - m_0\beta > \beta$$

and

$$\nu(\bar{G}_1) \geqslant \sum_{j=1}^{m_1} \nu(V_j) > \sum_{j=1}^{m_1} (s_j - \beta) = \mu(G_1) - m_1\beta > \beta$$

for every $\nu \in W$. Let γ be such that $\gamma^{n_0} r^{n_1} > \varepsilon/3$ and $\gamma > \mu(G_1) + 1$. Then

$$\beta < \nu(\bar{G}_i) < \gamma \qquad \text{for every} \quad \nu \in W \text{ and } i=0, 1. \tag{4}$$

Also, by the definition of W, we have

$$\nu(V_j) \leqslant \nu(\bar{G}_1) - \sum_{\substack{k=1\\k\neq j}}^{m_1} \nu(V_k) < \sum_{k=1}^{m_1} s_k + \beta - \sum_{\substack{k=1\\k\neq j}}^{m_1} (s_k - \beta) = s_j + m_1\beta$$

for every $\nu \in W$ and $j=1, ..., m_1$. Thus, setting $s = \max\{s_j : j=1, ..., m_1\}$, we have

$$\nu(V_j) < s + m_1\beta \qquad \text{for every} \quad \nu \in W \text{ and } j=1, ..., m_1. \tag{5}$$

From (4) we have $\nu(\bar{G}_i) > \beta > \alpha$ for every $\nu \in W$ and $i=0, 1$ and so by the definition of W

$$\left(\sum_{i=1}^{m_0} \nu(\{x_i\})\right)^{n_0} > (\nu(\bar{G}_0) - \alpha)^{n_0} > 0$$

and

$$\left(\sum_{j=1}^{m_1} \nu(V_j)\right)^{n_1} > (\nu(\bar{G}_1) - \alpha)^{n_1},$$

for every $v \in W$. Also, from (5) and property (i) of $\mu \in D_r$ we have

$$\sum_{\tau \in T \setminus T_1} \prod_{j \in I_1} v(V_{\tau(j)}) < (m_1^{n_1} - m_1(m_1-1)\cdots(m_1-n_1+1))(s+m_1\beta)^{n_1}$$
$$< \frac{r^{n_1}}{s^{n_1}}(s+m_1\beta)^{n_1} = r^{n_1}\left(1+\frac{m_1}{s}\beta\right)^{n_1}$$

for every $v \in W$. Therefore, it follows from (2) that for every $v \in W$

$$(v \times \cdots \times v)(V) > (v(\bar{G}_0)-\alpha)^{n_0} \cdot \left[(v(\bar{G}_1)-\alpha)^{n_1} - r^{n_1}\left(1+\frac{m_1}{s}\beta\right)^{n_1}\right]. \tag{6}$$

Finally, α and β are specified as follows. Let $\delta_1 > 0$ be such that $(1+r^{n_1})\,\gamma^{n_0}\delta_1 < \varepsilon/3$ and $(1+r^{n_1})\,\delta_1 < r^{n_1/2} - r^{n_1}$ and choose $\beta > 0$ satisfying (3) such that

$$\left(1+\frac{m_1}{s}\beta\right)^{n_1} < 1+\delta_1.$$

Next, let $\delta_0 > 0$ be such that $\delta_0\gamma^{n_1} < \varepsilon/3$ and $\delta_0 < \beta^{n_0}$ and choose $\alpha > 0$, $\alpha < \beta$, such that

$$t^{n_i} - (t-\alpha)^{n_i} < \delta_i \qquad \text{for every} \quad t \in [\beta, \gamma] \text{ and } i = 0, 1.$$

It now follows from (4) that for every $v \in W$

$$(v(\bar{G}_0)-\alpha)^{n_0} > v(\bar{G}_0)^{n_0} - \delta_0 > 0$$

and, using also the last condition of the definition of W,

$$(v(\bar{G}_1)-\alpha)^{n_1} - r^{n_1}\left(1+\frac{m_1}{s}\beta\right)^{n_1} > v(\bar{G}_1)^{n_1} - \delta_1 - r^{n_1}(1+\delta_1)$$
$$> r^{n_1/2} - r^{n_1} - \delta_1(1+r^{n_1}) > 0.$$

Thus, comparing with (6), we have

$$\begin{aligned}(v \times \cdots \times v)(V) &> (v(\bar{G}_0)^{n_0} - \delta_0)(v(\bar{G}_1)^{n_1} - \delta_1 - r^{n_1}(1+\delta_1)) \\ &> v(\bar{G}_0)^{n_0}\, v(\bar{G}_1)^{n_1} - \delta_0 v(\bar{G}_1)^{n_1} - (\delta_1 + r^{n_1} + r^{n_1}\delta_1)\, v(\bar{G}_0)^{n_0} \\ &> v(G_0)^{n_0}\, v(G_1)^{n_1} - \delta_0\gamma^{n_1} - r^{n_1} v(\bar{G}_0)^{n_0} - \delta_1(1+r^{n_1})\,\gamma^{n_0} \\ &> (v \times \cdots \times v)(S(\sigma(1), ..., \sigma(n))) - \varepsilon\end{aligned}$$

for every $v \in W$, i.e., inequality (1).

Case (b). Condition (b) is satisfied.

The proof of Claim III in this case follows the lines of the proof of Case (a) for $m_0 > 0$ (although here we may have $m_0 = 0$) and is simpler. Thus we choose V_j, $j = 1, ..., m_1$, as in case (a) (here $\tilde{R} = R$) and define in the same way, ignoring the last two conditions involving r, an open neighborhood W of μ depending on α and β, $0 < \alpha < \beta < 1$. Then we set

$$V = \bigcup_{\tau \in T_1} \prod_{j=1}^{n} V_{\tau(j)}$$

and equality (2) becomes

$$(v \times \cdots \times v)(V) = \left(\sum_{j=1}^{m_1} v(V_j) \right)^n - \sum_{\tau \in T \setminus T_1} \prod_{j=1}^{n} v(V_{\tau(j)})$$

for every $v \in M(X)$ (note that here $n_1 = n$ and $I_1 = \{1, ..., n\}$). Next we assume (3) for $i = 1$ and, setting $\gamma = \mu(G_1) + 1$, we have (4) for $i = 1$ and (5). Now (6) becomes

$$(v \times \cdots \times v)(V) > (v(\bar{G}_1) - \alpha)^n - r^n \left(1 + \frac{m_1}{s} \beta \right)^n$$

for every $v \in W$.

Finally, α and β are specified as follows. Let $\delta > 0$ be such that $\delta(1 + r^n) < \varepsilon - r^n$ and choose β satisfying (3) for $i = 1$ such that

$$\left(1 + \frac{m_1}{s} \beta \right)^n < 1 + \delta.$$

Then choose $\alpha < \beta$ such that

$$t^n - (t - \alpha)^n < \delta \qquad \text{for every} \quad t \in [\beta, \gamma].$$

It now follows that for every $v \in W$

$$\begin{aligned} (v \times \cdots \times v)(V) &> v(\bar{G}_1)^n - \delta - r^n(1 + \delta) \\ &> v(G_1)^n - \varepsilon \\ &= (v \times \cdots \times v)(S(\sigma(1), ..., \sigma(n))) - \varepsilon. \end{aligned}$$

As in case (a), this completes the proof of Claim III in Case (b).

Last, using Claims II and III, we show that C_ε is nowhere dense in $M(X)$, completing the proof of the theorem. First assume that $n_0 > 0$ and set

$$M_0 = \{\mu \in M(X) : \mu(G_1) > 0\}$$

and

$$M_r = \{\mu \in M(X) : \mu(G_1) > r^{1/2} \text{ and } r^{n_1}\mu(\bar{G}_0)^{n_0} < \varepsilon/3\}$$

for $0 < r < 1$. Then $M_0 = \bigcup_{0<r<1} M_r$, each M_r is open in $M(X)$, and M_0 is open dense in $M(X)$. By Claim II, $D_r \cap M_r$ is dense in M_r and so Claim III (Case (a)) implies that $C_\varepsilon \cap M_r$ is nowhere dense in M_r for every r with $0 < r < 1$. Thus $C_\varepsilon \cap M_0$ is nowhere dense in M_0 and so C_ε is nowhere dense in $M(X)$. Now assume that $n_0 = 0$ and fix an r with $0 < r < 1$ and $r^n < \varepsilon$. By Claim II, D_r is dense in $M(X)$ and so Claim III (Case (b)) implies that C_ε is nowhere dense in $M(X)$.

3. CONSEQUENCES

In this section we present some consequences of Theorem 2.1. The following result, part of which is Theorem 2.1 itself, is a characterization of those subsets of X^n with the Baire property that are of the first category (resp. of the second category).

THEOREM 3.1. *Let X be a Hausdorff space, M a dense subset of $M(X)$ such that M is of the second category in itself, $n \in \mathbb{N}$, and R a subset of X^n with the Baire property. Then*

(a) *R is of the first category in X^n if and only if $(\mu \times \cdots \times \mu)^*(R) = 0$ for all $\mu \in M$ except for a set of measures of the first category in M;*

(b) *R is of the second category in X^n if and only if $(\mu \times \cdots \times \mu)^*(R) > 0$ for all $\mu \in M$ except for a set of measures of the first category in M.*

Proof. The "only if" part of (a) is Theorem 2.1. The "only if" part of (b) is proved as follows: Assume that R is of the second category in X^n, so $R = G \Delta P$, where G is nonempty open and P is of the first category in X^n. Let V_i, $i = 1, \ldots, n$, be nonempty open subsets of X such that $\prod_{i=1}^n V_i \subset G$. Then $R \supset (\prod_{i=1}^n V_i) \setminus P$ and, by Theorem 2.1, there exists a subset C of M of the first category such that $(\mu \times \cdots \times \mu)^*(P) = 0$ for all $\mu \in M \setminus C$. It follows that

$$\{\mu \in M : (\mu \times \cdots \times \mu)^*(R) = 0\} \subset C \cup \bigcup_{i=1}^{n} \{\mu \in M : \mu(V_i) = 0\},$$

where the right side is a set of the first category in M. This completes the proof of the "only if" part of (b).

Finally, the "if" parts of (a) and (b) follow immediately from the above and the assumption that M is of the second category in itself.

The hypothesis of Theorem 3.1(a) and (b) that M is of the second category in itself cannot be dropped. Indeed, if M is of the first category in itself, then any subset of X^n trivially satisfies the conditions in (a) and (b). A similar remark holds for the hypothesis that R has the Baire property as the following example shows.

EXAMPLE 3.2. Assume the continuum hypothesis. Let $X = \mathbb{R}$, $M = M(X)$, and R a Lusin set in X, i.e., R is an uncountable subset of X such that $R \cap P$ is countable for every subset P of X of the first category; the existence of a Lusin set in X follows from the continuum hypothesis (see, e.g., [6, p. 525]). Then M is of the second category in itself (in fact, M is a Polish space) and R is of the second category in X. However, $\mu^*(R) = 0$ for all $\mu \in M$ except for a set of measures of the first category in M. Indeed, let E be the set of measures in M that are nonatomic. It is well known (and follows from the comments made after the corollary of the Introduction) that $M \backslash E$ is of the first category in M. Also, if $\mu \in E$ then there exists an F_σ subset P of X of the first category such that $\mu(X \backslash P) = 0$ (cf. Theorem 1.6 in [10]) and so $\mu^*(R) = \mu^*(R \cap P) = 0$.

Theorem 3.1 for $n = 2$ takes the following form which should be compared with the Kuratowski–Ulam theorem and its partial converse (Theorems 15.1 and 15.4 in [10]).

COROLLARY 3.3. *Let X be a Hausdorff space, M a dense subset of $M(X)$ such that M is of the second category in itself, and R a subset of $X \times X$ with the Baire property. Then R is of the first category* (*resp. of the second category*) *in $X \times X$ if and only if for all $\mu \in M$ except for a set of measures of the first category in M the following condition holds* (*resp. fails*):

$$\mu^*(R_x) = 0 \qquad \text{for} \quad \mu\text{-almost all } x \in X, \tag{$*$}$$

where $R_x = \{y \in X : (x, y) \in R\}$.

Proof. As in Theorem 3.1 it suffices to prove the "only if" parts.

By Fubini's theorem every $\mu \in M(X)$ with $(\mu \times \mu)^*(R) = 0$ satisfies condition $(*)$ of the corollary. Thus, if R is of the first category in $X \times X$, then by Theorem 2.1 (for $n = 2$) all $\mu \in M$ except for a set of measures of the first category in M satisfy $(*)$.

Now assume that R is of the second category in $X \times X$. Since R has the Baire property, it contains a set of the form $(U \times V) \backslash P$, where U and V

are nonempty open sets in X and P is of the first category in $X \times X$. By the above, there exists a subset Q_0 of M of the first category such that for every $\mu \in M \backslash Q_0$ we have $\mu^*(P_x) = 0$ for μ-almost all $x \in X$. We set $Q = Q_0 \cup \{\mu \in M : \mu(U)\,\mu(V) = 0\}$. Then Q is of the first category in M and for every $\mu \in M \backslash Q$, $(*)$ fails since $\mu^*(R_x) \geqslant \mu^*(V \backslash P_x) = \mu(V) > 0$ for μ-almost all $x \in U$ and $\mu(U) > 0$.

As in Theorem 3.1, the hypothesis of Corollary 3.3 that R has the Baire property cannot be dropped. This is shown in the following example without any use of set theoretic hypotheses. It follows from the same example that condition $(*)$ of Corollary 3.3 for some $R \subset X \times X$ is not equivalent to $(\mu \times \mu)^*\,(R) = 0$. Thus Corollary 3.3 is not just a special case of Theorem 3.1 (using Fubini's theorem).

EXAMPLE 3.4. Let $X = \mathbb{R}$ and $M = M(X)$. A slight modification of an example of Sierpinski (see Theorem 14.4 in [10]) shows that there exists a subset R of $X \times X$ such that (a) R meets every closed uncountable subset of $X \times X$ and (b) no three points of R are collinear. Let E be the set of nonatomic measures in M. Then $M \backslash E$ is of the first category in M and it follows from (a) that R is of the second category in $X \times X$ and $(\mu \times \mu)^*\,(R) > 0$ for all non-zero $\mu \in E$ and from (b) that $(*)$ of Corollary 3.3 holds for all $\mu \in E$.

If X is the space of Theorem 3.1 or Corollary 3.3, then $M(X)$ is of the second category since it contains a dense subset of the second category in itself. The following corollary gives some information about these spaces X (see also Remark 4 following Corollary 4.5).

COROLLARY 3.5. *Let X be a Hausdorff space such that $M(X)$ is of the second category. Then X^n is a Baire space for every $n = 1, 2, \ldots$.*

Proof. Let G be a nonempty open subset of X^n. We choose nonempty open subsets V_i, $i = 1, \ldots, n$, of X such that $\prod_{i=1}^{n} V_i \subset G$ and observe that

$$\{\mu \in M(X) : (\mu \times \cdots \times \mu)^*\,(G) = 0\} \subset \bigcup_{i=1}^{n} \{\mu \in M(X) : \mu(V_i) = 0\},$$

where the right side is a set of the first category in $M(X)$. By Theorem 3.1(b), G is of the second category in X^n. Therefore X^n is a Baire space.

In the next two results we consider products of different spaces and prove another version of Theorem 3.1 and a generalization of Corollary 3.5.

COROLLARY 3.6. *Let $X_1, ..., X_n$ be Hausdorff spaces, M_i a dense subset of $M(X_i)$ for $i=1, ..., n$ such that $\prod_{i=1}^n M_i$ is of the second category in itself and R a subset of $\prod_{i=1}^n X_i$ with the Baire property. Then R is of the first category (resp. of the second category) in $\prod_{i=1}^n X_i$ if and only if $(\mu_1 \times \cdots \times \mu_n)^*(R)=0$ (resp. $(\mu_1 \times \cdots \times \mu_n)^*(R)>0$) for all $(\mu_1, ..., \mu_n) \in \prod_{i=1}^n M_i$ except for a subset of $\prod_{i=1}^n M_i$ of the first category.*

Proof. Let $X=X_1 \oplus \cdots \oplus X_n$ be the topological sum of $X_1, ..., X_n$ and set $M=\{\sum_{i=1}^n \bar{\mu}_i : \mu_i \in M_i \text{ for } i=1, ..., n\}$, where $\bar{\mu}_i$ denotes the Borel measure on X given by $\bar{\mu}_i(B)=\mu_i(B \cap X_i)$. Then M is a dense subset of $M(X)$ and, since each X_i is closed and open in X, the function

$$h\colon \prod_{i=1}^n M_i \to M \qquad \text{with} \quad h(\mu_1, ..., \mu_n)=\sum_{i=1}^n \bar{\mu}_i$$

is a homeomorphism. So M is of the second category in itself. Also, $\prod_{i=1}^n X_i$ is considered as an open subspace of X^n and so R is a subset of X^n with the Baire property.

We set

$$E=\left\{(\mu_1, ..., \mu_n) \in \prod_{i=1}^n M_i : (\mu_1 \times \cdots \times \mu_n)^*(R)>0\right\}$$

and

$$F=\{\mu \in M : (\mu \times \cdots \times \mu)^*(R)>0\}$$

and observe that $h(E)=F$. So E is of the first category in $\prod_{i=1}^n M_i$ if and only if F is of the first category in M. By Theorem 3.1, this happens if and only if R is of the first category in X^n or, equivalently, in $\prod_{i=1}^n X_i$. This completes the proof of the first part of the theorem. The second part is completely analogous.

COROLLARY 3.7. *Let $X_1, ..., X_n$ be Hausdorff spaces such that $\prod_{i=1}^n M(X_i)$ is of the second category in itself. Then for every $m \in \mathbb{N}$ and every $\tau \in \{1, ..., n\}^m$, $\prod_{i=1}^m X_{\tau(i)}$ is a Baire space.*

Proof. If the Cartesian product of two spaces is a Baire space, then it is well known that both spaces are Baire spaces. Thus it suffices to show that for every $k \in \mathbb{N}$ $(\prod_{i=1}^n X_i)^k$ is a Baire space.

As in the Proof of Corollary 3.6, we see that $\prod_{i=1}^n M(X_i)$ is homeomorphic to $M(X)$ where X is the topological sum of $X_1, ..., X_n$. Thus $M(X)$ is of the second category in itself and, since $\prod_{i=1}^n X_i$ is open in X^n, it follows from Corollary 3.5 that $(\prod_{i=1}^n X_i)^k$ is a Baire space for every $k \in \mathbb{N}$.

Remarks. Corollary 3.5 was proved by Wójcicka [15] when X is a metric space. It should be noted that Wójcicka's proof can be adapted to yield Theorem 2.1 when X is a metric space without isolated points.

The hypothesis in Corollary 3.6 that $\prod_{i=1}^{n} M_i$ is of the second category in itself is stronger than the hypothesis that each M_i is of the second category in itself. Indeed, Wójcicka [15] showed that there exist metric spaces X_i, $i=1, 2$, such that $M(X_i)$ is of the second category in itself for $i=1, 2$, but $M(X_1) \times M(X_2)$ is of the first category in itself.

4. INVARIANT SETS OF MEASURES

Throughout this section X is a Hausdorff space. Let R be a subset of X^n for some $n \in \mathbb{N}$ and set

$$E = \{\mu \in M(X) : (\mu \times \cdots \times \mu)^* (R) > 0\}. \qquad (*)$$

It follows from Theorem 3.1 that if R has the Baire property in X^n, then either E or $M(X) \backslash E$ is of the first category in $M(X)$. A natural question is whether the same conclusion holds whenever E has the Baire property in $M(X)$. One may also ask whether E has always the Baire property. Note that if E has the Baire property, then R need not have the Baire property (see Example 3.2 or 3.4; also a Bernstein set [10, p. 24] gives an example similar to 3.2 without the continuum hypothesis).

In this section we introduce a class of invariant (under some equivalence relation) sets in $M(X)$, which contains the sets E given by $(*)$. The main result is Theorem 4.2 which provides an affirmative answer to the first question for all invariant sets in $M(X)$ (Corollary 4.5(a)). The second question is answered in the negative for invariant sets and, under the continuum hypothesis, for sets given by $(*)$ (Proposition 4.6).

Definitions. If $\mu \in M(X)$ and B is a Borel set in X, we denote by μ_B the Borel measure on X given by $\mu_B(A) = \mu(B \cap A)$. We define a relation $\sim$ on $M(X)$ by $\mu \sim \nu$ if and only if there exist $c_1, ..., c_n > 0$ $(n \in \mathbb{N})$ and Borel sets $B_1, ..., B_n$ in X such that $X = \bigcup_{i=1}^{n} B_i$ and $\nu = \sum_{i=1}^{n} c_i \mu_{B_i}$.

It is clear that the sets $B_1, ..., B_n$ can be chosen to be disjoint and that $\sim$ is an equivalence relation. Also, this relation can be defined equivalently as follows: $\mu \sim \nu$ if and only if there exists a simple Borel measurable function $f: X \to (0, \infty)$ such that $\nu(A) = \int_A f \, d\mu$ for every Borel set A in X.

If E is a subset of $M(X)$ we say that E is *invariant under* $\sim$ in $M(X)$ or simply *invariant* if for every $\mu, \nu \in M(X)$ with $\mu \sim \nu$ we have $\mu \in E$ if and only if $\nu \in E$. More generally, if r is an equivalence relation on a set S, the invariant under r subsets of S are similarly defined.

EXAMPLES 4.1. The following subsets of $M(X)$ are invariant: the sets of the form $\{\mu \in M(X) : (\mu \times \cdots \times \mu)^* (R) > 0\}$, where R is a subset of X^n and $n \in \mathbb{N}$, the spaces of measures $M_t(X)$, $M_\tau(X)$, and $M(X)$, the set of measures in $M(X)$ vanishing on singletons and the set of measures $\mu \in M(X)$ with full support (i.e., with $\mu(U) > 0$ for every nonempty open $U \subset X$). In fact, it can be proved that these sets are invariant under the weaker equivalence relation $\sim^*$ given by $\mu \sim^* \nu$ if and only if μ and ν have the same nullsets. A different example of invariant set is $\{\mu \in M(X): \lim_n \mu^*(R_n) = 0\}$, where R_n, $n = 1, 2, ...,$ is a given sequence of subsets of X.

THEOREM 4.2. *Let M be a dense subset of $M(X)$ and E a subset of M such that E is invariant in $M(X)$. Then either E is of the first category in M or E is of the second category at any point of M (i.e., $E \cap G$ is of the second category in M for every nonempty open set G in M).*

For the proof of Theorem 4.2 we need two lemmas and the following notation. If $V_1, ..., V_n$ $(n \in \mathbb{N})$ are disjoint nonempty open sets in X, $\alpha_1, ..., \alpha_n \geqslant 0$, and $\alpha > 0$ such that $\sum_{i=1}^n \alpha_i < \alpha$, we set

$$N(V_1, ..., V_n; \alpha_1, ..., \alpha_n, \alpha)$$
$$= \{\mu \in M(X) : \mu(V_i) > \alpha_i, i = 1, ..., n \text{ and } \mu(X) < \alpha\}.$$

We denote by $\mathcal{N}$ the family of all subsets of $M(X)$ of this form.

LEMMA 4.3. *$\mathcal{N}$ is a pseudobase for the topology of $M(X)$.*

Proof. Clearly $\mathcal{N}$ consists of nonempty open subsets of $M(X)$. Let $\nu = \sum_{i=1}^n t_i \delta_{x_i} \in M(X)$, where $t_i > 0$, $x_i \in X$, and $x_i \neq x_j$ for $i \neq j$. Since the set of these measures is dense in $M(X)$, it suffices to show that the family of all $N \in \mathcal{N}$ with $\nu \in N$ is a neighborhood base for ν.

Let G be a basic open set in $M(X)$, i.e., $G = \{\mu \in M(X) : \mu(U_j) > \beta_j$, $j = 1, ..., k$, and $\mu(X) < \beta\}$ for some nonempty open sets $U_1, ..., U_k$ in X and some $\beta_1, ..., \beta_k \geqslant 0$ and $\beta > 0$, such that $\nu \in G$. We choose disjoint open sets $V_1, ..., V_n$ in X such that $x_i \in V_i$, $i = 1, ..., n$, and if x_i belongs to some U_j then $V_i \subset U_j$. Next we choose $\alpha_1, ..., \alpha_k \geqslant 0$ and $\alpha > 0$ such that $t_i - \varepsilon/n < \alpha_i < t_i$, $i = 1, ..., n$, and $\sum_{i=1}^n t_i < \alpha < \sum_{i=1}^n t_i + \varepsilon$, where $\varepsilon > 0$ is such that $\varepsilon < \nu(U_j) - \beta_j$, $j = 1, ..., k$, and $\varepsilon < \beta - \nu(X)$. Finally we set $N = N(V_1, ..., V_n; \alpha_1, ..., \alpha_n, \alpha)$. It is clear that $\nu \in N \in \mathcal{N}$. Also, if $\mu \in N$ then

$$\mu(X) < \sum_{i=1}^n t_i + \varepsilon = \nu(X) + \varepsilon < \beta$$

and for every $j = 1, ..., k$

$$\begin{aligned}\mu(U_j) &\geqslant \sum \{\mu(V_i): V_i \subset U_j\} = \sum \{\mu(V_i): x_i \in U_j\} \\ &> \sum \{t_i - \varepsilon/n : x_i \in U_j\} \geqslant \sum \{t_i : x_i \in U_j\} - \varepsilon \\ &= \nu(U_j) - \varepsilon > \beta_j.\end{aligned}$$

Therefore $N \subset G$, completing the proof of the lemma.

In the next lemma we shall use the following concepts. Let $f: S \to T$ be a function between topological spaces. We say that f is *feebly continuous* if for every open subset V of T with $f^{-1}(V) \neq \varnothing$, the interior of $f^{-1}(V)$ in S is nonempty. If f is one to one and onto and both f and f^{-1} are feebly continuous, then we say that f is a *feeble homeomorphism*. We shall also use the simple fact that if f is a feeble homeomorphism and N is a subset of T, then N is of the first category in T (resp. the interior of N in T is empty) if and only if $f^{-1}(N)$ has the corresponding property in S (cf. Proposition 4.4 in [2]).

LEMMA 4.4. *Let $V_1, ..., V_n$ be disjoint nonempty open subsets of X, $\alpha_i, \beta_i \geqslant 0$ for $i = 1, ..., n$ and $\alpha > \sum_{i=1}^n \alpha_i$, $\beta > \sum_{i=1}^n \beta_i$. Then there is a feeble homeomorphism h: $N(V_1, ..., V_n; \alpha_1, ..., \alpha_n, \alpha) \to N(V_1, ..., V_n; \beta_1, ..., \beta_n, \beta)$ such that $h(\mu) \sim \mu$ for every $\mu \in N(V_1, ..., V_n; \alpha_1, ..., \alpha_n, \alpha)$.*

Proof. We set

$$N_\alpha = N(V_1, ..., V_n; \alpha_1, ..., \alpha_n, \alpha), \qquad N_\beta = N(V_1, ..., V_n; \beta_1, ..., \beta_n, \beta),$$

$$V = \bigcup_{i=1}^n V_i, \qquad \bar{\alpha} = \sum_{i=1}^n \alpha_i, \qquad \bar{\beta} = \sum_{i=1}^n \beta_i, \qquad \text{and} \qquad c = \frac{\beta - \bar{\beta}}{\alpha - \bar{\alpha}} > 0.$$

Then we define

$$\begin{aligned}&\varphi_i: \mathbb{R} \to \mathbb{R} \quad \text{with} \quad \varphi_i(x) = c(x - \alpha_i) + \beta_i \text{ for } i = 1, ..., n, \\ &\varphi: \mathbb{R} \to \mathbb{R} \quad \text{with} \quad \varphi(x) = c(x - \bar{\alpha}) + \bar{\beta}\end{aligned}$$

and for every $\mu \in N_\alpha$

$$h(\mu) = \sum_{i=1}^n \frac{\varphi_i(\mu(V_i))}{\mu(V_i)} \mu_{V_i} + c\mu_{(X \setminus V)}.$$

It is clear that $h(\mu) \in M(X)$ and $h(\mu) \sim \mu$. Next we show that $h(\mu) \in N_\beta$. Indeed, we have

$$h(\mu)(V_i) = \varphi_i(\mu(V_i)) > \varphi_i(\alpha_i) = \beta_i,$$

since $\mu(V_i) > \alpha_i$ for $i = 1, ..., n$, and

$$h(\mu)(X) = \sum_{i=1}^{n} \varphi_i(\mu(V_i)) + c\mu(X \backslash V) = c(\mu(V) - \bar{\alpha}) + \bar{\beta} + c\mu(X \backslash V)$$
$$= \varphi(\mu(X)) < \varphi(\alpha) = \beta,$$

since $\mu(X) < \alpha$. Thus h: $N_\alpha \to N_\beta$ and it is easy to verify that h is one to one and onto. In fact, h^{-1}: $N_\beta \to N_\alpha$ is given by

$$h^{-1}(v) = \sum_{i=1}^{n} \frac{\varphi_i^{-1}(v(V_i))}{v(V_i)} v_{V_i} + \frac{1}{c} v_{(X \backslash V)}$$

for every $v \in N_\beta$. Since h and h^{-1} are of the same form, it suffices to show that h is feebly continuous.

Since N_β is open in $M(X)$, by Lemma 4.3 every nonempty open subset of N_β contains an open set of the form

$$N = N(U_1, ..., U_k; \gamma_1, ..., \gamma_k, \gamma) \in \mathcal{N}.$$

Moreover, we can assume that for every $j = 1, ..., k$, either $U_j \subset V_i$ for some i or $U_j \subset X \backslash V$. (If $U_j \cap V \neq \varnothing$, we replace U_j by $U_j \cap V_i$ where i is such that $U_j \cap V_i \neq \varnothing$.) Now it is easy to check that

$$h^{-1}(N) = \left\{ \mu \in N_\alpha : \mu(U_j) > \frac{\gamma_j \mu(V_i)}{\varphi_i(\mu(V_i))} \text{ if } U_j \subset V_i, \mu(U_j) > \frac{\gamma_j}{c} \right.$$
$$\left. \text{if } U_j \subset X \backslash V, \text{ and } \mu(X) < \varphi^{-1}(\gamma) \right\}.$$

Thus $h^{-1}(N)$ is open in N_α and h is feebly continuous.

Proof of Theorem 4.2. Since M is dense in $M(X)$, if E is of the first category in $M(X)$ (resp. of the second category at any point of $M(X)$) then E is of the first category in M (resp. of the second category at any point of M). Thus if suffices to prove the theorem when $M = M(X)$.

Assume that E is of the second category in $M(X)$ and let $N = N(V_1, ..., V_n; \alpha_1, ..., \alpha_n, \alpha)$ be an element of $\mathcal{N}$. We set

$$N_k = N\left(V_1, ..., V_n; \frac{\alpha_1}{k}, ..., \frac{\alpha_n}{k}, k\alpha\right)$$

for $k=1,2,\ldots$. By Lemma 4.4 for every k there exists a feeble homeomorphism $h_k\colon N\to N_k$ such that $h_k(\mu)\sim\mu$ for every $\mu\in N$.

Since $\bigcup_{k=1}^{\infty} N_k=\{\mu\in M(X):\mu(V_i)>0 \text{ for } i=1,\ldots,n\}$ is open dense in $M(X)$ and E is of the second category in $M(X)$, there exists $k\in\mathbb{N}$ such that $E\cap N_k$ is of the second category in $M(X)$ and so in N_k. Since E is invariant $h_k(E\cap N)=E\cap N_k$. Thus $E\cap N$ is of the second category in N and so in $M(X)$ (because N is open in $M(X)$). It now follows from Lemma 4.3 that E is of the second category at any point of $M(X)$.

Corollary 4.5. (a) *Let M and E be as in Theorem 4.2. If E has the Baire property in M then either E or $M\backslash E$ is of the first category in M.*

(b) *Every second category invariant set in $M(X)$ is a Baire space.*

Proof. (a) Assume that $M\backslash E$ is of the second category in M. Since $M\backslash E$ has the Baire property in M, $M\backslash E$ contains a set of the form $G\backslash P$, where G is nonempty open and P is of the first category in M. It follows that $G\cap E\subset P$ and so $G\cap E$ is of the first category in M. By Theorem 4.2, E is of the first category in M.

(b) If E is a second category invariant set in $M(X)$, then by Theorem 4.2 (for $M=M(X)$) every nonempty relatively open subset of E is of the second category in $M(X)$ and so in E. Thus E is a Baire space.

Remarks. 1. Using the method of the proof of Theorem 4.2 we can prove that if E is an invariant set in $M(X)$, then

(a) E is either nowhere dense or dense in $M(X)$; and

(b) the interior of E in $M(X)$ is either empty or dense in $M(X)$.

Indeed, assume that for some $N=N(V_1,\ldots,V_n;\alpha_1,\ldots,\alpha_n,\alpha)\in\mathcal{N}$, E is dense in N (resp. E contains N). Let N_k and h_k, $k=1,2,\ldots$, be as in the proof of Theorem 4.2. Since $h_k(E\cap N)=E\cap N_k$ it follows that for every k, E is dense in N_k (resp. E contains N_k). But $\bigcup_{k=1}^{\infty} N_k$ is open dense in $M(X)$ and so E is dense in $M(X)$ (resp. E contains an open dense subset of $M(X)$).

2. As mentioned in the Introduction the reason for considering a dense subset M of $M(X)$ in the formulation of our results is to cover the cases where M is a space of measures ($M=M(X)$, $M_\tau(X)$, or $M_t(X)$). Since the spaces of measures are invariant (Examples 4.1), it follows that if M in Theorem 4.2 and Corollary 4.5(a) is a space of measures, then the hypothesis that E is invariant in $M(X)$ can be replaced by "E is invariant in M," that is, invariant under the restriction of $\sim$ to M.

3. Let S be a topological space and r be an equivalence relation on S. We say that (S,r) *satisfies the category zero-one law* if the following

category analogue of the zero-one law of Kolmogorov holds: for every invariant under r set E in S with the Baire property, either E or $S \setminus E$ is of the first category in S. Several category zero-one laws are known (see, e.g., [8, 9, 12] and the references given there). It follows from Corollary 4.5(a) and Remark 2 that every space of measures equipped with $\sim$ satisfies the category zero-one law.

4. Every space of measures of the second category in itself is a Baire space. This follows from Corollary 4.5(b) since the spaces of measures are invariant in $M(X)$. In particular, we have that $M(X)$ in Corollary 3.5 and $\prod_{i=1}^{n} M(X_i)$ in Corollary 3.7 are Baire spaces.

Next we prove the existence of invariant sets without the Baire property.

PROPOSITION 4.6. *Let X be a Polish space (i.e., a metrizable space by a metric for which it is separable and complete) without isolated points. Then there exists an invariant set E in $M(X)$ without the Baire property. Moreover, if we assume the continuum hypothesis, E can be chosen to be of the form $\{\mu \in M(X) : \mu^*(R) > 0\}$ for some $R \subset X$.*

Proof. Let $\{E_i : i \in I\}$ be the partition of $M(X)$ into the $\sim$-equivalence classes. First we prove that every E_i is of the first category in $M(X)$. For every $i \in I$, we choose $\mu_i \in E_i$. If $\mu_i = 0$, then $E_i = \{0\}$ is of the first category in $M(X)$. If $\mu_i \neq 0$, then there exists a subset A_i of X of the first category with $\mu_i^*(A_i) > 0$ (cf. Theorem 1.6 in [10]). It follows that E_i is included in the set $\{\mu \in M(X) : \mu^*(A_i) > 0\}$ which is of the first category in $M(X)$ by Theorem 2.1 (for $n = 1$). Therefore every E_i is of the first category.

Assume, if possible, that for every $J \subset I$, $\bigcup_{i \in J} E_i$ has the Baire property in $M(X)$. Since $\bigcup_{i \in J} E_i$ is invariant, it follows from Corollary 4.5(a) that either $\bigcup_{i \in J} E_i$ or $M(X) \setminus \bigcup_{i \in J} E_i$ is of the first category in $M(X)$. For every $J \subset I$ we set $\lambda(J) = 0$, if $\bigcup_{i \in J} E_i$ is of the first category, and $\lambda(J) = 1$, otherwise. It is clear that λ is a $\{0, 1\}$-valued measure defined on all subsets of I and vanishing on singletons. Also, $\lambda(I) = 1$ since $M(X)$ is of the second category. But this is a contradiction since I has the cardinal of the continuum (see [6, p. 533]). Thus, there exists some $J \subset I$ such that $\bigcup_{i \in J} E_i$ does not have the Baire property and clearly $\bigcup_{i \in J} E_i$ is the desired set E.

Now assume the continuum hypothesis and let $(M_\alpha)_{\alpha < \omega_1}$ be an enumeration of all second category G_δ subsets of $M(X)$. We construct by induction first category F_σ subsets P_α, Q_α of X and $\mu_\alpha, \nu_\alpha \in M(X) \setminus \{0\}$ for every $\alpha < \omega_1$ such that

$$P_\alpha \cup Q_\alpha \subset X \Big\backslash \bigcup_{\beta < \alpha} (P_\beta \cup Q_\beta), \qquad P_\alpha \cap Q_\alpha = \varnothing$$

$$\mu_\alpha, \nu_\alpha \in M_\alpha, \qquad \mu_\alpha(X \setminus P_\alpha) = 0, \qquad \text{and} \qquad \nu_\alpha(X \setminus Q_\alpha) = 0.$$

Assume that P_β, Q_β, μ_β, and ν_β have been constructed for $\beta<\alpha$. Since $\bigcup_{\beta<\alpha}(P_\beta\cup Q_\beta)$ is of the first category in X, the set $\{\mu\in M(X): \mu(\bigcup_{\beta<\alpha}(P_\beta\cup Q_\beta))>0\}$ is of the first category in $M(X)$ (by Theorem 2.1 for $n=1$). Thus we can choose $\mu_\alpha\in M_\alpha\backslash\{0\}$ such that $\mu_\alpha(\bigcup_{\beta<\alpha}(P_\beta\cup Q_\beta))=0$. Then we choose a first category F_σ subset P_α of X such that $P_\alpha\subset X\backslash\bigcup_{\beta<\alpha}(P_\beta\cup Q_\beta)$ and $\mu_\alpha(X\backslash P_\alpha)=0$. Similarly, using that $P_\alpha\cup(\bigcup_{\beta<\alpha}(P_\beta\cup Q_\beta))$ is of the first category, we choose Q_α and ν_α with the required properties. The construction is now complete.

We set $R=\bigcup_{\alpha<\omega_1}P_\alpha$ and $E=\{\mu\in M(X):\mu^*(R)>0\}$. Since $\mu_\alpha\in M_\alpha\cap E$ and $\nu_\alpha\in M_\alpha\backslash E$ for every $\alpha<\omega_1$, E and $M(X)\backslash E$ do not contain any second category G_δ subset of $M(X)$. Therefore, E does not have the Baire property (see Theorem 4.4 in [10]).

Finally, let us remark that the results of this paper remain valid if we replace $M(X)$ by the space $M^1(X)=\{\mu\in M(X):\mu(X)=1\}$ of probability measures. To see this one can check that the proofs with slight modifications apply to probability measures. However, it is much easier to use the results proved so far in conjunction with the following lemma, the simple proof of which is omitted.

LEMMA 4.7. *Let* φ: $(0,\infty)\times M^1(X)\to M(X)\backslash\{0\}$ *be given by* $\varphi(r,\mu)=r\mu$. *Then*

(a) φ *is a homeomorphism and for every* $\mu\in M(X)\backslash\{0\}$, $\varphi^{-1}(\mu)=(\mu(X),\mu(X)^{-1}\mu)$;

(b) *given a subset* E *of* $M(X)\backslash\{0\}$, *we have* $\varphi^{-1}(E)=(0,\infty)\times(E\cap M^1(X))$ *if and only if* $r\mu\in E$ *for every* $r>0$ *and* $\mu\in E$;

(c) *given a subset* E *of* $M^1(X)$, *we have that* E *is invariant in* $M^1(X)$ *if and only if* $\varphi((0,\infty)\times E)$ *is invariant in* $M(X)$.

First let us prove that Theorems 2.1 and 4.2 remain valid if we replace $M(X)$ by $M^1(X)$. We can assume that the dense subset M of $M^1(X)$ is $M^1(X)$ itself. Let φ be as in Lemma 4.7.

Let R be a first category subset of X^n and set $E=\{\mu\in M(X): (\mu\times\cdots\times\mu)^*(R)>0\}$. By Theorem 2.1, E is of the first category in $M(X)$. By Lemma 4.7(a) and (b), $\varphi^{-1}(E)=(0,\infty)\times(E\cap M^1(X))$ is of the first category in $(0,\infty)\times M^1(X)$ and so $E\cap M^1(X)=\{\mu\in M^1(X): (\mu\times\cdots\times\mu)^*(R)>0\}$ is of the first category in $M^1(X)$. Therefore, Theorem 2.1 holds for probability measures.

Now let E be an invariant set in $M^1(X)$. By Lemma 4.7(c), $\varphi((0,\infty)\times E)$ is invariant in $M(X)$. By Theorem 4.2, either $\varphi((0,\infty)\times E)$ is of the first category in $M(X)$ or $\varphi((0,\infty)\times E)$ is of the second category at any point of $M(X)$. Thus, using Lemma 4.7(a), we conclude that either E is of the first category in $M^1(X)$ or E is of the second category at any point of $M^1(X)$. Therefore, Theorem 4.2 holds for probability measures.

It is now clear that 3.1, 3.3, 3.5, 4.5, and 4.6 for probability measures can be proved either by the above method or as consequences of Theorems 2.1 and 4.2 for probability measures. Note also that by Lemma 4.7(a), the hypothesis of Corollary 3.5 that $M(X)$ is of the second category is equivalent to the hypothesis of Corollary 3.5 for probability measures that $M^1(X)$ is of the second category (in itself).

To prove Corollary 3.6 for probability measures we proceed as in the proof of 3.6 using a different homeomorphism. Namely, we set

$$\Delta_n = \left\{ (t_1, ..., t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = 1, t_i \geqslant 0 \text{ for } i = 1, ..., n \right\}$$

and

$$M = \left\{ \sum_{i=1}^{n} t_i \bar{\mu}_i : (t_1, ..., t_n) \in \Delta_n, \mu_i \in M_i \text{ for } i = 1, ..., n \right\}$$

and define the homeomorphism

$$h: \Delta_n \times \prod_{i=1}^{n} M_i \to M \text{ with } h(t_1, ..., t_n, \mu_1, ..., \mu_n) = \sum_{i=1}^{n} t_i \bar{\mu}_i.$$

If E and F are as in the proof of 3.6, then $h(\Delta_n \times E) = F$ and the result follows from Theorem 3.1 for probability measures. Finally, Corollary 3.7 for probability measures follows similarly since, by the above, $\Delta_n \times \prod_{i=1}^{n} M^1(X_i)$ is homeomorphic to $M^1(X)$.

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