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Traffic flow modelling with junctions

Magali Mercier *

Institut Camille Jordan, Université Claude Bernard Lyon I, 43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France

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ABSTRACT

Motivated by the modelling of a roundabout, we are led to study the traffic on a road with points of entry and exit. In this note, we would like to describe the modellisation of a junction and solve the Riemann problem for such a model. More precisely, between each point of discontinuity we use a multi-class extension of the LWR model to describe the evolution of the density of the vehicles, the 'multi-class' approach being used in order to distinguish the vehicles after their origin and destination. Then, we treat the points of entry and exit thanks to special boundary conditions that give bounds on the flows of the different types of vehicles. In the case of the one-T road we obtain a result of existence and uniqueness. This first step allows us to obtain a similar result for the n-T road. We describe these results and also some properties of the obtained solutions, in order to see how long this model is valid.

1. Introduction

Traffic modelling, in particular from a macroscopic point of view has been intensively investigated since the seminal paper by Lighthill and Whitham [15] and Richards [17], see for example [10,16]. This paper is motivated by the modelling of roundabouts. Some papers have already tackled this problem by considering the roundabouts as special networks, see [11] or [6]. Here, we want rather to consider a roundabout as an infinite road with points of entry and exit periodically distributed, so that a period corresponds to the roundabout's perimeter. In particular, we do not want to study here the traffic on the roads of arrival and exit.

More precisely, the model we introduce is such that between two points of entry and exit the traffic is governed by the LWR model, so that the total density of the vehicles, denoted by $r(t,x)$, verifies the equation:

$$\partial_t r + \partial_x\big(r v(r)\big) = 0, \tag{1.1}$$

where v is a given speed law. In fact, we will rather consider here a multi-class extension of the LWR model, as in [14], differentiating the vehicles after the place they come from and the place they are going to. As this distinction is quite artificial, we have to give the same speed law v for all the types of vehicles. Finally, if ρ_a is the density of one type of vehicle, it verifies the following equation on each open segment between two points of discontinuity:

$$\partial_t \rho_a + \partial_x\big(\rho_a v(r)\big) = 0.$$

By summation over all a, we re-find Eq. (1.1).

Besides, the points of entry and exit are treated thanks to special boundary conditions, inspired from the Bardos–Le Roux–Nédélec ones ([2], [19, Chapter 15]). These boundary conditions are given as inequalities and follow in fact the

* Temporary address: Dipartimento di Matematica, Università degli Studi di Brescia, Via Branze 38, 25123 Brescia, Italy.
E-mail address: mercier@math.univ-lyon1.fr.

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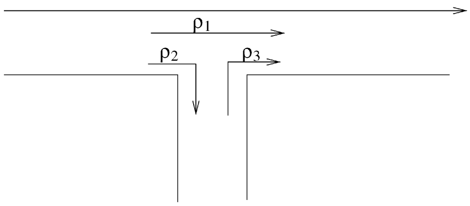

Fig. 1. A 'T-road'.

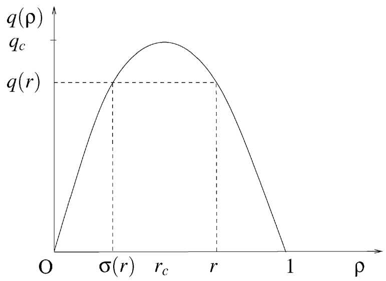

Fig. 2. Fundamental diagram for the flow density.

same ideas as R. Colombo and P. Goatin [7]: they consist in bounds on the flows of vehicles and give a constraint corresponding to the capacity of the secondary road the vehicles are entering or exiting.

We will first treat the case of the 'one-T road' (see Fig. 1), and then glue the solutions obtained for each point of entry and exit (in order to obtain a local in time solution). The idea is that, at one point of entry and exit, we can differentiate three types of vehicles: the vehicles that go straight, of density ρ_1, the vehicles that are about to exit the road, of density ρ_2, and the vehicles that have entered the road, of density ρ_3. Then we require that, across the point of discontinuity, the flow of ρ_1 is conserved, the flow of ρ_2 is less than some prescribed output function, and the flow of ρ_3 is less than some prescribed input function. We obtain in this way a unique weak entropy solution.

In order to treat the n-T case, we have only to collate the local solutions at each point of discontinuity, the finite propagation speed allowing in this case to obtain a unique local in time weak entropy solution. We can also give a lower bound on the time of existence of the solution.

This paper is structured as follows: in Section 2, we describe more precisely the model and we give our principal results; in Section 3, we give the details of the proof in the case of the 'one-T road'; and in Section 4, we give the details of the proof in the general case.

2. Description of the models and main results

2.1. The 'one-T' junction

General hypotheses. Throughout, we assume the following conditions on the speed law:

(**V**) All the vehicles have the same speed law $v : [0,1] \to \mathbb{R}_+$, which is $\mathcal{C}^{0,1}$, decreasing and vanishes at $r = 1$.
Here we denote $\mathcal{C}^{0,1}$ the set of continuous Lipschitz functions; $\mathbb{R}_+$ is the interval $[0,+\infty)$ and $\mathbb{R}_+^*$ is the interval $]0,+\infty)$.

(**F**) The flow $q : [r \mapsto rv(r)]$ is strictly concave and attains its maximum q_c at $r = r_c \in\,]0,1[$, see Fig. 2.

Below, we denote $\sigma : [0,1] \mapsto [0,1]$ the continuous map uniquely defined by

$$rv(r) = \sigma(r)v\big(\sigma(r)\big)$$

and $\sigma(r) \neq r$ for $r \neq r_c$.

Particular case. For a concrete example, we may take

$$v(r) = V_m(1-r).$$

In this case, we have $q(r) = V_m r(1-r)$, $r_c = 1/2$, $q_c = V_m/4$ and $\sigma(r) = 1 - r$.

Description of the 'one-T' model. Let ρ_1 be the density of the population that neither enters nor exits the road, ρ_2 the density of the population that exits and ρ_3 the density of the population that enters. Assuming that the behaviour of drivers, modelled by the speed law $r \to v(r)$, is independent from both their origin and their destination, we are led to the local conservation laws:

$$\begin{cases} \partial_t \rho_1 + \partial_x\big(\rho_1 v(\rho_1+\rho_2)\big) = 0, \\ \partial_t \rho_2 + \partial_x\big(\rho_2 v(\rho_1+\rho_2)\big) = 0 \end{cases} \text{for } x<0, \qquad \begin{cases} \partial_t \rho_1 + \partial_x\big(\rho_1 v(\rho_1+\rho_3)\big) = 0, \\ \partial_t \rho_3 + \partial_x\big(\rho_3 v(\rho_1+\rho_3)\big) = 0 \end{cases} \text{for } x>0. \tag{2.2}$$

Furthermore, we add the initial data:

$$\begin{aligned} &\rho_1(0,x) = \rho_{1,0}(x) \quad \text{for } x \in \mathbb{R}, \\ &\rho_2(0,x) = \rho_{2,0}(x) \quad \text{for } x < 0, \\ &\rho_3(0,x) = \rho_{3,0}(x) \quad \text{for } x > 0, \end{aligned} \tag{2.3}$$

and the following *'boundary'* conditions, that can be compared with the ones given in [7], where a toll-gate is considered:

$$\begin{aligned} &\rho_1 v(\rho_1+\rho_2)(t,0-) = \rho_1 v(\rho_1+\rho_3)(t,0+) && \max, \\ &\rho_2 v(\rho_1+\rho_2)(t,0-) \leqslant o(t) && \max, \\ &\rho_3 v(\rho_1+\rho_3)(t,0+) \leqslant i(t) && \max, \end{aligned} \tag{2.4}$$

where o and i are some prescribed output and input functions taking values in $\mathbb{R}_+$. In these equations, *max* means that the flows of ρ_1, ρ_2 and ρ_3 are maximised.

These conditions signify that the flow of ρ_1 is conserved across the points of discontinuity, whereas the flow of ρ_2 (respectively ρ_3) must be less than the capacity of the secondary road this kind of vehicles is entering (respectively exiting).

At this point, we have to add a priority rule; otherwise, it will not be possible to decide which flow is maximised first. Our goal being to treat the case of a roundabout, we choose to give priority to the vehicles that are already on the road. This means that we maximise the flows of ρ_2 and ρ_1 first, and only after the flow of ρ_3.

Remark 2.1. It is *a priori* not obvious that we can maximise at the same time the flows of ρ_1 and ρ_2, but it will turn out in the resolution that, except in the special cases where we have to treat null densities at time $t=0$, the maximisation of these densities is simultaneous.

Definitions.

Definition 2.2. Let $o, i \in \boldsymbol{BV}([0,+\infty); [0,q_c])$. A *solution* to (2.2)–(2.3)–(2.4) is a triple of functions

$$\rho_1 \in \boldsymbol{BV}\big(\mathbb{R}; [0,1]\big), \qquad \rho_2 \in \boldsymbol{BV}\big((-\infty,0); [0,1]\big), \qquad \rho_3 \in \boldsymbol{BV}\big((0,+\infty); [0,1]\big)$$

such that

1. (ρ_1,ρ_2) is a *weak entropy solution* to (2.2) for $(t,x) \in [0,+\infty) \times (-\infty,0)$;
2. (ρ_1,ρ_3) is a *weak entropy solution* to (2.3) for $(t,x) \in [0,+\infty) \times (0,+\infty)$;
3. for a.e. $x \in \mathbb{R}$, the traces of ρ_1, ρ_2, ρ_3 in $t=0$ satisfy (2.3);
4. for a.e. $t>0$, the traces of ρ_1, ρ_2, ρ_3 in $x=0$ satisfy (2.4).

We denote below $\mathcal{S} = \{\boldsymbol{\rho} = (\rho, \widetilde{\rho}) \in \mathbb{R}^2\colon\ \rho \geqslant 0,\ \widetilde{\rho} \geqslant 0,\ \rho + \widetilde{\rho} \leqslant 1\}$.

Definition 2.3. By *Riemann problem for* (2.2)–(2.3)–(2.4) we mean (2.2)–(2.3)–(2.4) with constant boundary values $i \geqslant 0$ and $o \geqslant 0$ for the inflow and the outflow, and with constant initial data, i.e.

$$\begin{aligned} &\rho_1(0,x) = \rho_1^- \quad \text{for } x \in (-\infty,0), \qquad \rho_2(0,x) = \rho_2^- \quad \text{for } x \in (-\infty,0), \\ &\rho_1(0,x) = \rho_1^+ \quad \text{for } x \in [0,+\infty), \qquad \rho_3(0,x) = \rho_3^+ \quad \text{for } x \in (0,+\infty). \end{aligned} \tag{2.5}$$

In this case, we obtain the following result:

Theorem 2.4. *Under the hypotheses* (**V**) *and* (**F**), *the Riemann problem for* (2.2)–(2.3)–(2.4)–(2.5) *admits a unique solution in the sense of Definition* 2.2.

Furthermore, when $o>0$, $i>0$, there exist some invariant sets $T_{a,b} = \{(\rho_1,\rho_2) \in \mathcal{S};\ a \leqslant \rho_1+\rho_2 \leqslant b\}$, for $a \geqslant 0$ small enough and $b \leqslant 1$ large enough.

On $\mathcal{S}$ the Riemann solver for the considered problem is not continuous. However, it is continuous on some subset: for $o \in [\varepsilon,1]$ with $\varepsilon>0$ and $\boldsymbol{\rho} \in T_{0,b}$, with $b<1$ ($T_{0,b}$ being an invariant set), the solution is obtained continuously.

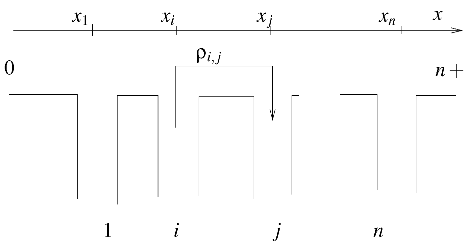

Fig. 3. An 'n-T road'.

Qualitative properties.

Remark 2.5. If $\rho_1 = 0$ at time $t = 0$, then $\rho_1(t, x) = 0$ for all t and x and (2.2)–(2.3)–(2.4) decouples in two independent IBVPs. From the traffic point of view, it means that those who exit do not interact with those who enter.

Remark 2.6. If $i \equiv 0$, $\rho_2 = 0$ and $\rho_3 = 0$ at time $t = 0$, then $\rho_2(t, x) = \rho_3(t, x) = 0$ for all t and x, while ρ_1 is the usual solution to a scalar conservation law, and we recover the classical LWR model, as exposed in Whitham's book [21].

We observe here that some discontinuities appear when $o, \rho_2^- \to 0$ and also when $r^+ \to 1$, $\rho_1^- \to 0$. We can explain this qualitatively as follows: when a road where vehicles can go is jammed (which corresponds to $o = 0$, respectively $r^+ = 1$), the behaviour of the traffic on the left-side of the road will depend dramatically on the presence or the absence of vehicles willing to go on this jammed road. In fact, in the case $o = 0$ (respectively $r^+ = 1$), if $\rho_2^- > 0$ (respectively $\rho_1^- > 0$) then the total density on the left will increase abruptly to 1; whereas if $\rho_2^- = 0$ (respectively $\rho_1^- = 0$) then the problem arising from the blocked road is simply ignored and nothing particular happens.

2.2. The '*n*-T' case

In order to modelise a roundabout we would like to address the problem of an infinite road with entries periodically distributed, so that a period corresponds to the perimeter of a roundabout. More generally, we can also consider an infinite road with a countable number of points of entry and exit, the points of discontinuity not accumulating in some point. In fact, we consider first the easier case of a road with n points of entry and exit and $n \in \mathbb{N}$, $n < \infty$. Indeed, thanks to the finite propagation speed, it is sufficient to see what happens in this case to obtain a local in time result, even in the case of a countable number of points periodically distributed. Later, if we want to study what happens for larger times on a roundabout, it will be necessary to consider the case of countable periodically distributed discontinuities.

We have to introduce here some notations. We assume that the points of entry and exit are located in the points $x_k \in \mathbb{R}$, for $k \in [\![1, n]\!]$, with $x_{k-1} < x_k < x_{k+1} < \cdots$. Furthermore, we number the main entry by 0, and the main exit by $n + 1$. In this context, we call $\rho_{i,j}$ the density of vehicles which enter in i and exit in j (see Fig. 3); in fact, the multi-class approach by origin–destination has already been introduced in the previous work [14]. As no vehicle enters in $n + 1$ or exits in 0, we finally have $(n + 1)^2$ unknowns: the $(\rho_{i,j})_{i \in [\![0,n]\!], j \in [\![1,n+1]\!]}$.

Remark 2.7. We do not allow vehicles to do more than one turn. This means that just after x_k, say in x_k^+, $\rho_{i,k} = 0$ for all $i \in [\![0, n]\!] \setminus \{k\}$; and just before x_k, say in x_k^-, $\rho_{k,j} = 0$ for all $j \in [\![1, n + 1]\!] \setminus \{k\}$.

We add here the following hypothesis:

(**P**) We know the numbers $p_{i,j}$ that represent the proportion of vehicles entering in i that are to exit in j.

We have the following (local) conservation laws on the intervals $(-\infty, x_1)$, $(x_n, +\infty)$ and (x_k, x_{k+1}), for $k \in [\![1, n]\!]$:

$$\forall i \in [\![0, n]\!],\ \forall j \in [\![1, n + 1]\!], \quad \partial_t \rho_{i,j} + \partial_x\left(\rho_{i,j} v\left(\sum_{l,m} \rho_{l,m}\right)\right) = 0. \tag{2.6}$$

For all $k \in [\![1, n]\!]$, we could prescribe as boundary conditions in x_k:

$$\begin{aligned}
&\forall i \neq k,\ \forall j \neq k, \quad \rho_{i,j} v\left(\sum_{l,m} \rho_{l,m}\right)(t, x_k^-) = \rho_{i,j} v\left(\sum_{l,m} \rho_{l,m}\right)(t, x_k^+) \quad \max,\\
&\forall i \in [\![0, n]\!], \quad \rho_{i,k} v\left(\sum_{l,m} \rho_{l,m}\right)(t, x_k^-) \leqslant o_k(t) \quad \max,\\
&\forall j \in [\![1, n + 1]\!], \quad \rho_{k,j} v\left(\sum_{l,m} \rho_{l,m}\right)(t, x_k^+) \leqslant i_k(t) \quad \max,
\end{aligned} \tag{2.7}$$

the flows of the $\rho_{i,j}$ being maximised in all these equations.

However, it is better to consider stronger boundary conditions in x_k that take into account the total flow for the exiting and entering vehicles:

$$\begin{aligned}
&\forall i \neq k,\ \forall j \neq k, \quad \rho_{i,j} v\Big(\sum_{l,m} \rho_{l,m}\Big)\big(t, x_k^-\big) = \rho_{i,j} v\Big(\sum_{l,m} \rho_{l,m}\Big)\big(t, x_k^+\big) && \max,\\
&\sum_{0\leqslant i\leqslant n} \rho_{i,k} v\Big(\sum_{l,m} \rho_{l,m}\Big)\big(t, x_k^-\big) \leqslant o_k(t) && \max,\\
&\sum_{1\leqslant j\leqslant n+1} \rho_{k,j} v\Big(\sum_{l,m} \rho_{l,m}\Big)\big(t, x_k^+\big) \leqslant i_k(t) && \max,
\end{aligned} \tag{2.8}$$

the flows being maximised first in x_k^- and then in x_k^+ because of the priority rule. As in the one-T case, o_k and i_k are some prescribed output and input functions, taking values in $\mathbb{R}^+$ and corresponding to the capacity of the secondary road located in x_k.

We denote below $\mathcal{S}_{(n+1)^2} = \{\boldsymbol{\rho} \in \mathbb{R}^{(n+1)^2}\colon\ \forall (i,j) \in [\![0,n]\!] \times [\![1,n+1]\!],\ \rho_{i,j} \geqslant 0,\ \sum_{i,j} \rho_{i,j} \leqslant 1\}$. We can remark that the notation of the previous section can be identified here by $\mathcal{S} = \mathcal{S}_2$.

The Riemann problem. We are interested in weak entropy solutions of the problem (2.6)–(2.8), when the functions o_k and i_k are taken to be constants and when we choose initial conditions that are constant on the intervals $(-\infty, x_1)$, $(x_n, +\infty)$ and (x_k, x_{k+1}), for $k \in [\![1,n]\!]$:

$$\begin{aligned}
&\rho_{i,j}|_{t=0, x\in(-\infty,x_1)} = \rho_{i,j}^-,\\
&\rho_{i,j}|_{t=0, x\in(x_k,x_{k+1})} = \rho_{i,j}^{k+1/2}, \quad k \in [\![1,n]\!],\\
&\rho_{i,j}|_{t=0, x\in(x_n,+\infty)} = \rho_{i,j}^+.
\end{aligned} \tag{2.9}$$

With the previous notations, we can announce the following:

Theorem 2.8. *Under the hypotheses* (**V**), (**F**) *and* (**P**), *there exists $T > 0$ such that the Riemann problem* (2.6)–(2.8)–(2.9) *admits a unique weak entropy solution for $t \in [0, T]$.*

Furthermore, we can give a lower bound for the time of existence: *let $L = \min_k\{x_{k+1} - x_k\} > 0$, then $T \geqslant \frac{L}{2V}$.*

Qualitative properties. As in the case $n = 1$, some discontinuity phenomena appear when we make the initial conditions vary.

3. Technical analysis for the 'one-T' case

In this section, we solve the Riemann problem for (2.2)–(2.3)–(2.4). In Section 3.1, we first study the Riemann problem on a standard road with two types of vehicles which have the same speed law. Then, in Section 3.2, we study *'half'* Riemann problems in the quarter planes $\{x \leqslant 0, t \geqslant 0\}$ and $\{x \geqslant 0, t \geqslant 0\}$. Finally, in Section 3.3, we will complete the proof of Theorem 2.4.

3.1. Riemann problem with two types of vehicles

First, we consider the classical Riemann problem, for a road with two typesof vehicles, of respective densities ρ_1 and ρ_2, which have the same speed law. We obtain the following result:

Proposition 3.1. *Let us consider the two-populations system*

$$\begin{cases} \partial_t \rho_1 + \partial_x\big(\rho_1 v(\rho_1 + \rho_2)\big) = 0,\\ \partial_t \rho_2 + \partial_x\big(\rho_2 v(\rho_1 + \rho_2)\big) = 0 \end{cases} \tag{3.10}$$

with the following piecewise constant initial data:

$$\begin{aligned}
&\rho_1(0,x) = \rho_1^- \quad \text{for } x \in \mathbb{R}_-^*, \qquad \rho_1(0,x) = \rho_1^+ \quad \text{for } x \in \mathbb{R}_+^*,\\
&\rho_2(0,x) = \rho_2^- \quad \text{for } x \in \mathbb{R}_-^*, \qquad \rho_2(0,x) = \rho_2^+ \quad \text{for } x \in \mathbb{R}_+^*.
\end{aligned} \tag{3.11}$$

Then, for all $\boldsymbol{\rho}^- = (\rho_1^-, \rho_2^-)$, $\boldsymbol{\rho}^+ = (\rho_1^+, \rho_2^+)$ in $\mathcal{S}$, there exists a unique weak entropy solution.

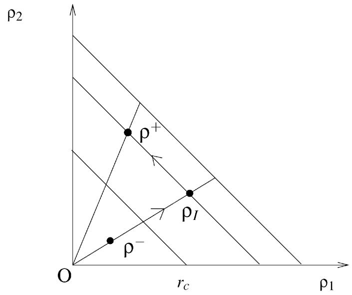

Fig. 4. Solution of the Riemann problem; waves curves.

Proof. The 2×2 system (3.10) is of a standard type ('straight-line systems' see [1]; see also rich or Temple systems, see [20], [19, Chapters 12 and 13]). We also based our study on [18] and on the article of Benzoni and Colombo [4].

Let $r=\rho_1+\rho_2$ and $s=\rho_1/\rho_2$ when $\rho_2\neq 0$; we also denote $r^+=\rho_1^++\rho_2^+$, $s^+=\rho_1^+/\rho_2^+$, etc. We easily obtain that, for smooth solutions with $\rho_2\neq 0$, the system (3.10) is equivalent to:

$$\begin{cases}\partial_t r+\big(v(r)+rv'(r)\big)\partial_x r=0,\\ \partial_t s+v(r)\partial_x s=0.\end{cases}$$

If $\rho_2=0$ and $\rho_1\neq 0$, we obtain something similar by considering ρ_2/ρ_1 instead of ρ_1/ρ_2, so the only problem is in $\boldsymbol{\rho}=0$. Consequently, we first solve the problem in $\mathcal{S}\backslash\{0\}$.

The characteristic speeds of (3.10) are $\lambda_1(\boldsymbol{\rho})=v(r)+rv'(r)$ and $\lambda_2(\boldsymbol{\rho})=v(r)$, with $\boldsymbol{\rho}=(\rho_1,\rho_2)$. We remark that $\lambda_1(\boldsymbol{\rho})<\lambda_2(\boldsymbol{\rho})$ when $\boldsymbol{\rho}=(\rho_1,\rho_2)\neq 0$, and that λ_2 is always non-negative (except in $r=1$). The associated eigenvectors are $v_1(\boldsymbol{\rho})=\binom{\rho_1}{\rho_2}=\boldsymbol{\rho}$ and $v_2=\binom{1}{-1}$. This allows us to see that the 1-characteristic field is genuinely nonlinear when $\boldsymbol{\rho}\neq 0$, as $d\lambda_1(\boldsymbol{\rho})\cdot v_1(\boldsymbol{\rho})$ does not vanish in $\mathcal{S}\backslash\{0\}$, q being strictly concave. The 2-characteristic field is linearly degenerate since for all $\boldsymbol{\rho}\in\mathcal{S}$ we have $d\lambda_2\cdot v_2\equiv 0$. We can remark here that the functions r and s are strong Riemann invariants. Let $\bar{\boldsymbol{\rho}}\in\mathcal{S}$. The wave curves are $\mathcal{O}_1(\bar{\boldsymbol{\rho}})=\mathcal{O}_1(\bar\rho_1,\bar\rho_2)=\{(\rho_1,\rho_2)\in\mathcal{S};\ \rho_1/\rho_2=\bar\rho_1/\bar\rho_2\}$, when $\bar\rho_2\neq 0$, and $\mathcal{O}_2(\bar{\boldsymbol{\rho}})=\{(\rho_1,\rho_2)\in\mathcal{S};\ \rho_1+\rho_2=\bar\rho_1+\bar\rho_2\}$. In particular, the shock-curves and the rarefaction-curves coincide and are straight lines.

The 1-waves are made of shocks or rarefaction waves, whereas the 2-waves are contact discontinuities. More precisely, we have the following:

If $\bar\rho_1/\bar\rho_2=\rho_1^+/\rho_2^+$ **and** $r^+\leqslant\bar r$**,** the 1-waves that go from $\bar{\boldsymbol{\rho}}=(\bar\rho_1,\bar\rho_2)$ to $\boldsymbol{\rho}^+=(\rho_1^+,\rho_2^+)$ are rarefaction waves, which are between the lines of equations $x/t=q'(\bar r)$ and $x/t=q'(r^+)$.
If $\bar\rho_1/\bar\rho_2=\rho_1^+/\rho_2^+$ **and** $r^+>\bar r$**,** the 1-waves that go from $\bar{\boldsymbol{\rho}}$ to $\boldsymbol{\rho}^+$ are shocks of speed

$$c=\frac{r^+v(r^+)-\bar r v(\bar r)}{r^+-\bar r}.$$

In order to solve the Riemann problem, we must find an intermediate state $\boldsymbol{\rho_I}=(\rho_{1,I},\rho_{2,I})$ so that $s(\boldsymbol{\rho_I})=s^-$ and $r(\boldsymbol{\rho_I})=r^-$ (see Fig. 4). Clearly, we have $\boldsymbol{\rho_I}=\frac{r^+}{r^-}\boldsymbol{\rho}^-$, when $r^-\neq 0$.

Now, we want to see what happens when $\boldsymbol{\rho}^-$ or $\boldsymbol{\rho}^+$ is 0.

If we make $\boldsymbol{\rho}^-$ tend to 0 along a line $\rho_1/\rho_2=cst$, it looks like we could obtain different solutions: a 1-wave from 0 to whatever point in $\Delta_{r^+}=\{\boldsymbol{\rho}\in\mathcal{S};\ \rho_1+\rho_2=r^+\}$, and then a 2-wave from this intermediate state to $\boldsymbol{\rho}^+$. However, all these intermediate states correspond in fact to the same solution, because the 1-shocks are of speed $c=\frac{q(r^+)-q(r^-)}{r^+-r^-}=v(r^+)$ and the 2-waves are of speed $\lambda_2(\boldsymbol{\rho}^+)=v(r^+)$, so the waves are attached. If we do not accept fictitious waves between $\boldsymbol{\rho}^-=0$ and $\boldsymbol{\rho}^+$, we take only a shock of speed $c=v(r^+)$. To summarise:

if $\boldsymbol{\rho}^-=\boldsymbol{\rho}^+=0$**,** we define the solution as $\boldsymbol{\rho}\equiv 0$;
if $\boldsymbol{\rho}^-=0$**,** we define the solution by linking $\boldsymbol{\rho}^-$ to $\boldsymbol{\rho}^+$ by a 1-shock;
if $\boldsymbol{\rho}^+=0$**,** we define the solution by linking $\boldsymbol{\rho}^-$ to $\boldsymbol{\rho}^+$ by a 1-wave. □

Remark 3.2. The set $\mathcal{S}$ is invariant under the flow of the system (3.10). More precisely, the trapezoids whose boundaries are the Hugoniot loci are also invariant under the flow of the system (3.10). (For some general results on invariant sets, see [12].)

3.2. Half-Riemann problems

We call 'half-Riemann problem' the simple case of an initial–boundary value problem in the quarter of plane $\{x\leqslant 0;\ t\geqslant 0\}$ or $\{x\geqslant 0; t\geqslant 0\}$ when the initial condition is a constant. The problem here is to find the acceptable boundary

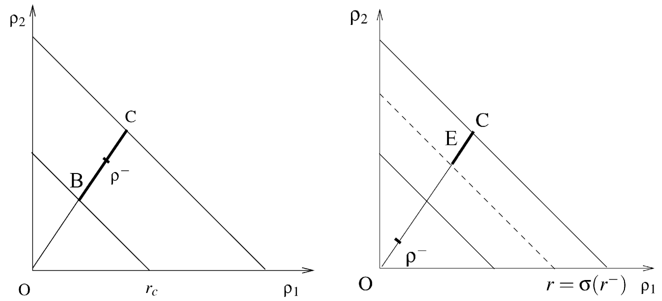

Fig. 5. Left, $r^- \geqslant r_c$ and $N(\boldsymbol{\rho}^-) = [B, C]$; right, $r^- < r_c$ and $N(\boldsymbol{\rho}^-) = \{\boldsymbol{\rho}^-\} \cup [E, C]$.

conditions in $x = 0$. Some general criteria have been introduced in the literature in an attempt to characterise the set of attainable states (see [2,3,9]).

3.2.1. Left-half problem

We fix a left state and we look for the right states attainable by a wave of negative speed.

Lemma 3.3. *Fix $\boldsymbol{\rho}^- = (\rho_1^-, \rho_2^-) \in \mathcal{S}$ and denote $r^- = \rho_1^- + \rho_2^-$. Then the set $N(\boldsymbol{\rho}^-)$ of points $\widehat{\boldsymbol{\rho}} = (\widehat{\rho}_1, \widehat{\rho}_2) \in \mathcal{S}$ such that the solution to the Riemann problem*

$$\begin{cases} \partial_t \rho_1 + \partial_x\big(\rho_1 v(\rho_1 + \rho_2)\big) = 0, \\ \partial_t \rho_2 + \partial_x\big(\rho_2 v(\rho_1 + \rho_2)\big) = 0, \\ (\rho_1, \rho_2)(0, x) = \begin{cases} (\rho_1^-, \rho_2^-) & \text{if } x < 0, \\ (\widehat{\rho}_1, \widehat{\rho}_2) & \text{if } x > 0 \end{cases} \end{cases}$$

contains only waves with negative speed is:

If $r^- \geqslant r_c$**:** *the segment with extreme points* $\frac{r_c}{r^-}\boldsymbol{\rho}^-$ *and* $\frac{1}{r^-}\boldsymbol{\rho}^-$ (*see Fig.* 5, *left*);

If $r^- < r_c$**:** *the segment with extreme points* $\frac{\sigma(r^-)}{r^-}\boldsymbol{\rho}^-$ *and* $\frac{1}{r^-}\boldsymbol{\rho}^-$, *together with the point* $\boldsymbol{\rho}^-$ (*see Fig.* 5, *right*).

In both cases, for all $\boldsymbol{\rho}^- \in \mathcal{S}$, $\min_{N(\boldsymbol{\rho}^-)} \rho_1 v(\rho_1 + \rho_2) = 0$.

Remark 3.4. When we want not to consider shocks of zero speed in the left-half problem (that are fictitious as they are located on the axis $x = 0$), we have to modify a little the set $N(\boldsymbol{\rho}^-)$ in the case $r^- < r_c$ and take $N'(\boldsymbol{\rho}^-) = \{\boldsymbol{\rho}^-\} \cup \{\lambda\boldsymbol{\rho}^-;\ \lambda \in]\frac{\sigma(r^-)}{r^-}, \frac{1}{r^-}]\}$.

Proof of Lemma 3.3. The study of the left-half problem is equivalent to searching an artificial right state $\widehat{\boldsymbol{\rho}} = (\widehat{\rho}_1, \widehat{\rho}_2)$ in a problem like (3.10)–(3.11). We are here only interested in the waves with negative speed, in order to know what are the states attainable on the line $\{x = 0\}$. We have seen in the proof of Proposition 3.1, that the 2-waves always have positive speed, so we can only have a 1-wave: either a shock or a rarefaction wave, depending on the sign of $\widehat{r} - r^-$, which means in particular that $N(\boldsymbol{\rho}^-) \subset \{\boldsymbol{\rho} \in \mathcal{S};\ \rho_1/\rho_2 = \rho_1^-/\rho_2^-\}$.

If $\widehat{r} > r^-$**,** we have a shock of speed $c = \frac{q(\widehat{r}) - q(r^-)}{\widehat{r} - r^-}$, which is negative if and only if $q(\widehat{r}) - q(r^-) \leqslant 0$. There are two cases:
- when $r^- \geqslant r_c$, $q(\widehat{r}) \leqslant q(r^-) \Leftrightarrow \widehat{r} \geqslant r^-$,
- when $r^- < r_c$, $q(\widehat{r}) \leqslant q(r^-) \Leftrightarrow \widehat{r} \geqslant \sigma(r^-)$.

If $\widehat{r} \leqslant r^-$**,** we have a rarefaction wave which is located between the lines of equations $x/t = q'(r^-)$ and $x/t = q'(\widehat{r})$. These two lines have negative slopes if and only if $\widehat{r} \geqslant r_c$, because q' is non-increasing. Finally, we obtain $r^- \geqslant \widehat{r} \geqslant r_c$ (in particular $r^- \geqslant r_c$).

We can summarise the situation as follows:

- if $r^- \geqslant r_c$, we have a wave of negative slope when $r_c \leqslant \widehat{r} \leqslant 1$,
- if $r^- \leqslant r_c$, we have a wave of negative slope when $\sigma(r^-) \leqslant \widehat{r} \leqslant 1$.

In order to complete the proof, we note that we can always have the state $\boldsymbol{\rho}^-$ as artificial right state. In this case, nothing happens and the solution is taken to be constant, equal to $\boldsymbol{\rho}^-$.

In the special case $\boldsymbol{\rho}^- = 0$, we define $N(0) = \{0\} \cup \Delta_1$, where $\Delta_1 = \{\boldsymbol{\rho} \in \mathcal{S};\ \rho_1 + \rho_2 = 1\}$. $\square$

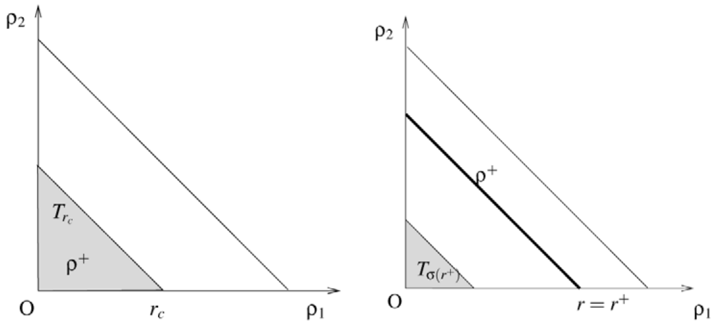

Fig. 6. Let $P = P(\boldsymbol{\rho}^+)$: left, $r^+ \leqslant r_c$ and $P = T_{r_c}$; right, $r^+ > r_c$ and $P = T_{\sigma(r^+)} \cup \Delta_{r^+}$.

3.2.2. Right-half problem

We fix now a right state and we search the states on the left attainable by a wave of positive speed.

Lemma 3.5. *Fix $\boldsymbol{\rho}^+ = (\rho_1^+, \rho_3^+) \in \mathcal{S}$ and denote $r^+ = \rho_1^+ + \rho_3^+$. Then, the set $P(\boldsymbol{\rho}^+)$ of points $\check{\boldsymbol{\rho}} = (\check{\rho}_1, \check{\rho}_3) \in \mathcal{S}$ such that the solution to the Riemann problem*

$$\begin{cases} \partial_t \rho_1 + \partial_x\big(\rho_1 v(\rho_1 + \rho_3)\big) = 0, \\ \partial_t \rho_3 + \partial_x\big(\rho_3 v(\rho_1 + \rho_3)\big) = 0, \\ (\rho_1, \rho_3)(0, x) = \begin{cases} (\check{\rho}_1, \check{\rho}_3) & \text{if } x < 0, \\ (\rho_1^+, \rho_3^+) & \text{if } x > 0 \end{cases} \end{cases}$$

contains only waves with positive speed is:

If $r^+ \leqslant r_c$: *the triangle T_{r_c} of points $\check{\boldsymbol{\rho}} \in \mathcal{S}$ such that $\check{\rho}_1 + \check{\rho}_3 \leqslant r_c$ (see Fig. 6, left).*

If $r^+ > r_c$: *the triangle $T_{\sigma(r^+)}$ of points $\check{\boldsymbol{\rho}} \in \mathcal{S}$ such that $\check{\rho}_1 + \check{\rho}_3 \leqslant \sigma(r^+)$, together with the line $\Delta_{r^+} = \{\boldsymbol{\rho} \in \mathcal{S};\ \check{\rho}_1 + \check{\rho}_2 = r^+\}$ (see Fig. 6, right).*

In both cases, for all $(\rho_1^+, \rho_3^+) \in \mathcal{S}$, $\min_{P(\boldsymbol{\rho}^+)} \rho_1 v(\rho_1 + \rho_3) = 0$.

Remark 3.6. As before, when we do not want to consider shocks of zero speed, we have to change a little the definition of P in the case $r^+ > r_c$ into $P'(\boldsymbol{\rho}^+) = \Delta_{r^+} \cup \{\boldsymbol{\rho} \in \mathcal{S};\ \sigma(r^+)/r^+ > r \geqslant 0\}$.

Proof of Lemma 3.5. The ideas of the proof are essentially the same as for Lemma 3.3, up to replacing 'negative speed' by 'positive speed'. For this reason, the 2-waves are now always allowed, that is why the line Δ_{r^+} of equation $\rho_1 + \rho_3 = r^+$ is always in $P(\boldsymbol{\rho}^+)$. It also implies that we only have to search the admissible 1-waves and then add whatever 2-wave. That is why we search the states $\check{\boldsymbol{\rho}}$ that can be linked to $\boldsymbol{\rho}^+$ by a 1-wave. This 1-wave is a shock or a rarefaction wave, depending on the sign of $\check{r} - r^+$.

If $\check{r} \geqslant r^+$, we have a rarefaction wave located in the quarter plane $\{x \geqslant 0, t \geqslant 0\}$ if and only if $\check{r} \leqslant r_c$. So, in this case, we have $r^+ \leqslant \check{r} \leqslant r_c$.

If $\check{r} \leqslant r^+$, we have a shock of speed $c = \frac{q(\check{r}) - q(r^+)}{\check{r} - r^+}$, which is positive if and only if $q(\check{r}) \leqslant q(r^+)$.

- If $r^+ > r_c$, $c \leqslant 0$ if $\check{r} \leqslant \sigma(r^+)$.
- If $r^+ \leqslant r_c$, $c \leqslant 0$ if $\check{r} \leqslant r^+$.

Finally,

- if $r^+ > r_c$, then $(\check{\rho}_1, \check{\rho}_3) \in T_{\sigma(r^+)} \cup \Delta_{r^+}$,
- if $r^+ \leqslant r_c$, then $(\check{\rho}_1, \check{\rho}_3) \in T_{r_c}$. $\square$

3.3. The Riemann problem

3.3.1. Proof of Theorem 2.4

Now we prove Theorem 2.4, according to which the Riemann problem for (2.2)–(2.3)–(2.4) admits a unique solution.

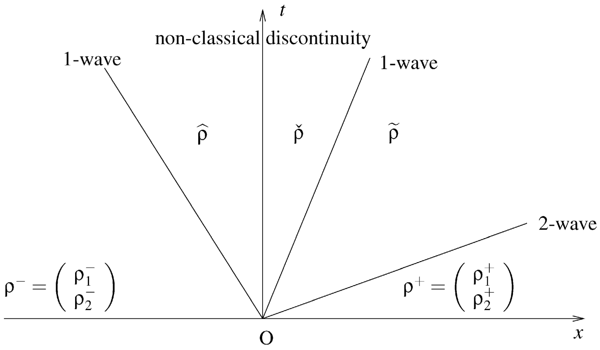

Fig. 7. Solution.

Proof of Theorem 2.4. Let $r^- = \rho_1^- + \rho_2^-$ and $r^+ = \rho_1^+ + \rho_3^+$. The solution consists in (see Fig. 7):

1. the state (ρ_1^-, ρ_2^-);
2. a (possibly null) 1-wave with negative speed;
3. the state $(\widehat{\rho}_1, \widehat{\rho}_2)$ in $N(\rho_1^-, \rho_2^-)$;
4. a non-classical discontinuity with 0 speed;
5. the state $(\check{\rho}_1, \check{\rho}_3)$ in $P(\rho_1^+, \rho_3^+)$;
6. a (possibly null) 1-wave with positive speed;
7. the state $(\widetilde{\rho}_1, \widetilde{\rho}_3)$;
8. a (possibly null) 2-wave with positive speed.

The first wave and the last two waves are the restrictions of a standard solution to the Riemann problem (3.10); the states $(\widehat{\rho}_1, \widehat{\rho}_2)$, $(\check{\rho}_1, \check{\rho}_3)$ are obtained thanks to Lemma 3.3 and Lemma 3.5. Now we only have to attach these states in $x = 0$, which is possible if the boundary conditions (2.4) are realised. The uniqueness will come from the maximisation of these conditions.

There are several cases, depending on the position of r^- and r^+ with respect to r_c.

If $\boldsymbol{r^+ \leqslant r_c}$, we note

$$\mathcal{N} = \mathcal{N}(\rho_1^-, \rho_2^-) = N(\rho_1^-, \rho_2^-) \cap \{(\rho_1, \rho_2),\ \rho_2 v(\rho_1 + \rho_2) \leqslant o\}$$

and

$$\mathcal{P} = \mathcal{P}(\rho_1^+, \rho_3^+) = P(\rho_1^+, \rho_3^+) \cap \{(\rho_1, \rho_3),\ \rho_1 v(\rho_1 + \rho_3) = M\} \cap \{(\rho_1, \rho_3),\ \rho_3 v(\rho_1 + \rho_3) \leqslant i\},$$

with $M = \widehat{\rho}_1 v(\widehat{\rho}_1 + \widehat{\rho}_2)$, so that $\mathcal{N}$ and $\mathcal{S}$ are the sets where the boundary conditions are verified. These sets are non-empty: for example the point $\frac{1}{r^-}(\rho_1^-, \rho_2^-)$ is in $\mathcal{N}$, and the point $(q|_{[0,r_c]}^{-1}(M), 0)$ is in $\mathcal{P}$ since q is bijective from $[0, r_c]$ to $[0, q_c]$ and $M \leqslant q_c$ for all $(\widehat{\rho}_1, \widehat{\rho}_2) \in \mathcal{N}$. Now, we need to maximise the flows of ρ_1, ρ_2 and ρ_3 on $\mathcal{N}$ and $\mathcal{P}$. The question is: is the maximum uniquely attained?

First, we maximise the flow of $\widehat{\rho}_2$ on $\mathcal{N} \cap \{\rho_1 + \rho_2 \geqslant r_c\}$, which is the same as maximising the function $f_2 : (t \mapsto t\rho_2^- v(tr^-))$, with $t \geqslant r_c/r^-$.

If $\boldsymbol{\rho_2^- \neq 0}$, we can write $f_2(t) = \frac{\rho_2^-}{r^-} q(tr^-)$, and we obtain a unique maximum for this function as we have here $tr^- \geqslant r_c$ and q is strictly decreasing on $[r_c, 1]$. Moreover, we have $f_2(t_0) = o \Leftrightarrow t_0 = \frac{1}{r^-} q|_{[r_c,1]}^{-1}(\frac{or^-}{\rho_2^-})$. Thus we take $\widehat{\boldsymbol{\rho}} = t\boldsymbol{\rho}^-$ with $t = \max\{\frac{\sigma(r^-)}{r^-}, \frac{r_c}{r^-}, t_0\}$.

Throughout, we can remark that the maximum of the flow of $\widehat{\rho}_1$ is attained at the same point that realises also the maximum of the flow of $\widehat{\rho}_2$. This comes from the fact that we obtain the maximum of the flow of $\widehat{\rho}_1$ by maximising $f_1 : (t \mapsto t\rho_1^- v(tr^-))$, f_1 and f_2 being defined on the same set.

If $\boldsymbol{\rho_2^- = 0, \rho_1^- \neq 0}$, then we have $f_2 \equiv 0$ and we cannot maximise the flow of ρ_2. Instead we maximise the flow of ρ_1. We get that f_1 is maximised for $t_1 = \frac{r_c}{\rho_1^-}$ and then we take $t = \max\{\frac{\sigma(r^-)}{r^-}, \frac{r_c}{r^-}\}$.

If $\boldsymbol{\rho_1^- = \rho_2^- = 0}$, then we take $\widehat{\boldsymbol{\rho}} = \boldsymbol{\rho}^-$ or $\widehat{\boldsymbol{\rho}} \in \Delta_1 = \{\boldsymbol{\rho} \in \mathcal{S};\ r = 1\}$. It seems here that the maximum is not uniquely defined, but in fact from $(0, 0)$ to whatever point in Δ_1, there is a shock of speed zero, hence it does not appear as this shock is located on the axis $x = 0$. Finally all these points correspond to the same solution.

Now, we would like to know if we have continuity of the solution with respect to the initial conditions. Essentially, we have to examine what happens when $\rho_2^- \to 0$, since the definition of t in the cases $\boldsymbol{\rho} \neq 0$ differs in respect to t_0, and more generally what happens when $\boldsymbol{\rho}^- \to 0$.

We assume at first that $o > 0$; then, t_0 is well defined when $\frac{or^-}{\rho_2^-} \leqslant q_c$.

If $\rho_1^- = 0$, then $\frac{or^-}{\rho_2^-} = o$ and t_0 is invariant when $\rho_2^- \to 0$; thus we have $\widehat{r} = tr^- = \max\{\sigma(\rho_2^-), r_c, q^{-1}(o)\} = \sigma(\rho_2^-)$ for ρ_2^- small enough. Consequently, $\widehat{r} \to 1$ when $\rho_2^- \to 0$ and we find the same solution as in the case $\boldsymbol{\rho}^- = 0$.

If $\rho_1^- \neq 0$, we have that $t_0 r^- \to r_c$ when $\rho_2^- \to 0$, and even, for ρ_2^- small enough, t_0 is no longer well defined: for example for $\rho_2^- < \frac{\rho_1^- o}{q_c}$, we have $f_2(t) < o$ $\forall t \geqslant r_c/r^-$, and the condition is automatically verified. Hence, o does not intervene any longer. In this case, we have $t = \max\{\frac{\sigma(r^-)}{r^-}, \frac{r_c}{r^-}\}$ and we have the same solution as in the case $\rho_2^- = 0, \rho_1^- \neq 0$.

Finally, let us see what happens if we make $r^- \to 0$. As before, t_0 will not realise the maximum and we have only to see that t becomes $\frac{\sigma(r^-)}{r^-}$ for r^- small enough. Consequently, $\widehat{\boldsymbol{\rho}}$ tends to a point of Δ_1, and we are done since all the points of Δ_1 correspond to the same solution $\widehat{\boldsymbol{\rho}} = 0$.

Finally, in the case $o > 0$, there is no problem of discontinuity when $\boldsymbol{\rho}^- \to 0$.

Now, if $o = 0$ we have $t_0 = \frac{1}{r^-}$ when $\rho_2^- \neq 0$ and then $t = t_0$, $\widehat{\boldsymbol{\rho}} \in \Delta_1$. But, if $\rho_1^- \neq 0$ and $\rho_2^- = 0$, then we obtain a very different solution $\widehat{\boldsymbol{\rho}} = \max\{\sigma(\rho_1^-)/\rho_1^-, r_c/\rho_1^-\}(\rho_1^-, 0)$, and we have a discontinuity in the solution when $\rho_2^- \to 0$. That is why we have to impose $o > 0$ if we want to have a continuity in the Riemann solver.

At this point there are two cases:

1. if $r^- \geqslant r_c$, $\mathcal{N} = \mathcal{N} \cap \{\rho_1 + \rho_2 \geqslant r_c\}$, and the work is finished;
2. if $r^- < r_c$, either $\mathcal{N} = \mathcal{N} \cap \{\rho_1 + \rho_2 \geqslant r_c\}$ and the work is finished, or $\mathcal{N} = \{(\rho_1^-, \rho_2^-)\} \cup (\mathcal{N} \cap \{\rho_1 + \rho_2 \geqslant r_c\})$, and the maximum could be obtained in two points. We have to observe what happens in this last case: the maximum is obtained in $t(\rho_1^-, \rho_2^-)$ (with $t \geqslant r_c/r^-$) and in (ρ_1^-, ρ_2^-) and we have:

$$\rho_2^- v(r^-) = t\rho_2^- v(tr^-)$$
$$\text{i.e.} \quad tr^- = \sigma(r^-).$$

Finally, in this case, we have two solutions:
- $(\widehat{\rho}_1, \widehat{\rho}_2) = (\rho_1^-, \rho_2^-)$,
- $(\widehat{\rho}_1, \widehat{\rho}_2) = \frac{\sigma(r^-)}{r^-}(\rho_1^-, \rho_2^-)$, and we have between (ρ_1^-, ρ_2^-) and $(\widehat{\rho}_1, \widehat{\rho}_2)$ a shock of speed $c = \frac{q(\sigma(r^-)) - q(r^-)}{\sigma(r^-) - r^-} = 0$, so this shock is in fact fictitious as it is on the axis $x = 0$.

Moreover, the flows of $\widehat{\rho}_1$ are the same in the two cases. Consequently, we have in fact two times the same half solution in the quarter $\{x \leqslant 0, t \geqslant 0\}$.

Secondly, we maximise the flow of $\check{\rho}_3$ on $\mathcal{P}$. As v is strictly non-increasing, when $M \neq 0$ the elements of $\mathcal{P}$ can be written as $(\check{\rho}_1, v^{-1}(\frac{M}{\check{\rho}_1}) - \check{\rho}_1)$ so that the flow of $\check{\rho}_3$ is:

$$f_3(\rho_1) = v^{-1}\left(\frac{M}{\rho_1}\right)\frac{M}{\rho_1} - M$$
$$= q\left(v^{-1}\left(\frac{M}{\rho_1}\right)\right) - M.$$

As we are on T_{r_c} and q is strictly increasing on $[0, r_c]$, we obtain a unique maximum for f_3 that is attained when ρ_1 is maximum.

Besides, $f_3(\rho_1) = i \Leftrightarrow \rho_1 = \frac{M}{i+M} q|_{[0,r_c]}^{-1}(i + M)$, which is defined when $M + i$ is small enough, more precisely when $M + i \leqslant q_c$. For $M + i$ larger we take $\check{\boldsymbol{\rho}} \in \Delta_{r_c}$ and in this case, we have $\rho_1 = \frac{M}{v(r_c)}$.

If $M = 0$, then we have $\rho_1 = 0$ and $f_3(\rho_3) = q(\rho_3)$, or $r = 1$ and $f_3 = 0$. As we want to maximise f_3, we take $\rho_1 = 0$ and $\rho_3 = \min\{q|_{[0,r_c]}^{-1}(i), r_c\}$. We see here that the choice of $\check{\boldsymbol{\rho}}$ is then continuous, if we assume that M varies continuously.

If $\boldsymbol{r^+ > r_c}$ the ideas are the same, but we now denote

$$\mathcal{N} = \mathcal{N}(\rho_1^-, \rho_2^-) = N(\rho_1^-, \rho_2^-) \cap \{(\rho_1, \rho_2),\ \rho_2 v(\rho_1 + \rho_2) \leqslant o\} \cap \{(\rho_1, \rho_2),\ \rho_1 v(\rho_1 + \rho_2) \leqslant q(r^+)\},$$

whereas the definition of $\mathcal{P}$ is unchanged. This new definition guarantees that these sets are non-empty. Indeed, if we reconsider the same examples as in the first case, we see now that $M \leqslant q(r^+) \Rightarrow q|_{[0,r_c]}^{-1}(M) \leqslant \sigma(r^+)$ and the point $(q|_{[0,r_c]}^{-1}(M), 0)$ is in $T_{\sigma(r^+)}$, thus it is in $\mathcal{P}$.

The way to maximise the flows of $\widehat{\rho}_1$ and $\widehat{\rho}_2$ is the same as in the first case, so we will not rewrite all. Now we have $\widehat{\boldsymbol{\rho}} = t\boldsymbol{\rho}^-$, with $tr^- = \max\{\sigma(r^-), r_c, t_0 r^-, t_1 r^-\}$ and $t_0 r^- = q|_{[r_c,1]}^{-1}(\frac{or^-}{\rho_2^-})$, $t_1 r^- = q|_{[r_c,1]}^{-1}(\frac{q(r^+)r^-}{\rho_1^-})$. We have

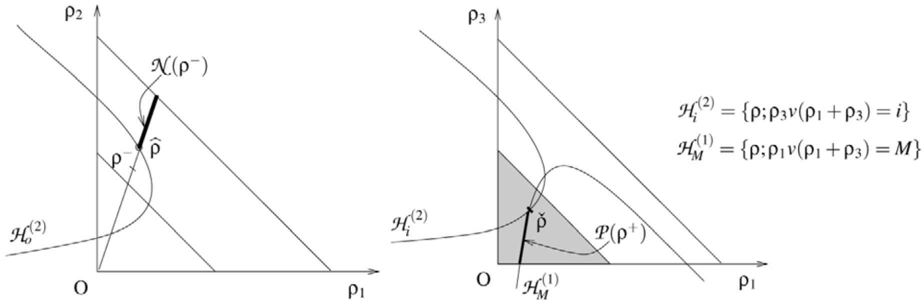

Fig. 8. Maximisation of the flows of ρ_2 and ρ_1 (left) and of ρ_3 (right), when $r^- \geqslant r_c$ and $r^+ \leqslant r_c$.

seen before that there is a lack of continuity when $o, \rho_2^- \to 0$. This phenomenon is always to consider, but slightly changed: when $o, \rho_2^- \to 0$, $\widehat{\boldsymbol{\rho}} \to q_+^{-1}(\kappa)(1,0), \forall \kappa \in [0, \min\{q(r^+), q_c, q(\rho_1^-)\}]$ when $\rho_1^- \neq 0$ and $\widehat{\boldsymbol{\rho}}$ tends to $(0,0)$ if also $\rho_1^- \to 0$.

Besides, here an other phenomenon of discontinuity can appear when we make $r^+ \to 1$ and $\rho_1^- \to 0$, when ρ_2^- tends to something strictly positive. In this case, $t = t_1$ for $q(r^+)$ and ρ_1^- small enough, and we obtain that for $\rho_2^- \neq 0$, $\widehat{\boldsymbol{\rho}} \to q_+^{-1}(\kappa)(0,1)$, $\forall \kappa \in [0, \min\{o, q(\rho_2^-), q_c\}]$. However, the flow of $\widehat{\rho}_2$ is not maximised for all these solutions, but only for $\kappa = \min\{o, q(\rho_2^-), q_c\}$. We obtain these limits by making $\frac{q(r^+)\rho_2^-}{\rho_1^-} \to \kappa$. However, the discontinuity disappears if we make also $\rho_2^- \to 0$, since in this case the only choice possible for $\widehat{\boldsymbol{\rho}}$ is $(0,0)$.

The way to maximise the flow of $\check{\rho}_3$ on $\mathcal{P}$ is slightly changed.

As before, we obtain a unique maximum of the flow of $\check{\rho}_3$ on $\mathcal{P} \cap \{\rho_1 + \rho_3 \leqslant r_c\}$. However an other point may attain the maximum on the line Δ_{r^+}. In this case, we have $(\check{\rho}_1^1, \check{\rho}_3^1) \in \mathcal{P} \cap \{\rho_1 + \rho_3 \leqslant r_c\}$ and $(\check{\rho}_1^2, \check{\rho}_3^2) = (\widetilde{\rho}_1^2, \widetilde{\rho}_3^2) \in \Delta_{r^+}$, so we have

$$\begin{aligned} &\check{\rho}_1^1 v(\check{r}^1) = M, \\ &\check{\rho}_3^1 v(\check{r}^1) \leqslant i, \\ &\check{r}^1 \leqslant \sigma(r^+) \end{aligned}$$

and $\check{\rho}_1^2 v(\check{r}^2) = M$, $\check{\rho}_3^2 v(\check{r}^2) = \check{\rho}_3^1 v(\check{r}^1)$, this last condition giving:

$$\begin{aligned} &(\check{r}^1 - \check{\rho}_1^1) v(\check{r}^1) = (\check{r}^2 - \check{\rho}_1^2) v(\check{r}^2) \\ &\quad \text{i.e.} \quad q(\check{r}^1) - M = q(r^+) - M \\ &\quad \text{so} \quad \check{r}^1 = \sigma(r^+). \end{aligned}$$

Thus, we have between $(\check{\rho}_1^1, \check{\rho}_3^1)$ and $(\widetilde{\rho}_1^1, \widetilde{\rho}_3^1) = \frac{r^+}{\sigma(r^+)}(\frac{M}{v(\sigma(r^+))}, \sigma(r^+) - \frac{M}{v(\sigma(r^+))})$ a shock of speed $c' = 0$, meaning that it is fictitious. Moreover $(\widetilde{\rho}_1^1, \widetilde{\rho}_3^1) = (\widetilde{\rho}_1^2, \widetilde{\rho}_3^2)$, and we have in fact the same solution. $\square$

Remark 3.7. Here, we have given two different definitions of the set $\mathcal{N}$, depending on r^+ greater or less than r_c. In fact, this corresponds to giving only one definition:

$$\mathcal{N}(\rho_1^-, \rho_2^-) = N(\rho_1^-, \rho_2^-) \cap \{(\rho_1, \rho_2),\ \rho_2 v(\rho_1 + \rho_2) \leqslant o\} \cap \{(\rho_1, \rho_2),\ \rho_1 v(\rho_1 + \rho_2) \leqslant d(r^+)\},$$

where d is a function equal to q_c on $[0, r_c]$ and coinciding with q on $[r_c, 1]$.

3.3.2. Study of the point of discontinuity

We observed in the proof just above that there is a lack of continuity of the Riemann solver when o and ρ_2^- tend together to 0. We have obtained the following:

Proposition 3.8. *Some discontinuities appear when o and ρ_2^- tend together to 0, and also when $r^+ \to 1$, $\rho_1^- \to 0$.*

Now, we would like to see what are the different limits obtained, depending on the manner that ρ_2^- and o tend to 0 and how we can have access to them in the (ρ_2^-, o) plane.

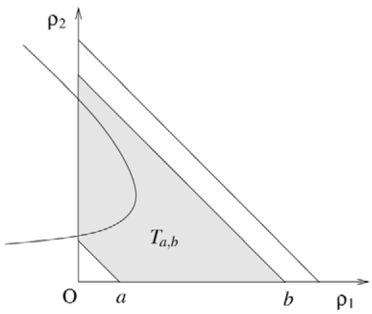

Fig. 9. Example of an invariant set.

Proposition 3.9. *In the case $r^+ \leqslant r_c$, when o and ρ_2^- tend together to 0, we have the following*:

- *if $\rho_1^- \geqslant r_c$ and $\frac{o}{\rho_2^-} \to \frac{\kappa}{\rho_1^-}$, with $\kappa \leqslant q_c$, then $\widehat{\boldsymbol{\rho}} \to q|_{[r_c,1]}^{-1}(\kappa)(1,0)$;*
- *if $0 < \rho_1^- < r_c$ and $\frac{o}{\rho_2^-} \to \frac{\kappa}{\rho_1^-}$, with $\kappa \leqslant q(\rho_1^-)$, then $\widehat{\boldsymbol{\rho}} \to q|_{[r_c,1]}^{-1}(\kappa)(1,0)$;*
- *if o, ρ_2^- and ρ_1^- tend together to 0, then $\widehat{\boldsymbol{\rho}} \to 0$.*

Proof. We have seen that, when $\rho_2^- \neq 0$ the solution is given by

$$\widehat{\boldsymbol{\rho}} = \frac{t}{r^-}\boldsymbol{\rho}^- \quad \text{with } t = \max\left\{\sigma(r^-), r_c, q_+^{-1}\left(\frac{or^-}{\rho_2^-}\right), q_+^{-1}\left(\frac{d(r^+)r^-}{\rho_1^-}\right)\right\},$$

where we have denoted $q_+^{-1} = q|_{[r_c,1]}^{-1}$, and the definition of t has been slightly changed by a multiplicative constant.

We assume first that ρ_1^- stays strictly positive, so the problem is essentially the behaviour of t when $o, \rho_2^- \to 0$. If $q_+^{-1}(\frac{or^-}{\rho_2^-})$ has not to be taken into account at the limit (so that the maximum has to be taken between $\sigma(r^-)$ and r_c), then the solution will be $\widehat{\boldsymbol{\rho}} = \max\{\sigma(\rho_1^-), r_c\}(1,0)$. The question is now to know if this are the only possible limits.

We can see that the limit of t will be $\lim_{\rho_2^- \to 0} q_+^{-1}(\frac{or^-}{\rho_2^-})$ if $\lim_{\rho_2^- \to 0} q_+^{-1}(\frac{or^-}{\rho_2^-}) \geqslant \max\{\sigma(r^-), r_c\}$. Here we have to study two different cases depending on the situation of ρ_1^- with respect to r_c.

- If $\rho_1^- \geqslant r_c$, then the condition is $o(1 + \frac{\rho_1^-}{\rho_2^-}) < q_c$. Consequently, if $o\frac{\rho_1^-}{\rho_2^-} \to \kappa \leqslant q_c$, the limit will be $\widehat{\boldsymbol{\rho}} = q_+^{-1}(\kappa)(1,0)$.
- If $0 < \rho_1^- < r_c$, then for ρ_2^- small enough, we have also $r^- < r_c$ and the condition is $o < \rho_2^- v(r^-)$. Then, if $\frac{o}{\rho_2^- v(r^-)} \to \frac{\kappa}{q(\rho_1^-)} \leqslant 1$, we get $\widehat{\boldsymbol{\rho}} \to q_+^{-1}(\kappa)(1,0)$.

Now, we want to see what happens when we have also $\rho_1^- \to 0$ (and so $r^- \to 0$). In this case, $\frac{1}{r^-}\boldsymbol{\rho}^-$ tends to whatever points in Δ_1. Besides, $t > \sigma(r^-) \Rightarrow o < \rho_2^- v(r^-)$ and as before, we take o, ρ_2^-, r^- such that $\frac{o}{\rho_2^- v(r^-)} \to \kappa \leqslant 1$. In this case we have $\frac{o}{\rho_2^- v(r^-)} r^- v(r^-) \sim \kappa q(r^-) \to 0$, because $q(0) = 0$ and $r^- \to 0$. Then $t \to 1$ and $\widehat{\boldsymbol{\rho}}$ tends to whatever point in Δ_1, which corresponds in fact to the same solution and it is equivalent to take $\widehat{\boldsymbol{\rho}} = (0,0)$. So, we keep continuity by making ρ_1^- tend to 0. $\square$

Remark 3.10. Among all the possible limits, only one limit realises the maximum condition on the flow.

Remark 3.11. We have not treated here the cases $r^+ > r_c$ and $r^+ \to 1$ because they are very similar.

3.4. Invariant sets

We would like to describe here some invariant sets of this problem, that is to say that we want to find the sets $\mathcal{U} \subset \mathcal{S}$ such that $(\boldsymbol{\rho}^-, \boldsymbol{\rho}^+) \in \mathcal{U}^2 \Rightarrow (\widehat{\boldsymbol{\rho}}, \check{\boldsymbol{\rho}}, \widetilde{\boldsymbol{\rho}}) \in \mathcal{U}^3$. We will prove the following:

Proposition 3.12. *We assume that $o, i > 0$ and we introduce $m = q|_{]0,r_c[}^{-1}(i)$, $M = q|_{]r_c,1[}^{-1}(o)$ and $T_{a,b} = \{\boldsymbol{\rho} \in \mathcal{S};\ a \leqslant r \leqslant b\}$ for $0 \leqslant a \leqslant b \leqslant 1$. Then $\forall a \leqslant m$ and $\forall b \geqslant M$, $T_{a,b}$ is an invariant set for the Riemann problem* (2.2)–(2.3)–(2.4)*, see Fig.* 9.

Proof. First, we highlight that we have automatically $a \leqslant r_c$ and $b \geqslant r_c$. Then, we have seen before that there are in fact only few cases for $\widehat{\boldsymbol{\rho}}$: $\widehat{r} \leqslant r_c$ and $\widehat{\boldsymbol{\rho}} = \boldsymbol{\rho}^-$; otherwise $\widehat{r} \geqslant r_c$ and $\widehat{\boldsymbol{\rho}} \in \Delta_{r_c}$ or $\widehat{\boldsymbol{\rho}} \in \mathcal{H}_o^{(2)} = \{\boldsymbol{\rho},\ \rho_2 v(\rho_1+\rho_2) = o\}$ or $\widehat{\boldsymbol{\rho}} \in \mathcal{H}^{(1)}_{q(r^+)} = \{\boldsymbol{\rho},\ \rho_1 v(\rho_1+\rho_2) = q(r^+)\}$. Now we examine these cases:

1. if $\widehat{\boldsymbol{\rho}} = \boldsymbol{\rho}^-$, it is obvious as we have taken $\boldsymbol{\rho}^- \in T_{a,b}$;
2. if $\widehat{\boldsymbol{\rho}} \in \Delta_{r_c}$: we have $\Delta_{r_c} \subset T_{a,b}$, since $a \leqslant r_c$ and $b \geqslant r_c$ so $\widehat{\boldsymbol{\rho}} \in T_{a,b}$;
3. if $\widehat{\boldsymbol{\rho}} \in \mathcal{H}_o^{(2)}$: here we have $\widehat{\rho}_2 v(\widehat{\rho}_1 + \widehat{\rho}_2) = o$ so $\widehat{r} v(\widehat{r}) \geqslant o$. As $\widehat{r} \geqslant r_c$ and q decreases on $]r_c, 1[$, then $\widehat{r} \leqslant M$, and we are done;
4. if $\widehat{\boldsymbol{\rho}} \in \mathcal{H}^{(1)}_{q(r^+)}$: as above, we see that $\widehat{\rho}_1 v(\widehat{r}) = q(r^+)$ implies $q(\widehat{r}) \geqslant q(r^+)$ and consequently $\widehat{r} \leqslant r^+ \leqslant M$, and we have finished.

Then we have to do the same thing for $\check{\boldsymbol{\rho}}$. Either $\check{r} > r_c$ and $\check{\boldsymbol{\rho}} \in \Delta_{r^+}$; or $\check{r} \leqslant r_c$ and $\check{\boldsymbol{\rho}} \in \mathcal{H}_i^{(2)} = \{\boldsymbol{\rho};\ \rho_2 v(\rho_1+\rho_2) = i\}$ or $\check{\boldsymbol{\rho}} \in \Delta_{r_c}$. As before, we examine very briefly the different cases:

1. if $\check{\boldsymbol{\rho}} \in \Delta_{r^+}$, then $\check{r} = r^+$ and then $\check{\boldsymbol{\rho}} \in T_{a,b}$;
2. if $\check{\boldsymbol{\rho}} \in \mathcal{H}_i^{(2)}$, then $\check{\rho}_2 v(\check{r}) = i$ so $q(\check{r}) \geqslant i$ and consequently, as $\check{r} \leqslant r_c$ then $\check{r} \geqslant m$.

In order to finish the proof, we have only to say that $\widetilde{\boldsymbol{\rho}} \in \Delta_{r^+}$, and we have directly that $\widetilde{\boldsymbol{\rho}} \in T_{a,b}$. $\square$

3.5. Particular case

In the case $v(r) = V_m r(1-r)$, we have $q(r) = V_m r(1-r)$; and $\rho_1 v(\rho_1+\rho_3) = i$ becomes $\rho_1^2 + \rho_1\rho_3 - \rho_1 + i/V_m = 0$. We are thus led to study the curve H of equation $x^2 + xy - x + C/V_m = 0$ in the plane (x, y). This is the equation of a hyperbola of centre $(0, 1)$ and of asymptotes the lines of equations $x = 0$ and $y = 1 - x$; and if $C/V_m \leqslant 1/4$, then $H \cap \mathcal{S} \neq \emptyset$.

If $r^- \geqslant r_c$ and $r^+ \leqslant r_c$, then we obtain Fig. 8.

4. Resolution of the '*n*-T' problem

We want to consider an infinite road with only one point of entry and exit. That is why, in order to treat the half-Riemann problems, we first deal with a Riemann problem on a road without extra entry or exit but with M types of vehicles, with the same speed law.

4.1. Riemann problem with M types of vehicles on an infinite road

Proposition 4.1. *The Riemann problem with M types of vehicles*:

$$\forall i \in [\![1, M]\!], \quad \partial_t \rho_i + \partial_x\left(\rho_i v\left(\sum_{1\leqslant i\leqslant M} \rho_i\right)\right) = 0 \tag{4.12}$$

with constant initial conditions:

$$\forall i \in [\![1, M]\!], \quad \begin{cases} \rho_i(0, x) = \rho_i^- & \text{for } x < 0, \\ \rho_i(0, x) = \rho_i^+ & \text{for } x > 0, \end{cases} \tag{4.13}$$

where $\boldsymbol{\rho}^-, \boldsymbol{\rho}^+ \in \mathcal{S}_M$ admits a unique solution.

Proof. Let $r = \sum_{1\leqslant i\leqslant M} \rho_i$, $s_i = \frac{\rho_1}{\rho_i}$, for $i \in [\![2, M]\!]$. For regular solutions, the system is equivalent to:

$$\begin{cases} \partial_t r + \partial_x\big(r v(r)\big) = 0, \\ \partial_t s_i + v(r)\partial_x s_i = 0 \quad \text{for } i \in [\![2, M]\!]. \end{cases}$$

We see here that the characteristic speeds of the system are $\lambda_1 = v(r) + r v'(r)$, which is of order 1, and $\lambda_2 = v(r) \geqslant 0$, which is of order $M - 1$. We also obtain the wave set:

$$\mathcal{O}_1(\overline{\boldsymbol{\rho}}) = \{t\overline{\boldsymbol{\rho}},\ 0 \leqslant t \leqslant 1/\bar{r}\},$$

$$\mathcal{O}_2(\overline{\boldsymbol{\rho}}) = \{r = \bar{r}\}.$$

The solution of the Riemann problem consists consequently of a 1-wave followed by a 2-wave, the intermediate state being $\widetilde{\boldsymbol{\rho}} = \frac{r_+}{r_-}\boldsymbol{\rho}^-$. $\square$

4.2. Half-Riemann problem

We use the same ideas as for the half-Riemann problem with only two types of vehicles, as the second characteristic speed is always positive, so we have only to take the first characteristic speed into account: only the total density is seen.

Lemma 4.2. *Fix $\boldsymbol{\rho}^- \in \mathcal{S}_{(n+1)^2}$ and denote $r^- = \sum_{i,j} \rho^-_{i,j}$. Then the set $N(\boldsymbol{\rho}^-)$ of points $\widehat{\boldsymbol{\rho}} \in \mathcal{S}_{(n+1)^2}$ such that the solution to the Riemann problem* (4.12)–(4.13) *contains only waves with negative speed is*:

If $r^- \geqslant r_c$**:** *the segment with extreme points $\frac{r_c}{r^-}\boldsymbol{\rho}^-$ and $\frac{1}{r^-}\boldsymbol{\rho}^-$;*
If $r^- < r_c$**:** *the segment with extreme points $\frac{\sigma(r^-)}{r^-}\boldsymbol{\rho}^-$ and $\frac{1}{r^-}\boldsymbol{\rho}^-$, together with the point $\boldsymbol{\rho}^-$.*

Lemma 4.3. *Fix $\boldsymbol{\rho}^+ \in \mathcal{S}_{(n+1)^2}$ and denote $r^+ = \sum_{i,j} \rho^+_{i,j}$. Then, the set $P(\boldsymbol{\rho}^+)$ of points $\check{\boldsymbol{\rho}} \in \mathcal{S}_{(n+1)^2}$ such that the solution to the Riemann problem* (4.12)–(4.13) *contains only waves with positive speed is*:

If $r^+ \leqslant r_c$**:** *the set T_{r_c} of points $\check{\boldsymbol{\rho}} \in \mathcal{S}_{(n+1)^2}$ such that $r \leqslant r_c$;*
If $r^+ > r_c$**:** *the set $T_{\sigma(r^+)}$ of points $\check{\boldsymbol{\rho}} \in \mathcal{S}_{(n+1)^2}$ such that $r \leqslant \sigma(r^+)$, together with the line $\Delta = \{r = r^+\}$.*

4.3. Local resolution in x_k

We have now to stick the two half problems in $x = x_k$, taking the boundary conditions into account.

Proof of Theorem 2.8. Locally in a neighbourhood of x_k, the solution consists of:

1. the state $\boldsymbol{\rho}^-$;
2. a (possibly null) 1-wave with negative speed;
3. the state $\widehat{\boldsymbol{\rho}}$ in $N(\boldsymbol{\rho}^-)$;
4. a non-classical discontinuity with 0 speed;
5. the state $\check{\boldsymbol{\rho}}$ in $P(\boldsymbol{\rho}^+)$;
6. a (possibly null) 1-wave with positive speed;
7. the state $\widetilde{\boldsymbol{\rho}}$;
8. a (possibly null) 2-wave with positive speed.

The first wave and the last two waves are a standard solution to the Riemann problem; the states $\widehat{\boldsymbol{\rho}}$, $\check{\boldsymbol{\rho}}$ are obtained thanks to Lemma 4.2 and Lemma 4.3. Now we only have to stick these states in $x = x_k$, which is possible if the boundary conditions (2.7)–(2.8) are realised. The uniqueness will come from the maximisation of these conditions.

In order to do this we are coming back to a Riemann problem for a 'one-T' road with only three types of vehicles. Let $\rho_1 = \sum_{i\neq k,\, j\neq k} \rho_{i,j}$, $\rho_2 = \sum_{0\leqslant i\leqslant n} \rho_{i,k}$ and $\rho_3 = \sum_{1\leqslant i\leqslant n+1} \rho_{k,j}$. Thanks to the preceding work, we obtain $(\widehat{\rho}_1, \widehat{\rho}_2)$ and $(\check{\rho}_1, \check{\rho}_3)$.

We get:

$$\widehat{\rho}_1 = \sum_{i\neq k,\ j\neq k} \widehat{\rho}_{i,j}, \qquad \check{\rho}_1 = \sum_{i\neq k,\ j\neq k} \check{\rho}_{i,j},$$
$$\widehat{\rho}_2 = \sum_{0\leqslant i\leqslant n} \widehat{\rho}_{i,k}, \qquad \check{\rho}_3 = \sum_{1\leqslant j\leqslant n+1} \check{\rho}_{k,j}.$$

In particular, we know $\widehat{r} = \sum_{i,j} \widehat{\rho}_{i,j} = \widehat{\rho}_1 + \widehat{\rho}_2$. The $\widehat{\rho}_{k,j}$, for $j \neq k$, seem to miss, but in fact the correspondent vehicles are disappearing before the road $k-1$; their densities are consequently null. Otherwise, it is sufficient to know $\widehat{r}$ in order to know $\widehat{\boldsymbol{\rho}}$, as $\widehat{\boldsymbol{\rho}} \in N(\boldsymbol{\rho}^-)$ and so it is proportional to $\boldsymbol{\rho}^-$. Hence we have: $\widehat{\boldsymbol{\rho}} = \frac{\widehat{r}}{r^-}\boldsymbol{\rho}^-$.

It is the same for $\check{r} = \sum_{i,j} \check{\rho}_{i,j} = \check{\rho}_1 + \check{\rho}_3$. The missing species have indeed their densities null, as they disappear in x_k. This fact allows us to determine the $\check{\rho}_{i,j}$ for $i \neq k$, $j \neq k$, if $\check{r} \neq 1$, as the flow conservation gives: $\check{\rho}_{i,j} = \frac{v(\widehat{r})}{v(\check{r})}\widehat{\rho}_{i,j}$.

We now have only to determine the $\check{\rho}_{k,j}$ for $j \in [\![1, n+1]\!]$, knowing that $\check{\rho}_3 = \sum_{1\leqslant j\leqslant n+1} \check{\rho}_{k,j}$. As we know the numbers $p_{k,j}$ giving the probability for a vehicle entering in k to go in j, we can conclude:

$$\check{\rho}_{k,j} = p_{k,j}\check{\rho}_3,$$

and we are done.

The case $\check{r} = 1$ arrives if and only if $r^+ = 1$, so nothing happens on the right because the traffic is blocked. That is why, in this case, we take $\check{\boldsymbol{\rho}} = \boldsymbol{\rho}^+$. In fact, all the points of maximal densities are equivalent, as they are linked by a 2-wave of speed $v(1) = 0$.

Then, it remains to collate these local solutions, which is possible as long as the waves do not cross each other. As the points of discontinuity are separated by at least L, the time of existence is $T \geqslant L/(2V)$, where $V = v(0)$. $\square$

5. Conclusion

Thanks to this work, we have obtained a new modellisation of the traffic on a roundabout. The main point in this new modellisation is the introduction of special boundary conditions in order to treat the points of entry and exit. Thanks to classical tools of hyperbolic systems and maximisation of the boundary conditions, we first obtain a result of existence and uniqueness of a weak entropy solution in the case of the one-T road. Then, collating the solutions obtained locally, we have been able to derive a result of existence and uniqueness of a weak entropy solution for the Riemann problem in the case of a roundabout; the time of existence of this solution is finite, but we can give a lower bound on it. A more painful result is that the obtained Riemann solver is not continuous, essentially in two points that correspond to the fact that a road is blocked, and for example the maximal densities lead us to discontinuity phenomena. However, there exists some invariant sets, that can avoid these maximal densities.

Qualitatively, this model seems quite coherent, and the points of discontinuity can be understood since we have not at all the same behaviour in two very close situations: if a road is jammed and nobody wants to enter it, the problem is ignored, but if only one car wants to go there, then it will stop and block all the traffic.

Finally, we can consider some new problems: if we never consider maximal density in the initial condition, and if the output functions have a lower bound strictly positive, then we are allowed to think to methods such as front tracking (see [5,8,13]) in order to address the Cauchy problem, and perhaps prolong the time of existence.

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