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On the Lower Semicontinuity of the Set-Valued Metric Projection

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1. INTRODUCTION

Let X be a real normed linear space and V a proximinal subset of X. To each element f in X we associate the set

$$P_V(f) := \{v_0 \in V : \|f - v_0\| = \inf_{v \in V} \|f - v\|\},$$

which is called the set of best approximations for f by elements of V. Thus we obtain a set-valued mapping P_V, which carries the normed linear space X into the set of the closed nonvoid subsets of V. This set-valued mapping is called the *metric projection associated with* V.

For set-valued mappings concepts of continuity are defined as follows (cf. Hahn [5]):

DEFINITION 1. (a) The metric projection P_V is called upper semicontinuous (usc) if the set

$$\{f \in X : P_V(f) \cap K \neq \varnothing\}$$

is closed whenever K is a closed subset of V.

(b) The metric projection is called lower semicontinuous (lsc) if the set

$$\{f \in X : P_V(f) \cap U \neq \varnothing\}$$

is open whenever U is an open subset of V. (The topology on V is understood to be induced by the norm-topology of X).

The metric projection P_V is usc or lsc only for restricted classes of subsets V. Singer [8], for example, has proved that the metric projection associated with an approximatively compact subset V of a normed linear space is usc. Hence, in particular, P_V is usc whenever V is a linear subspace of finite dimension. But even if V is a linear subspace of finite dimension, P_V may fail to be lsc, as Blatter, Morris, and Wulbert [2] have shown.

In this paper, we first prove a general criterion which is sufficient for the lower semicontinuity of the metric projection associated with certain subspaces of a normed linear space. As a consequence of this criterion, we obtain a result of Brosowski *et al.* [4]. Further we apply our criterion to derive a sufficient condition for the lower semicontinuity of the metric projection associated with certain linear subspaces of $C_0(T, X)$, where T is a locally compact Hausdorff-space, X is a strictly convex normed linear space, and $C_0(T, X)$ is the set of all continuous functions $f: T \to X$ which vanish at infinity, provided with the norm $\|f\| := \max_{t\in T} \|f(t)\|_X$.

We shall show that in $C_0(T, X)$ the criterion thus obtained is also necessary for the lower semicontinuity of P_V. This generalizes the results of Blatter [1], Blatter, Morris, and Wulbert [2], and Brosowski *et al.* [3].

Furthermore, we apply our general sufficient criterion to prove the sufficiency of a criterion stated by Lazar, Wulbert, and Morris [6] for the lower semicontinuity of the metric projection associated with finite-dimensional linear subspaces of $L_1(T, \mathfrak{K}, \mu)$, where $(T, \mathfrak{K}, \mu)$ is a σ-finite measure space.

2. A Sufficient Condition For the Lower Semicontinuity of P_V

We first state some necessary definitions. For X a normed linear space, we define

$$S_X := \{x \in X : \| x \| \leqslant 1\}$$

and

$$\mathscr{E}_X := \mathrm{Ep}(S_X)$$

where $\mathrm{Ep}(A)$ denotes the set of extreme points of a set A. For $f \in X$, we set

$$\Sigma_f := \{x' \in S_{X'} : x'(f) = \|f\|\}$$

and

$$\mathscr{E}_f := \{x' \in \mathscr{E}_{X'} : x'(f) = \|f\|\}.$$

As is well known, $\mathcal{E}_f = \Sigma_f \cap \mathcal{E}_{X'}$. We use the terms σ-topology and σ_{Ep}-topology to denote the restrictions of the weak topology $\sigma(X', X)$ on the sets $S_{X'}$ and $\mathcal{E}_{X'}$, respectively. For V a proximinal linear subspace of X and f in X with $0 \in P_V(f)$, we define the set

$$\mathfrak{N}_{f,V} := \bigcap_{v \in P_V(f)} \{x' \in \mathcal{E}_{X'} : x'(v) = 0\}.$$

The subscript V will be omitted if it is understood from the context. Finally, we define

$$E(f; V) := \inf_{v \in V} \|f - v\|.$$

Now we prove the following.

LEMMA 2. *Let A be a subset of $\mathcal{E}_{X'}$ and f an element in X. Then the inequality*

$$\sup_{x' \in A} x'(f) < \|f\|$$

holds if and only if there exists a $\sigma(X', X)$-open convex subset U of X' such that

$$\mathcal{E}_{X'} \setminus A \supset U \cap \mathcal{E}_{X'} \supset \mathcal{E}_f .$$

Proof. Let A be a subset of $\mathcal{E}_{X'}$. Whenever

$$s := \sup_{x' \in A} x'(f) < \|f\|,$$

then there is an $\epsilon > 0$ so that $s \leq \|f\| - \epsilon$. For the $\sigma(X', X)$-open convex set

$$U := \{x' \in X' : x'(f) > \|f\| - \epsilon\},$$

we have $A \cap U = \varnothing$, and hence $A \cap (U \cap \mathcal{E}_{X'}) = \varnothing$. Thus we obtain

$$\mathcal{E}_{X'} \setminus A \supset U \cap \mathcal{E}_{X'} \supset \mathcal{E}_f .$$

To prove the inverse implication, we assume that U is a $\sigma(X', X)$-open convex subset of X' such that

$$\mathcal{E}_{X'} \setminus A \supset U \cap \mathcal{E}_{X'} \supset \mathcal{E}_f .$$

Then

$$\sup_{x' \in A} x'(f) \leq \sup_{x' \in \mathcal{E}_{X'} \setminus (U \cap \mathcal{E}_{X'})} x'(f) = \sup_{x' \in \mathcal{E}_{X'} \setminus U} x'(f) \leq \sup_{x' \in S_{X'} \setminus U} x'(f).$$

The Krein–Milman theorem yields $\Sigma_f = \overline{\text{con}\, \mathscr{E}_f}$. Since U is convex, and contains $\mathscr{E}_f$, there follows $U \supset \text{con}\, \mathscr{E}_f$. We show that $U \supset \Sigma_f$. In fact, suppose that there is an element x_0' in Σ_f which is not in U. Then x_0' is in the boundary of U, and, since U is open and convex, there exists an element $g \in X$ such that $x_0'(g) = \inf_{y' \in U} y'(g)$. Define

$$H := \{x' \in X' : x'(g) = x_0'(g)\}.$$

Since U is open, $H \cap U = \varnothing$. Using $\Sigma_f \subset \bar{U}$ we obtain

$$x_0'(g) = \inf_{y' \in \Sigma_f} y'(g),$$

whence $H \cap \Sigma_f$ is an extremal subset of Σ_f. Therefore $H \cap \Sigma_f$ contains an extreme point x_1' of Σ_f. In view of $\mathscr{E}_f \subset U$ we have $x_1' \in H \cap U$, which contradicts $H \cap U = \varnothing$. Thus we have $U \supset \Sigma_f$ and finally, since $S_{X'} \backslash U$ is compact,

$$\sup_{x' \in S_{X'} \backslash U} x'(f) < \|f\|.$$

This completes the proof.

We are now ready to prove the following.

THEOREM 3. *Let V be a proximinal linear subspace of a normed linear space X with the properties*:

(1) *for each $f \in X$*, $\dim P_V(f) < \infty$,

(2) *the metric projection P_V is usc.*

Whenever for each element f in X with $0 \in P_V(f)$ there exists a $\sigma(X', X)$-open convex subset U of X' such that

$$\mathfrak{N}_f \supset U \cap \mathscr{E}_{X'} \supset \bigcap_{v \in P_V(f)} \mathscr{E}_{f-v}$$

then P_V is lsc.

Proof. We assume that there exists a point f in X so that P_V is not lsc in f. Then there exists a sequence $\{f_n\}$ of elements f_n in X, an element v_0 in $P_V(f)$ and a neighborhood U_0 of v_0 such that $\{f_n\}$ converges to f and

$$P_V(f_n) \cap U_0 = \varnothing$$

for each n. The set $P_V(f)$ is convex and, since P_V is usc, consists of more than one point. We may assume without loss of generality that v_0 is a relative interior point of $P_V(f)$. Replacing f by $f - v_0$ if necessary, we may assume that $v_0 = 0$.

Then it follows that

$$\mathfrak{N}_f \supset \bigcap_{v\in P_V(f)} \mathcal{E}_{f-v} = \mathcal{E}_f .$$

For each $g \in X$ the set $P_V(g)$ is closed, bounded and finite-dimensional, and hence compact. Since P_V is usc, the set

$$P_V(f) \cup \bigcup_{n\in\mathbb{N}} P_V(f_n)$$

is compact (cf. Michael [7]). Consequently there is a subsequence of $\{f_n\}$ (which we denote again by $\{f_n\}$), so that there exist elements v_n in $P_V(f_n)$ in such a way that the sequence $\{v_n\}$ converges to an element v' in V. Then $v' \in P_V(f)$, and furthermore $v' \neq 0$ since $P_V(f_n) \cap U_0 = \varnothing$ for all $n \in \mathbb{N}$.

Following Brosowski *et al.* [4], we can now construct a sequence $\{u_n\}$ of elements u_n in X with the following properties:

(a) the sequence $\{u_n\}$ converges to v';

(b) for each $n \in \mathbb{N}$ there holds

$$\|f - u_n\| = E(f; V);$$

(c) there exists an integer $n_0 \in \mathbb{N}$ so that

$$\|f - u_n + v'\| > E(f; V) \qquad \text{for all } n > n_0 .$$

Such a sequence may be defined explicitly by

$$u_n := f - (E(f; V)/E(f_n ; V))(f_n - v_n).$$

Obviously this sequence has properties (a) and (b). To show that $\{u_n\}$ also shares property (c), we note that $\|f - u_n + v'\| \geq E(f; V)$, since

$$f - u_n + v' = (E(f; V)/E(f_n ; V))(f_n - (v_n - (E(f_n ; V)/E(f; V))\, v')). \quad (1)$$

If there is some subsequence of $\{u_n\}$ (again denoted by $\{u_n\}$) such that $\|f - u_n + v'\| = E(f; V)$, then, by (1), the element

$$\bar{v}_n := v_n - \frac{E(f_n ; V)}{E(f; V)}\, v'$$

is in $P_V(f_n)$, and the sequence $\{\bar{v}_n\}$ converges to the zero element in V. This is impossible since $P_V(f_n) \cap U_0 = \varnothing$ for each n. Hence (c) is proved.

For $n > n_0$ and $x' \in \mathcal{E}_{f-u_n+v'}$ we have

$$\begin{aligned} E(f; V) &< \|f - u_n + v'\| \\ &= x'(f - u_n + v') = x'(f - u_n) + x'(v') \\ &\leq E(f; V) + x'(v'). \end{aligned}$$

which yields $x'(v') > 0$. Hence we obtain

$$\mathfrak{N}_f \cap \mathscr{E}_{f-u_n+v'} = \emptyset \qquad \text{for} \quad n > n_0 . \tag{2}$$

On the other hand, by hypothesis, there exists a $\sigma(X', X)$-open convex subset U of X' so that

$$\mathfrak{N}_f \supset U \cap \mathscr{E}_{X'} \supset \mathscr{E}_f ,$$

and, by Lemma 2,

$$\sup_{x' \in \mathscr{E}_{X'} \setminus U} x'(f) < \|f\| = E(f; V).$$

Hence there exists a real number $\epsilon > 0$ such that

$$x'(f) \leqslant E(f; V) - \epsilon$$

for each $x' \in \mathscr{E}_{X'} \setminus U$. The sequence $\{u_n\}$ converges to v'. Therefore we have $\| u_n - v' \| < \epsilon$ for sufficiently large n, and consequently

$$x'(f - u_n + v') < E(f; V)$$

for all x' in $\mathscr{E}_{X'} \setminus U$, in particular for x' in $\mathscr{E}_{X'} \setminus \mathfrak{N}_f$. Thus we obtain for sufficiently large n the inclusion $\mathfrak{N}_f \supset \mathscr{E}_{f-u_n+v'}$ which contradicts (2). The theorem is thus proved.

A consequence of Theorem 3 is the following result of Brosowski *et al.* [4].

THEOREM 4. *Let V be a proximinal linear subspace of a normed linear space X with the properties* (1), (2) *of Theorem* 3 *and the following additional property*:

$$\textit{The set } \mathscr{E}_{X'} \textit{ is } \sigma\textit{-closed.} \tag{3}$$

Whenever for each element f in X with $0 \in P_V(f)$ there exists a σ-open subset A of $S_{X'}$ such that

$$\mathfrak{N}_f \supset A \cap \mathscr{E}_{X'} \supset \bigcap_{v \in P_V(f)} \mathscr{E}_{f-v}$$

then the metric projection P_V is lsc.

Proof. In view of property (3), the set $\mathscr{E}_{X'}$ is σ-compact. Hence we obtain

$$\sup_{x' \in \mathscr{E}_{X'} \setminus A} x'(f) < \|f\|,$$

and, using the hypothesis,

$$\sup_{x' \in \mathscr{E}_{X'} \setminus \mathfrak{N}_f} x'(f) < \|f\|.$$

Thus, by Lemma 2, there is a $\sigma(X', X)$-open convex subset U of X' such that $\mathfrak{N}_f \supset U \cap \mathscr{E}_{X'} \supset \mathscr{E}_f$, and Theorem 3 yields the lower semicontinuity of P_V.

3. Applications in Some Special Spaces

Let T be a locally compact Hausdorff space, and X a strictly convex real normed linear space. We denote by $C_0(T, X)$ the set of continuous functions $f: T \to X$ vanishing at infinity, that is, a continuous function f is in $C_0(T, X)$ if and only if, for each $\epsilon > 0$, the set

$$\{t \in T : \|f(t)\| \geq \epsilon\}$$

is compact. If addition and multiplication with scalars are defined for elements in $C_0(T, X)$ in the same way as for vector-valued functions, and the norm

$$\|f\| := \max_{t \in T} \|f(t)\|_X$$

is introduced, then $C_0(T, X)$ is a normed linear space. Whenever it is necessary to distinguish between the norms in $C_0(T, X)$ and X we denote the latter by $\|\cdot\|_X$.

For V a proximinal subspace of $C_0(T, X)$, f an element in $C_0(T, X)$ with $0 \in P_V(f)$, and v_0 an element in $P_V(f)$, we define

$$N_{f,V} := \bigcap_{v \in P_V(f)} \{t \in T : v(t) = 0\}$$

and

$$M_{f-v_0} := \{t \in T : \|f(t) - v_0(t)\|_X = \|f - v_0\|\}.$$

The subscript V will be omitted if it is understood from the context.

Now we prove the following.

Lemma 5. *Let V be a proximinal linear subspace of $C_0(T, X)$, where X is strictly convex. Then*

$$N_f \supset \bigcap_{v \in P_V(f)} M_{f-v}$$

for each element f in $C_0(T, X)$ with 0 in $P_V(f)$.

Proof. Let t be in $\bigcap_{v \in P_V(f)} M_{f-v}$. Since $0 \in P_V(f)$, for each v in $P_V(f)$ the element $\frac{1}{2}v$ is also in $P_V(f)$. Hence

$$\|f(t) - \tfrac{1}{2}v(t)\| = \tfrac{1}{2}\|f(t)\| + \tfrac{1}{2}\|f(t) - v(t)\|.$$

In view of the strict convexity of X, there is a positive real number μ such that

$$f(t) = \mu(f(t) - v(t)).$$

Using $\|f(t)\| = \|f(t) - v(t)\|$, we obtain $\mu = 1$ and finally $v(t) = 0$, whence $t \in N_f$.

Now we give a sufficient criterion for the lower semicontinuity of P_V in $C_0(T, X)$.

THEOREM 6. *Let V be a proximinal linear subspace of the space $Z := C_0(T, X)$, where T is a locally compact Hausdorff space, and X is a strictly convex space. Assume that the following requirements hold:*

(1) *for each f in Z*, $\dim P_V(f) < \infty$;

(2) *the metric projection P_V is usc.*

Whenever for each f in Z with $0 \in P_V(f)$ the set N_f is open then the metric projection is lsc.

Proof. The set N_f is defined to be the intersection of the closed sets $\{t \in T : v(t) = 0\}$, $v \in P_V(f)$, hence it is closed. Since N_f is also open by hypothesis, the function g defined by

$$g(t) := \begin{cases} f(t) & \text{for} \quad t \text{ in } N_f, \\ 0 & \text{for} \quad t \text{ in } T \backslash N_f, \end{cases}$$

is in $C_0(T, X)$. The set

$$U := \{z' \in Z' : z'(g) > 0\}$$

is a $\sigma(Z', Z)$-open and convex subset of Z'.

The functionals in $\mathscr{E}_{Z'}$ are generalized evaluation functionals $L_{x',t}$, that is, there exist elements $x' \in \mathscr{E}_{X'}$ and $t \in T$ such that

$$L_{x',t}(h) = x'(h(t)) \qquad \text{for} \qquad h \in C_0(T, X).$$

By definition of U, for each functional $L_{x',t} \in \mathscr{E}_{Z'} \cap U$, the equality

$$L_{x',t}(v) = x'(v(t)) = x'(0) = 0$$

holds for all v in $P_V(f)$, whence we conclude $\mathfrak{N}_f \supset \mathcal{E}_{Z'} \cap U$. Using Lemma 5, we obtain the inclusion

$$\mathcal{E}_{Z'} \cap U \supset \bigcap_{v \in P_V(f)} \mathcal{E}_{f-v} .$$

Thus the requirements of Theorem 3 are fulfilled, whence the lower semicontinuity of P_V follows.

Remarks. Special cases of Theorem 6 have appeared in the literature. For T compact and X the real axis, the theorem was proved by Blatter, Morris, and Wulbert [2]. For T locally compact and X a pre-Hilbert space, the result was obtained by Brosowski *et al.* [4].

Now let X be the space $L_1(T, \mathfrak{K}, \mu)$, where $(T, \mathfrak{K}, \mu)$ is a σ-finite measure space. The dual space X' is identical to $L_\infty(T, \mathfrak{K}, \mu)$. For f a real-valued function defined on T, we set

$$\mathrm{supp}(f) := \{t \in T : f(t) \neq 0\},$$

$$Z(f) := \{t \in T : f(t) = 0\}$$

$$S(f) := \{t \in T : |f(t)| = \sup_{s \in T} |f(s)|\}.$$

These sets are defined only up to sets of zero measure.

In addition, we define for each linear subspace V of X the orthogonal space

$$V^\perp := \{x' \in X' : x'(v) = 0 \text{ for each } v \in V\}.$$

Lazar, Wulbert, and Morris [6] proved the following criterion.

THEOREM 7. *Let $(T, \mathfrak{K}, \mu)$ be a σ-finite measure space, and let V be an n-dimensional linear subspace of $L_1(T, \mathfrak{K}, \mu)$. The metric projection P_V is lsc if and only if there does not exist an x' in $V^\perp$, $x' \neq 0$, and a v in V for which*

(1) *$S(x')$ is purely atomic, and contains at most $n - 1$ atoms,*

(2) *$Z(v)$ contains $S(x')$,*

(3) supp (v) *is not the union of a finite family of atoms.*

We give a new proof for the sufficiency of this criterion by showing that the condition of Lazar, Wulbert, and Morris implies the condition of Theorem 3. From this, it follows in the case $X = L_1(T, \mathfrak{K}, \mu)$ that the condition of Theorem 3 is also necessary for the lower semicontinuity of P_V.

Proof of the sufficiency of the criterion in Theorem 7. We suppose that the condition of Theorem 7 holds but that there exists an element $f \in L_1$ with $0 \in P_V(f)$ (without loss of generality we may even assume that 0 is a relative interior point of $P_V(f)$) such that $\mathfrak{N}_f$ does not contain $U \cap \mathcal{E}_{X'}$ whenever U is a $\sigma(X', X)$-open convex subset of X' with $U \cap \mathcal{E}_{X'} \supset \mathcal{E}_f$.

First we show that there exists an element $\tilde{v}$ in $P_V(f)$ such that supp($\tilde{v}$) is not the union of a finite family of atoms.

To prove this, we suppose that for each v in $P_V(f)$, the support supp(v) is a union of a finite number of atoms. Then there exist atoms $A_1, \ldots, A_N$ such that

$$\operatorname{supp}(v) \subset A := A_1 \cup \cdots \cup A_N$$

for each v in $P_V(f)$.

For every $v \in P_V(f)$ and every $x' \in \mathcal{E}_f$, we have $x'(v) = 0$. Since the functionals $x' \in \mathcal{E}_f$ (interpreted as functions in L_∞) may be chosen outside supp (f) arbitrarily retaining only the requirement $|x'(t)| = 1$, there must be $\operatorname{supp}(v) \subset \operatorname{supp}(f)$ for all $v \in P_V(f)$. Therefore one may assume $A \subset \operatorname{supp}(f)$.

Corresponding to the set

$$W := \{x' \in \mathcal{E}_{X'} : x'(t) = \operatorname{sign}(f(t)) \text{ for } t \in A\}$$

there exists a $\sigma(X', X)$-open convex subset U of X' so that $W = U \cap \mathcal{E}_{X'}$, namely, e.g.

$$U := \left\{x' \in X' : \int_T x'(t)(\chi_{A_\nu}(t) \cdot f(t))\, d\mu > 0 \text{ for } \nu = 1, \ldots, N\right\}$$

where χ_{A_ν} denotes the characteristic function of the set A_ν. In addition, we have

$$\mathfrak{N}_f \supset W = U \cap \mathcal{E}_{X'} \supset \mathcal{E}_f .$$

Since such a relation was excluded by the choice of f, we have proved our assertion that there exists some element $\tilde{v}$ in $P_V(f)$ such that supp($\tilde{v}$) is not the union of a finite number of atoms.

Now let

$$\begin{aligned} Y' := \{x' \in S_{X'} : x'(v) = 0 \text{ for all } v \in V, \\ x'(t) = \operatorname{sign}(f(t)) \text{ for } t \in \operatorname{supp}(f)\}, \end{aligned}$$

then Y' is convex and σ-compact. By the well-known theorem of characterization of best approximations, there exists a functional $x' \in \Sigma_f$ such that $x'(v) = 0$ for all v in V. Since this x' is in Y', the set Y' is nonvoid. Hence there exists some extreme point y' of Y'. By construction, we have $|y'(t)| = 1$ for $t \in \operatorname{supp}(f)$. Since $\operatorname{supp}(\tilde{v}) \subset \operatorname{supp}(f)$ it follows that $Z(\tilde{v}) \supset S(y')$.

Now we show that $S(y')$ is purely atomic and consists of at most $n - 1$ atoms. Let $v^1, v^2, \ldots, v^n$ be a basis for V with $v^1 = \tilde{v}$.

First we exclude that $S(y')$ contains a nonatomic part. In fact, let $B \subset S(y')$ be a nonatomic subset of $S(y')$ with $\mu(B) \neq 0$. Then there exists $\epsilon > 0$ so that

$$B_1 := \{t \in B : |y'(t)| < 1 - \epsilon\}$$

has $\mu(B_1) \neq 0$. Then there exists a function $z' \in S_{X'}$, $z' \neq 0$, such that $\operatorname{supp}(z') \subset B_1$ and $\int_{B_1} z'(t)\, v^i(t)\, d\mu = 0$ for $i = 1,..., n$. For either sign, $y' \pm \epsilon z'$ is in Y'. This contradicts the fact that y' is an extreme point of Y'.

Now let us suppose $S(y') \supset B = B_1 \cup \cdots \cup B_n$, where B_ν are atoms. It follows that $\epsilon := 1 - \text{ess-sup}_{t \in B} |y'(t)| > 0$. Let v_ν^i be the value of v^i on the atom B_ν. The system of equalities

$$\sum_{\nu=1}^{n} \alpha_\nu v_\nu^i \mu(B_\nu) = 0, \qquad i = 2,..., n, \tag{4}$$

has a nonzero solution $\alpha_1^0,..., \alpha_n^0$ with $|\alpha_\nu^0| \leq 1$ for all ν. Hence for either sign the function z', defined by

$$z'(t) := \begin{cases} y'(t) \pm \epsilon\alpha_\nu^0 & \text{for} \quad t \in B_\nu \\ y'(t) & \text{for} \quad t \notin B \end{cases},$$

is in Y'. This is impossible since y' is an extreme point of Y'. Therefore $S(y')$ contains at most $n - 1$ atoms.

So far, we have constructed elements $\tilde{v} \in V$ and $y' \in Y' \subset V^\perp$ which fulfil the requirements (1), (2), and (3) of Theorem 7. But the condition of Theorem 7 states that such elements do not exist. Hence our assumption is not correct. Thus we have proved, that the condition of Lazar, Wulbert, and Morris implies the condition of Theorem 3. Since the latter is sufficient for the lower semicontinuity of P_V the sufficiency part of Theorem 7 is proved.

4. The Necessity of the Criterion in the Space $C_0(T, X)$

In this paragraph, we show that the sufficient condition of Theorem 6 is also necessary if the space under consideration is $C_0(T, X)$, where X is strictly convex. First we prove the following lemma.

Lemma 8. *Let V be a linear subspace of $C_0(T, X)$, and let f and g be elements in $C_0(T, X)$ with*

(a) $\|f\| = \|g\|$;

(b) $0 \in P_V(f)$ *and* $0 \in P_V(g)$;

(c) *there is a neighbourhood U of M_f such that $f(t) = g(t)$ for $t \in U$.*

Then $P_V(g)$ is contained in span $P_V(f)$, i.e., in the linear subspace of V which is spanned by $P_V(f)$.

Proof. Given an element $v \neq 0$ in $P_V(g)$, let λ be the positive number

$$\lambda := \min(1, (\|f\| - E^*)/\|v\|),$$

with

$$E^* := \sup_{t \in T \setminus U} \|f(t)\|_X < \|f\|.$$

By virtue of hypothesis (b), the element $v_1 := \lambda v$ is also in $P_V(g)$. We have for t in U

$$\|f(t) - v_1(t)\| = \|g(t) - v_1(t)\| \leq \|g\| = \|f\|.$$

and for $t \in T \setminus U$

$$\|f(t) - v_1(t)\| \leq \|f(t)\| + \|v_1(t)\| \leq E^* + (\|f\| - E^*) = \|f\|.$$

Hence the element v_1 is a best approximation for f, and $v = (1/\lambda) v_1$ is in span $P_V(f)$.

We are now in position to prove the main result of this paragraph.

THEOREM 9. *Let T be a locally compact Hausdorff space, X a strictly convex normed linear space, and let V be a proximinal linear subspace of $C_0(T, X)$ such that $\dim P_V(f) < \infty$ for all f in $C_0(T, X)$. Whenever the metric projection P_V is lsc, then, for each f in $C_0(T, X)$ with 0 in $P_V(f)$, the set*

$$N_f := \bigcap_{v \in P_V(f)} \{t \in T : v(t) = 0\}$$

is open.

Proof. We suppose the theorem is false, that is, there exists an element f_1 in $C_0(T, X)$ with $0 \in P_V(f_1)$ such that N_{f_1} is not open. Then f_1 is not in V, since otherwise $P_V(f_1) = \{f_1\}$ and $N_{f_1} = T$ would be open. Without loss of generality we may assume $\|f_1\| = 1$.

Now let $v_1, ..., v_k$ be linear independent elements in $P_V(f_1)$ which span the linear subspace $V_1 :=$ span $P_V(f_1)$ of V.

Since N_{f_1} is not open, there is a point t_0 in N_{f_1} with the property:

(E) every neighborhood U of t_0 contains some point t_U such that $v_\kappa(t_U) \neq 0$ for at least one $\kappa \in \{1, ..., k\}$.

Now we construct a function f as follows. In the case t_0 is in M_{f_1}, we define $f := f_1$. If t_0 is not in M_{f_1}, then we first choose an element r in X such that

$\|r\| = 1$ and $\|r - f_1(t_0)\|_X < 1 - \|f_1(t_0)\|_X$. A possible choice is, for instance,

$$r = f_1(t_0)/\|f_1(t_0)\|_X$$

in the case $f_1(t_0) \neq 0$. If $f_1(t_0) = 0$, each r with $\|r\| = 1$ will do. Since t_0 is not in the closed set M_{f_1}, there is a compact neighborhood U of t_0 such that $M_{f_1} \cap U = \varnothing$. From $t_0 \in N_{f_1}$ there follows for $\kappa = 1,\ldots, k$

$$v_\kappa(t_0) = 0$$

and

$$\|f_1(t_0) - v_\kappa(t_0)\| = \|f_1(t_0)\| < 1.$$

By reducing U, if necessary, we can ensure that, for all $t \in U$,

$$\|f_1(t) - v_\kappa(t)\| < 1, \qquad \kappa = 1,\ldots, k$$

and

$$\|r - f_1(t)\| > 0.$$

There exists a continuous function $\rho_1(t)$ such that $0 \leqslant \rho_1(t) \leqslant 1$ for all $t \in T$, $\rho_1(t_0) = 1$, and $\rho_1(t) = 0$ for $t \notin U$. In addition, we put

$$\rho_2(t) := \min_{1 \leqslant \kappa \leqslant k} ((1 - \|f_1(t) - v_\kappa(t)\|)/\|r - f_1(t)\|)$$

and

$$\rho_3(t) := \min(\rho_1(t), \max(0, \rho_2(t))).$$

We complete the definition of ρ_2 and ρ_3 by setting $\rho_2(t) = \rho_3(t) = 0$ for those t which have $r - f_1(t) = 0$, and thus obtain a continuous function ρ_3 with $0 \leqslant \rho_3(t) \leqslant 1$ for all $t \in T$, $\rho_3(t_0) = 1$, and $\rho_3(t) = 0$ for $t \in T\backslash U$. Now we define the function f by

$$f(t) := (1 - \rho_3(t)) f_1(t) + \rho_3(t) \cdot r.$$

This function has the property that $f(t) = f_1(t)$ for all t in $T\backslash U$, and $f(t_0) = r$, whence $\|f(t_0)\| = 1$ and $t_0 \in M_f$. Since $T\backslash U$ is, by construction, an open set containing M_{f_1}, it follows that $M_{f_1} \subset M_f$, $0 \in P_V(f)$, and finally from Lemma 8, $P_V(f) \subset V_1$. For each t in T and each $\kappa = 1,\ldots, k$ we have

$$\begin{aligned}\|f(t) - v_\kappa(t)\| &= \|f_1(t) - v_\kappa(t) + \rho_3(t)(r - f_1(t))\| \\ &\leqslant \|f_1(t) - v_\kappa(t)\| + \rho_3(t)\|r - f_1(t)\| \\ &\leqslant \|f_1(t) - v_\kappa(t)\| + \max(0, \rho_2(t))\|r - f_1(t)\| \leqslant 1.\end{aligned}$$

Hence all $v_1, \ldots, v_k$ are in $P_V(f)$ and, since 0 is also in $P_V(f)$, the element

$$v_0 := (v_1 + \cdots + v_k)/(k + 1)$$

is a relative interior point of $P_V(f)$. Then 0 is a relative interior point of $P_V(g)$ where $g := f - v_0$.

For each t in T, we have

$$\begin{aligned} \| g(t) \| &= \| f(t) - v_0(t) \| \\ &= \frac{1}{k+1} \left\| f(t) + \sum_{\kappa=1}^{k} (f(t) - v_\kappa(t)) \right\| \\ &\leqslant \frac{1}{k+1} \left\{ \| f(t) \| + \sum_{\kappa=1}^{k} \| f(t) - v_\kappa(t) \| \right\} = 1 \end{aligned}$$

with equality if and only if $v_\kappa(t) = 0$ for all $\kappa = 1, \ldots, k$, that is $t \in N_f$. Hence there follows

$$M_g \subset N_g = N_f .$$

Because t_0 is not an interior point of N_g, there is a net $(t_\lambda : \lambda \in \Lambda)$ of points t_λ in T such that (t_λ) converges to t_0 and, for each λ, $v_\kappa(t_\lambda) \neq 0$ at least for one κ. Then there exists an index κ_0 and a subnet $(t_\lambda : \lambda \in \Lambda_1)$ such that $v_{\kappa_0}(t_\lambda) \neq 0$ for all t_λ with $\lambda \in \Lambda_1$. We may assume $\kappa_0 = 1$.

Now we consider two cases.

First case. There is a subnet $(t_\lambda : \lambda \in \Lambda_2)$ of $(t_\lambda : \lambda \in \Lambda_1)$ such that for every $\lambda \in \Lambda_2$ there exists some functional $x_\lambda' \in \mathscr{E}_{g(t_0)}$ with $x_\lambda'(v_1(t_\lambda)) \neq 0$. By passing once more to a subnet $(t_\lambda : \lambda \in \Lambda_3)$, we can ensure that there exist signs $\epsilon_\kappa \in \{-1, +1\}$ such that the inequalities

$$\epsilon_1 x_\lambda'(v_1(t_\lambda)) < 0$$

and

$$\epsilon_\kappa x_\lambda'(v_\kappa(t_\lambda)) \leqslant 0, \qquad \text{for} \quad \kappa = 2, \ldots, k,$$

hold.

For each $\delta > 0$, the set

$$A_\delta := \{ t \in T : \| g(t_0) - g(t) \| < \delta \}$$

is a neighborhood of t_0. Hence there exists $\lambda \in \Lambda_3$ such that $t_\lambda \in A_\delta$. Since $M_g \subset N_g$ and $t_\lambda \notin N_g$, it follows that $t_\lambda \notin M_g$. Since M_g is closed, there exists a compact neighborhood W of t_λ such that $M_g \cap W = \varnothing$. Without loss of generality, we assume $W \subset A_\delta$. Now let ρ be a continuous function such that $0 \leq \rho(t) \leq 1$ for $t \in T$, $\rho(t_\lambda) = 1$, $\rho(t) = 0$ for $t \in T \setminus W$, and define

$$g_\delta(t) := \rho(t)\, g(t_0) + (1 - \rho(t))\, g(t).$$

Then the function g_δ is in $C_0(T, X)$, and $\| g_\delta - g \| < \delta$. Furthermore, we have for all $t \in T$

$$\| g_\delta(t) \| \leq \rho(t) \| g(t_0) \| + (1 - \rho(t)) \| g(t) \| \leq \| g \|,$$

and for $t \in T \setminus W$ the equality $g_\delta(t) = g(t)$. The set $T \setminus W$ is a neighborhood of M_g. Therefore we have $M_{g_\delta} \supset M_g$, and hence $0 \in P_V(g_\delta)$. Using Lemma 8, we obtain $P_V(g_\delta) \subset V_1$. For each element u in the set

$$B_k := \left\{ \sum_{\kappa=1}^{k} a_\kappa \epsilon_\kappa v_\kappa \in V_1 : a_\kappa > 0 \text{ for } \kappa = 1,\dots, k \right\}$$

there follows

$$\begin{aligned} \| g_\delta - u \| &\geq \| g_\delta(t_\lambda) - u(t_\lambda) \| \\ &\geq x_\lambda' \left(g_\delta(t_\lambda) - \sum_{\kappa=1}^{k} a_\kappa \epsilon_\kappa v_\kappa(t_\lambda) \right) \\ &= x_\lambda'(g(t_0)) - \sum_{\kappa=1}^{k} a_\kappa \epsilon_\kappa x_\lambda' v_\kappa(t_\lambda) \\ &> \| g \| = \| g_\delta \|, \end{aligned}$$

and consequently $B_k \cap P_V(g_\delta) = \varnothing$.

If the net $(t_\lambda : \lambda \in \Lambda_1)$ does not satisfy the conditions of the first case, then we must consider the alternative possibility.

Second case. There is a subnet $(t_\lambda : \lambda \in \Lambda_4)$ of $(t_\lambda : \lambda \in \Lambda_1)$ such that $x'(v_1(t_\lambda)) = 0$ for all $\lambda \in \Lambda_4$ and $x' \in \mathscr{E}_{g(t_0)}$. Let x_λ' be an arbitrary functional in $\mathscr{E}_{g(t_0)+v_1(t_\lambda)}$. Then x_λ' is not in $\mathscr{E}_{g(t_0)}$, since X is strictly convex, and $v_1(t_\lambda)$ is not proportional to $g(t_0)$. Hence it follows that

$$x_\lambda'(g(t_0)) < \| g(t_0) \|.$$

On the other hand we have, for $y' \in \mathscr{E}_{g(t_0)}$,

$$\| g(t_0) + v_1(t_\lambda) \| \geq y'(g(t_0)) + y'(v_1(t_\lambda)) = \| g(t_0) \|,$$

and therefore

$$\| g(t_0) \| \leq \| g(t_0) + v_1(t_\lambda) \| = x_\lambda'(g(t_0) + v_1(t_\lambda)) < \| g(t_0) \| + x_\lambda'(v_1(t_\lambda)),$$

whence it follows that

$$\| g(t_0) + v_1(t_\lambda) \| \geqslant 1 \tag{5}$$

and $x_\lambda'(v_1(t_\lambda)) > 0$ for each λ in Λ_4.

There are signs $\epsilon_1 := -1$ and $\epsilon_2, \epsilon_3, \ldots, \epsilon_k \in \{-1, +1\}$ such that for a suitable subnet $(t_\lambda : \lambda \in \Lambda_5)$ the inequalities

$$\epsilon_1 x_\lambda'(v_1(t_\lambda)) < 0$$

and

$$\epsilon_\kappa x_\lambda'(v_\kappa(t_\lambda)) \leqslant 0, \qquad \text{for} \quad \kappa = 2, 3, \ldots, k,$$

hold. For $\delta > 0$, the set

$$A_\delta := \{t \in T : \| g(t_0) - g(t) \| < \delta/3, \| v_1(t) \| < \delta/3$$
$$\text{and } | \| g(t_0) + v_1(t) \| - 1 | < \delta/3\}$$

is an open neighborhood of t_0. Hence there is $\lambda \in \Lambda_5$ such that $t_\lambda \in A_\delta$. Furthermore, there exists a compact neighborhood W of t_λ such that $M_g \cap W = \varnothing$ and $W \subset A_\delta$, and there is a continuous function ρ such that $0 \leqslant \rho(\ ) \leqslant 1$ for $t \in T$, $\rho(t_\lambda) = 1$, and $\rho(t) = 0$ for $t \in T \backslash W$. The mapping

$$g_\delta(t) := \rho(t) \frac{g(t_0) + v_1(t_\lambda)}{\| g(t_0) + v_1(t_\lambda) \|} + (1 - \rho(t))\, g(t)$$

is in $C_0(T, X)$. By using (5), we obtain for t in A_δ

$$\begin{aligned} \| g_\delta(t) - g(t) \| &= \rho(t) \left\| \frac{g(t_0) + v_1(t_\lambda)}{\| g(t_0) + v_1(t_\lambda) \|} - g(t) \right\| \\ &= \frac{\rho(t)}{\| g(t_0) + v_1(t_\lambda) \|} \| g(t_0) - g(t) + v_1(t_\lambda) \\ &\quad + (1 - \| g(t_0) + v_1(t_\lambda) \|)\, g(t) \| \\ &\leqslant \| g(t_0) - g(t) \| + \| v_1(t_\lambda) \| + | 1 - \| g(t_0) + v_1(t_\lambda) \| | < \delta. \end{aligned}$$

For t not in A_δ the equality $g_\delta(t) = g(t)$ holds, and hence $\| g_\delta - g \| < \delta$. By construction, we have $\| g_\delta \| = 1$. Since $g_\delta(t) = g(t)$ for t in the neighborhood $T \backslash W$ of M_g, it follows that $M_{g_\delta} \supset M_g$, hence 0 is in $P_V(g_\delta)$, and, by Lemma 8, $P_V(g_\delta) \subset V_1$.

For each element u in

$$B_k := \left\{ \sum_{\kappa=1}^{k} a_\kappa \epsilon_\kappa v_\kappa \in V_1 : a_\kappa > 0 \text{ for all } \kappa \right\}$$

there follows

$$\| g_\delta - u \| \geq \| g_\delta(t_\lambda) - u(t_\lambda) \|$$

$$\geq x_\lambda' \left(g_\delta(t_\lambda) - \sum_{\kappa=1}^{k} a_\kappa \epsilon_\kappa v_\kappa(t_\lambda) \right)$$

$$= x_\lambda' \left(\frac{g(t_0) + v_1(t_\lambda)}{\| g(t_0) + v_1(t_\lambda) \|} \right) - \sum_{\kappa=1}^{k} a_\kappa \epsilon_\kappa x_\lambda' v_\kappa(t_\lambda) \geq 1,$$

whence $B_k \cap P_V(g_\delta) = \varnothing$.

Thus in either case we have the following situation. The set B_k is open (in V_1), and contains the zero element in its boundary. Since 0 is an interior point of $P_V(g)$ (relative to V_1), it follows that $P_V(g) \cap B_k \neq \varnothing$. On the other hand for each $\delta > 0$ there exists a g_δ in $C_0(T, X)$ with $\| g - g_\delta \| < \delta$ and $P_V(g_\delta) \cap B_k = \varnothing$. This contradicts the lower semicontinuity of P_V. Thus the theorem is proved.

Remarks. In the case T compact and X the real axis, Theorem 9 specializes to a result of Blatter, Morris, and Wulbert [2]. The special case of Theorem 9, when X is a pre-Hilbert space, was proved by Brosowski *et al.* [3], but only for locally compact spaces T which have additional properties.

References

1. J. Blatter, Zur Stetigkeit von mengenwertigen metrischen Projektionen, *Schr. Rheinisch-Westfälischen Inst. Instrum. Math. Univ. Bonn, Ser. A*, Nr. 16, 1967.
2. J. Blatter, P. D. Morris, and D. E. Wulbert, Continuity of the set-valued metric projection, *Math. Ann.* **178** (1968), 12–24.
3. B. Brosowski, K.-H. Hoffmann, E. Schäfer, and H. Weber, Stetigkeitssätze für metrische Projektionen I. Metrische Projektionen auf lineare Teilräume von $C_0(T, H)$, Max-Planck-Institut für Physik und Astrophysik München, MPI-PAE/Astro 18, 1969.
4. B. Brosowski, K.-H. Hoffmann, E. Schäfer, and H. Weber, Stetigkeitssätze für metrische Projektionen IV. Eine hinreichende Bedingung für die $\mathfrak{T}_u$-Stetigkeit der metrischen Projektion, Max-Planck-Institut für Physik und Astrophysik München, MPI-PAE/Astro 20, 1969.
5. H. Hahn, "Reelle Funktionen I," Chelsea, New York, 1948.
6. A. J. Lazar, D. E. Wulbert, and P. D. Morris, Continuous selections for metric projections, *J. Functional Analysis* **3** (1969), 193–216.
7. E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.* **71** (1951), 152–182.
8. I. Singer, Some remarks on approximative compactness, *Rev. Roumaine Math. Pures Appl.* **9** (1964), 167–177.

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