On error estimation of finite element approximations to the elliptic
equations in nonconvex polygonal domains

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Abstract

Numerical verification methods, so-called Nakao’s methods, on existence or uniqueness of solutions to PDEs have been developed by Nakao and his group including the authors. They are based on the error estimation of approximate solutions which are mainly computed by FEM.

It is a standard way of the error estimation of FEM to estimate the projection errors by elementwise interpolation errors. There are some constants in the error estimation, which depend on the mesh size parameters $h$. The explicit values of the constants are necessary in order to use Nakao’s method. However, there were not so many researches for the computation of the explicit values of the constants. Then we had to develop the computation by ourselves, especially with guaranteed accuracy. Note that the methods of the computation depend on the dimension, the degree of bases, and the shape of the domain, etc.

The present paper shows how we have developed the methods to calculate the constants and describes new results for nonconvex domains.

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1. Introduction

Among the numerical verification methods on existence or uniqueness of solutions to PDEs, there are methods so-called Nakao’s methods which have been developed by Nakao and his group including one of the authors [3–5,12]. They are based on the error estimation of approximate solutions which are mainly computed by FEM. The error estimation to the FEM solution of Poisson equation, equivalent to the error estimation of the $H^1_0$-projection to FEM subspace, is especially important since it derives the verification methods for the elliptic equations.

It is a standard way to estimate the $H^1_0$-projection errors through the error estimation of the interpolation errors. There appear some constants (we call them error constants) in the error estimation, which depend on the mesh size parameters $h$. Nakao’s method needs the explicit values of the constants.

However, there were not so many researches for the computation of the explicit values of the constants. Then we had to develop the methods to compute the constants by ourselves. Note that the methods of the computation depend on the dimension, the degree of bases of FEM subspaces, and the shape of the domain etc, and that the computed results

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should be validated. The story how we have developed the methods for the error constants in various situations is shown in [11].

When the domain $\Omega$ is a nonconvex polygon, the solutions of Poisson equations do not belong to $H^2(\Omega)$, which causes severe difficulties to the computation of the error constants. A device has been developed against the difficulties [12,11]. It gives explicit and validated values of the error constants. We measured the order of decreasing of the errors with respect to the mesh-size parameter $h$ using uniform meshes, and found that it corresponds to the theoretical order [11]. But when we use non-uniform meshes, the results show that the decreasing order no longer corresponds to the theoretical one.

This paper reports the numerical experiments on the error constants in case of nonconvex polygonal domains with non-uniform meshes, and gives some consideration on the gap between the numerical results and the theoretical order.

In section 2, we give an outline of Nakao’s methods in order to explain our motivation. The methods to estimate the error constants are summarized in Sections 3 and 4. Some knowledge from a theoretical point of view is given in Section 5. In Section 6, we specify the numerical experiments and give the consideration.

2. Nakao’s methods and our motivation

Consider a nonlinear elliptic equation with the Dirichlet boundary condition.

$$\begin{cases} -\Delta u = f(x, u, \nabla u) & x \in \Omega, \\ u = 0 & x \in \partial \Omega, \end{cases}$$

where $\Omega$ denotes a polygonal domain $\subset R^2$, $\partial \Omega$ the boundary of $\Omega$, and $f$ is a bounded continuous mapping $R \times H^1 \times (L^2)^2 \rightarrow L^2$.

The weak form of the problem is as follows:

$$(\nabla u, \nabla v) = (f(\cdot, u, \nabla u), v), \quad \forall v \in H^1_0(\Omega).$$

Here $(\cdot, \cdot)$ indicates the inner product of $L^2(\Omega)$ or $(L^2(\Omega))^2$.

Note that every Poisson equation defined in a polygonal domain has a unique solution $z \in H^1_0(\Omega)$ in its weak form for any $g \in L^2(\Omega)$.

$$(\nabla z, \nabla v) = (g, v), \quad \forall v \in H^1_0(\Omega). \quad (1)$$

Consider a mapping $K : L^2(\Omega) \ni g \mapsto z \in H^1_0(\Omega)$ and write

$$z = K g.$$ 

Then $K$ is a compact operator from $L^2(\Omega)$ to $H^1_0(\Omega)$.

Using the mapping $K$, define the operator $F$ by

$$Fu = K f(\cdot, u, \nabla u),$$

and we have a fixed point formula

$$u = Fu$$

for the nonlinear elliptic equation. Using an appropriate fixed point theorem, e.g. Schauder’s fixed point theorem, we obtain the following sufficient condition to have a solution within a non-empty bounded convex closed subset $U \subset H^1_0(\Omega)$.

$$FU \subset U,$$

where

$$FU = \{ Fu \mid u \in U \}.$$ 

The set $U$ is so-called a candidate set and usually defined as a neighborhood of an approximate solution $\tilde{u}_h$. Once we have verified that $U$ includes a true solution, then the radius of $U$ gives an error bound of the approximation $\tilde{u}_h$. 
In Nakao’s method, the condition \( FU \subset U \) is treated as follows.

Let \( S_h \subset H^1_0(\Omega) \) be a finite element subspace, \( S_\perp \subset H^1_0(\Omega) \) be the orthogonal complement of \( S_h \) with respect to the inner product \( \langle \nabla \cdot, \nabla \cdot \rangle \) of \( H^1_0 \), and \( \bar{u}_h \in S_h \) be an approximate solution by FEM. The candidate set is taken as

\[
U_h = \{ u_h \in S_h \mid \| \nabla(u_h - \bar{u}_h) \| \leq \beta \},
\]

\[
U_\perp = \{ u_\perp \in S_\perp \mid \| u_\perp \| \leq \beta \},
\]

\[
U = U_h + U_\perp = \{ u = u_h + u_\perp \mid u_h \in U_h, u_\perp \in U_\perp \},
\]

where \( \beta \) and \( \gamma \) are given positive constants.

Note that the following condition is a sufficient condition to \( FU \subset U \).

\[
\begin{cases}
P_h FU \subset U_h, \\
(I - P_h) FU \subset U_\perp,
\end{cases}
\]

where \( I \) is the identity mapping and \( P_h \) denotes the orthogonal projection \( H^1_0 \rightarrow S_h \) defined by

\[
(\nabla P_h z, \nabla v_h) = (\nabla z, \nabla v_h), \quad \forall v_h \in S_h.
\]

(2)

In order to check the first inclusion in the above condition, we usually transform it into a form using a Newton-like operator and derive a condition for \( \beta \) and \( \gamma \). See [3] for details.

We need estimation of the projection error to check the second inclusion. Let us put

\[
z = Fu, \quad u \in U.
\]

Recall that

\[
Fu = Kf(\cdot, u, \nabla u),
\]

then \( z \) is the solution of Poisson equation

\[
(\nabla z, \nabla v) = (f(\cdot, u, \nabla u), v), \quad \forall v \in H^1_0(\Omega).
\]

(3)

If the domain \( \Omega \) is bounded and convex, and have piecewise smooth boundary, the weak solution \( z \) of the Poisson equation satisfies

- \( z \in H^2(\Omega) \cap H^1_0(\Omega) \), and
- \( |z|_{H^2} \leq \| \Delta z \| = \| f(\cdot, u, \nabla u) \|
\]

from the well-known property of the Poisson equations. Here \( |\cdot|_{H^2} \) denotes the \( H^2 \)-seminorm.

When \( z \in H^2(\Omega) \cap H^1_0(\Omega) \), the projection error can be estimated as

\[
\| \nabla z - \nabla P_h z \| \leq C'_h \| \Delta z \|
\]

\[
= C'_h \| f(\cdot, u, \nabla u) \|,
\]

(3)

where \( C'_h \) is a positive constant independent of \( z \), but dependent on the mesh size \( h \).

If the domain \( \Omega \) is a nonconvex polygon, the solution \( z \) of Poisson equation no longer belongs to \( H^2(\Omega) \). In this case we have to estimate the projection error without \( H^2 \)-seminorms.

\[
\| \nabla z - \nabla P_h z \| \leq C_h \| \Delta z \|
\]

\[
= C_h \| f(\cdot, u, \nabla u) \|
\]

(4)

The details are shown in Section 4.
Once we obtain an explicit value of $\hat{C}_h = C'_h$ or $C_h$, we have
\[ \| \nabla (I - P_h) Fu \| \leq \hat{C}_h \| f(\cdot, u, \nabla u) \| \]
for an arbitrary $u \in U$. The right-hand side can be estimated using $x$ and $\beta$. If we can find an appropriate value of $x$ and $\beta$ such that
\[ \hat{C}_h \| f(\cdot, u, \nabla u) \| \leq \beta \] (5)
holds for any $u \in U$, then we have verified
\[ (I - P_h) Fu \subset U \perp. \]
In order to check (5) numerically, we need an explicit value of $\hat{C}_h$. And the value of $\hat{C}_h$ should decrease as $h$ goes to be small. This is our motivation to have developed the methods to estimate the constant $C'_h$ and $C_h$.

3. Constants in error estimation for convex polygonal domains

First we mention some existing results on estimation of the constants $C'_h$ in (3) for convex domains, because they give the base to the method in estimating $C_h$ for nonconvex domains.

Let $\Omega$ be a convex polygonal domain. Consider its triangulation $\mathcal{T}$ and the linear triangular element subspace $S_h \subset H^1_0(\Omega)$. Define $I_h u \in S_h$ by the interpolation of $u \in H^1_0(\Omega) \cap H^2(\Omega)$. From
\[ \| \nabla u - \nabla P_h u \| \leq \| \nabla u - \nabla I_h u \|, \]
it is sufficient to estimate $C'_h$ such that
\[ \| \nabla u - \nabla I_h u \| \leq C'_h |u|_{H^2} \]
holds. Note that
\[ C'_h \leq \max_{\tau \in \mathcal{T}} C_\tau, \] (6)
where $C_\tau$ is a constant in the following estimation on each element $\tau \in \mathcal{T}$.
\[ \| \nabla u - \nabla I_h u \|_{L^2(\tau)} \leq C_\tau |u|_{H^2(\tau)}. \] (7)
Here $\| \cdot \|_2$ means the $L^2$-norm in $L^2(\tau)$.

Let $a$, $b$ and $c$ be the edges of the triangle $\tau$ ($a, b \leq c$), and let $|\tau|$ denote the area of $\tau$. Natterer [6] gave a formula to estimate $C_\tau$ and a bound of a constant $\tilde{C}$ such that
\[ C_\tau = \tilde{C} \frac{1 + \sqrt{1 - d^2}}{\sqrt{1 - \sqrt{1 - d^2}}} h, \] (8)
where
\[ h = \sqrt{\frac{a^2 + b^2}{2}}, \quad d = \frac{2|\tau|}{h^2} \quad (0 < d \leq 1). \]

But one may hesitate to use his result that $\tilde{C} < 0.81$, because it is rather overestimated.

Arbenz [1] calculated an approximate value to $\tilde{C} \approx 0.4888$, which may be close to the optimal value. But when we need a mathematically rigorous bound for $\tilde{C}$, we cannot use his result of course. In this situation, we have to use the techniques which have been developed for numerical calculations with guaranteed accuracy. According to the results of [4] and [5], an upper bound of $\tilde{C}$ is given by
\[ \tilde{C} \leq 0.4939 \] (9)
with rigorous calculations considering the influence of truncation and rounding errors. On a lower bound, we have
\[ \tilde{C} \geq 0.488719 \]
which comes from rigorous calculations of two kinds of norms of an appropriate polynomial function [7].

4. The upper bounds of the constants for nonconvex polygonal domains

The above arguments are based on the fact that the projection error of the finite element approximation is less than the error of the interpolation, and that the interpolation error can be estimated by the $H^2$-seminorm of the function $u$. But in nonconvex domains, the solutions to Poisson equations do not necessarily belong to $H^2$, then we cannot use the arguments with respect to the $H^2$-seminorms. It is necessary to construct methods for estimation of the constant $C_h$ in (4) without the interpolations nor $H^2$-seminorms.

The constant $C_h$ is no longer of $O(h)$. The order depends on the largest angle of the vortices of the domain $\Omega$. This situation compels us to give up finding an explicit relation between $C_h$ and $h$, which can be found in case of convex domains. Instead, we calculate a rigorous upper bound of $C_h$ for each triangulation.

There are some results for the methods which give upper bounds of $C_h$ for given triangulations of nonconvex polygonal domains. The reader may refer to our paper [12] and its improved version [11]. Let us show the results of [11] without the proofs.

For a given triangulation of a nonconvex polygonal domain $\Omega$, we take a convex polygon $\Omega'$ which includes $\Omega$. Divide $\Omega'$ into triangles to get another triangulation such that the second triangulation is identical with the first triangulation a s a r i n. Hereafter we put $\Omega' = \Omega$. We define the finite element subspaces $S_h \subset H^1_0(\Omega)$ and $S'_h \subset H^1_0(\Omega')$ with linear triangular elements such that

\[ S_h \subset S'_h \]

holds. The projections

\[ P_h : H^1_0(\Omega) \rightarrow S_h, \]
\[ P'_h : H^1_0(\Omega') \rightarrow S'_h \]

are defined similarly to (2). We also define the constants $C_h$ and $C'_h$ similarly to (4) and (3) corresponding to $P_h$ and $P'_h$, respectively.

Note that an arbitrary $u \in H^1_0(\Omega)$ can be extended to a function $E[u] \in H^1_0(\Omega')$ as follows.

\[ E[u] = \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \Omega'. \end{cases} \]

To $E[u]$ we can apply the following lemma.

**Lemma 1.** Let $\hat{\Omega}$ be $\Omega$ or $\Omega'$ and $\hat{P}_h$ be the corresponding $H^1_0$-projection. We can take a constant $\hat{C}_h$ independent of $v \in H^1_0(\hat{\Omega})$ such that

\[ \| v - \hat{P}_h v \|_{\hat{\Omega}} \leq \hat{C}_h \| \nabla v - \nabla \hat{P}_h v \|_{\hat{\Omega}}, \] (10)

where the constant $\hat{C}_h$ is the same that appears in

\[ \| \nabla v - \nabla \hat{P}_h v \|_{\hat{\Omega}} \leq \hat{C}_h \| \Delta v \|_{\hat{\Omega}} \] (11)

which holds for any $v \in H^1_0(\hat{\Omega})$ providing that $\Delta v \in L^2(\hat{\Omega})$.

In case that $\hat{\Omega} = \Omega'$ and $\hat{C}_h = C'_h$, the constant $C'_h$ can be estimated by Natterer’s formula (6) and (8) since $\Omega'$ is a convex polygon.

Using Lemma 1, we have the following theorem.
**Theorem 1.** Define the set \( U \) by

\[
U = \{ u \in H^1_0(\Omega) \mid P_h^t E[u] \neq P_h u \text{ in } \Omega, \}
\]

and the quantity \( K_h \) by

\[
K_h = \sup_{u \in U} \frac{\| P_h^t E[u] - P_h u \|_\Omega}{\| P_h^t E[u] \|_{\Omega^*}}.
\]

Then the constant \( C_h \) for any \( u \in H_0^1(\Omega) \) in

\[
\| \nabla u - \nabla P_h u \|_\Omega \leq C_h \| \Delta u \|_\Omega
\]

is estimated as follows.

\[
C_h^2 \leq (C_h')^2 (1 + K_h^2).
\]

Note that the denominator of \( K_h \) does not take 0 because of the definition of the set \( U \) and the fact that \( P_h^t E[u] = 0 \) implies \( P_h u = 0 \). The proofs of Lemma 1 and Theorem 1 are given in [11].

As we have mentioned, the constant \( C_h \) can be estimated by the Natterer’s formula. What remains is to show how to calculate the value of \( K_h \).

Let \( A', B'_Q \) be \( n \times n \) matrices whose \( ij \) elements are defined by

\[
A'_{ij} = (\nabla \phi'_i, \nabla \phi'_j),
\]

\[
(B'_Q)_{ij} = (\phi'_i, \phi'_j)_\Omega,
\]

\[
(B'_Q)_{ij} = (\phi'_i, \phi'_j)_{\Omega^*},
\]

where \( \{ \phi'_i \}_{i=1}^n \) is a basis of the finite element subspace \( S'_h \). Note that

\[
\| P_h^t E[u] - P_h u \|_\Omega = \| P_h^t E[u - P_h u] \|_\Omega,
\]

\[
\| P_h^t E[u] \|_{\Omega^*} = \| P_h^t E[u - P_h u] \|_{\Omega^*}.
\]

Define the vector \( g \) by

\[
g_i = (\nabla \phi'_i, \nabla E[u - P_h u])_{\Omega^*}
\]

where \( g_i \) denotes the \( i \)th element of \( g \). Then we have

\[
\| P_h^t E[u - P_h u] \|_\Omega = g^T A'^{-1} B'_Q A'^{-1} g,
\]

\[
\| P_h^t E[u - P_h u] \|_{\Omega^*} = g^T A'^{-1} B'_Q A'^{-1} g.
\]

We take \( \phi'_{i_1}, \phi'_{i_2}, \ldots, \phi'_{i_q} \) as the base functions which correspond to the nodes on \( \Gamma^* \), the boundaries between \( \Omega \) and \( \Omega^* \) except \( \partial \Omega' \). Considering the properties of the projections and the inclusion \( S_h \subset S'_h \), we find that \( g_i = 0 \) for \( i \notin Q = \{i_1, i_2, \ldots, i_q \} \). This implies the following.

Let \((M)_Q\) denote a \( q \times q \) matrix whose \( kl \) element equals the \( i_k j_l \) element of the given matrix \( M \) for \( i_k, j_l \in Q \). Then we have

\[
K_h^2 = \sup_{z \in R^q, z \neq 0} \frac{z^T (A'^{-1} B'_Q A'^{-1})_Q z}{z^T (A'^{-1} B'_Q A'^{-1})_Q z}.
\]

Note that the manner of constructing the matrix \((A'^{-1} B'_Q A'^{-1})_Q\) makes it positive definite, which corresponds to the definition of the set \( U \). Therefore \( K_h^2 \) can be calculated as the largest eigenvalue of the following problem.

\[
(A'^{-1} B'_Q A'^{-1})_Q z = \lambda (A'^{-1} B'_Q A'^{-1})_Q z.
\]

The reader may refer to [11] for more details.
5. Theoretical order of the constants

We can hardly find works which give some explicit bounds for the constants $C_h$ for nonconvex polygonal domains except the works by ourselves. But concerning the order of $C_h$ with respect to $h$, there are theoretical results [2,9].

Let $\omega$ be the largest angle of the boundary $\partial \Omega$, and define $\gamma$ as

$$\gamma = \frac{\pi}{\omega}.$$

In the following picture, $\theta$ equals $2\pi - \omega$. See Fig. 1.

Around the reentrant corner, the solution $z$ of Poisson equation (1) with $g$ smooth basically has the following form close to the corner [2].

$$z(r, \theta) = r^\gamma \alpha(\theta) + \beta(r, \theta),$$

where $\alpha$ and $\beta$ are smooth and polar coordinates $(r, \theta)$ with the pole at the corner are used.

This means $z \in H^{\gamma+1-\epsilon}$ for any $\epsilon > 0$. Let $\tau$ be any element adjoins the reentrant corner, and $h_\tau$ be the mesh size of $\tau$. We have

$$\|\nabla z - \nabla P_h z\|_\Omega \leq \hat{C}_h h_\tau^{\gamma-\epsilon} |z|_{H^{\gamma+1-\epsilon}(\tau)}$$

with a constant $\hat{C}_h$. Here again $P_h z$ denotes the interpolation of $z$ by functions in $S_h$. By similar arguments to the cases of convex polygonal domains, the following estimation can be obtained with respect to the $H^1_0$ projection $P_h$.

$$\|\nabla z - \nabla P_h z\|_\Omega \leq \sum_v \|\nabla z - \nabla P_h z\|_v + \sum_\tau \|\nabla z - \nabla P_h z\|_\tau$$

$$\leq \sum_v C_v h_v |z|_{H^2(v)} + \sum_\tau \hat{C}_h h_\tau^{\gamma-\epsilon} |z|_{H^{\gamma+1-\epsilon}(\tau)},$$

where $\sum_\tau$ takes the summation over the elements which adjoin the reentrant corner, and $\sum_v$ takes the summation over the other elements.

For a uniform triangulation, we have

$$\|\nabla z - \nabla P_h z\|_\Omega \leq \hat{C} h^{\gamma-\epsilon}$$ (14)

where the constant $\hat{C}$ depends on $|z|_{H^{\gamma+1-\epsilon}(\Omega)}$.

Therefore if we refine the uniform triangulation successively by the midpoint division, then we might have the constants $C_h$ of almost $O(h^\gamma)$.

And if we can take a non-uniform mesh in which the size of $h_v$ is chosen small where $|z|_{H^2(v)}$ is large, in particular the size of $h_\tau$ is taken very small, then we might have the constants $C_h$ of almost $O(h)$. This is what Johnson [2] discusses, and he recommends to construct the mesh as follows. Let

$h_k$: the longest edge of an element $k$ at a distance $d_k$ from the reentrant corner, where $d_k < d$. 
\( h \): the longest edge of an element enough away from the reentrant corner at a distance \( d \), and define the mesh in order that
\[
h_k = h \left( \frac{d_k}{d} \right)^{1-\gamma},
\]
holds. We show examples of such meshes in the next section.

6. The order of the constants in numerical experiments

We will compare the theoretical order with the order obtained by numerical experiments. All the calculations were carried out using Intlab [8]. In order to calculate \( K_h \), we used a method for verified computation of generalized eigenvalue problems which were developed by Yamamoto [10].

The domain is taken L-shaped, which gives \( \gamma = \frac{2}{3} \). Firstly we show experiments for uniform meshes, and secondly for non-uniform meshes.

6.1. Uniform meshes

In order to estimate the order with respect to \( h \), refinement of meshes are taken by midpoint division. We show the initial triangulation. See Fig. 2.

Here the domain \( \Omega' \) is taken as \((0, 1) \times (0, 1)\).

The results are as follows.

\[
\begin{array}{ccc}
  h & C'_h & C_h \\
  0.2500000000 & 0.1234750000 & 0.3653665286 \\
  0.1250000000 & 0.0617375000 & 0.2250096551 \\
  0.0625000000 & 0.0308687500 & 0.1401721175 \\
  0.0031250000 & 0.0154343750 & 0.0878737431 \\
\end{array}
\]

The order of \( C_h \) with respect to \( h \) is 0.68503. On the other hand, the theoretical order is
\[
\frac{2}{3} = 0.666666 \cdots,
\]
thus it can be said that they agree well. For other domains, the experiments also show the agreement with the theoretical orders [11].

6.2. Non-uniform mesh

We adopt a series of non-uniform meshes. Three figures of different mesh size are shown. See Fig. 3.

The results are as follows.

\[
\begin{array}{cccc}
  h & C_h & h \text{ (uniform)} & C_h \text{ (uniform)} \\
  3.341e-01 & 1.0616e+00 & 3.333e-01 & 4.4894e-01 \\
  2.034e-01 & 8.5566e+00 & 2.000e-01 & 3.1207e-01 \\
  1.473e-01 & 7.3799e-01 & 1.429e-01 & 2.4677e-01 \\
  1.153e-01 & 6.5874e-01 & 1.111e-01 & 2.0749e-01 \\
  8.042e-02 & 5.5568e-01 & 8.333e-02 & 1.7042e-01 \\
  6.984e-02 & 5.1953e-01 & 7.143e-02 & 1.5346e-01 \\
  6.172e-02 & 4.8966e-01 & 6.250e-02 & 1.4017e-01 \\
  5.529e-02 & 4.6442e-01 & 5.556e-02 & 1.2944e-01 \\
  5.007e-02 & 4.4273e-01 & 5.000e-02 & 1.2055e-01 \\
\end{array}
\]
Fig. 2. Initial triangulation.

Fig. 3. Non-uniform mesh refinement.

Fig. 4. log $C_h$ against log $h$. 

$y = 0.46309x + 0.57752$
We trace the graph of $\log C_h$ against $\log h$ (the blue line). The red line is the result of linear fitting. See Fig. 4. The order of $C_h$ with respect to $h$ is 0.46309. Moreover, the values of $C_h$ are very larger than the corresponding $C_h$’s for the uniform meshes.

The following is the graph of $\log C_h$ against $\log 1/\sqrt{\text{the number of elements}}$. See Fig. 5.

The order of $C_h$ is similar. These results implies that our non-uniform meshes do not improve the values and the order of $C_h$, which seems to show a difference from what Johnson says.

One possible reason would be that the transformation of the shapes between the uniform elements and the non-uniform elements causes the difference through the Natterer’s formular (8).

In order to check whether it is true, we show the values of $h \times \sqrt{1 + K_h^2}$, which is independent of the Natterer’s formular.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$h \sqrt{1 + K_h^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.341e - 01$</td>
<td>1.1276e+00</td>
</tr>
<tr>
<td>$2.034e - 01$</td>
<td>8.8849e - 01</td>
</tr>
<tr>
<td>$1.473e - 01$</td>
<td>7.6373e - 01</td>
</tr>
<tr>
<td>$1.153e - 01$</td>
<td>6.8010e - 01</td>
</tr>
<tr>
<td>$9.477e - 02$</td>
<td>6.1909e - 01</td>
</tr>
<tr>
<td>$8.042e - 02$</td>
<td>5.7206e - 01</td>
</tr>
<tr>
<td>$6.984e - 02$</td>
<td>5.3437e - 01</td>
</tr>
<tr>
<td>$6.172e - 02$</td>
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<td>$5.529e - 02$</td>
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</tr>
<tr>
<td>$5.007e - 02$</td>
<td>4.5459e - 01</td>
</tr>
</tbody>
</table>

The log–log graph of $h$ and $h \sqrt{1 + K_h^2}$ indicates that the order is still worse than the uniform meshes. See Fig. 6.

At the present time, it is not clear why there seems to be the difference between our results and Johnson’s theoretical discussion. Our list of other possible reasons is as follows.

1. The constant $\hat{C}$ in the right-hand side of the theoretical estimation (14) depends on $|z|_{H^{2+1-\epsilon}(\Omega)}$. On the other hand, we intend to have an estimation bounded by $\|\Delta z\|_{\Omega}$. Note that $z \notin H^2(\Omega)$. The relation between $|z|_{H^{2+1-\epsilon}(\Omega)}$ and $\|\Delta z\|_{\Omega}$ may give the gap of the order with respect to $h$. 

![Fig. 5. log $C_h$ against log $1/\sqrt{\text{the number of elements}}$.](image-url)
2. The estimation (14) should hold for a fixed \( z \). On the other hand, our estimation (12) allows the ‘worst’ \( z \) which maximizing \( K_h \) for each triangulation. This may cause the gap.

3. Our method may not suit well to non-uniform triangulations. The form (12) may give a considerable overestimation for non-uniform meshes because of some reason.

It should be our future work to solve these conjectures.

References