Right 2-Engel Elements and Commuting Automorphisms of Groups

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It is shown that there is a close connection between the right 2-Engel elements of a group and the set of the so-called commuting automorphisms of the group. As a consequence, the following general theorem is proved: If $G$ is a group and if $R(G)$ denotes the subgroup of right 2-Engel elements, then the factor group $R(G) \cap C_G(G')/Z_2(G)$ is a group of exponent at most 2.

Key Words: commuting automorphisms; central automorphisms; right 2-Engel elements

INTRODUCTION

Let $G$ be a group and consider the set $R_2(G) = \{g \in G \mid [g, x, x] = 1$ for all $x \in G\}$ of right 2-Engel elements of $G$. It is well known, see Kappe [5], that $R_2(G)$ is a characteristic subgroup of $G$. It is also known, see Heineken [3], that the inverse of a right 2-Engel element is a left 3-Engel element.

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In this paper we are interested in the structure of the subgroup $R_2(G)$ for centerless groups $G$. Results from [1] and [2] imply that if $G$ is a finite group with trivial center, then $R_2(G) = 1$. One would expect that whenever $G$ is a group with trivial center $R_2(G)$ is either trivial, or has a structure which is hard to control; indeed, the structure of $R_2(G)$ is known to be under control provided that some rather strong conditions are imposed on $G$, as, for example, the condition that $G$ satisfies the maximum condition on subgroups—see, for example, Baer [1] or the discussion in [9, Sect. 12.3]. In a certain sense, centerless groups are as far from being nilpotent as one could get; recent results, see [7], show that even in nilpotent groups there may exist 5-Engel elements none of whose powers is a left 5-Engel element. The following result comes then as a surprise; it asserts that all groups possess a certain canonic section which has a very special structure:

**Theorem 1.1.** Let $G$ be a group. Then $R_2(G) \cap C_G(G')/Z_2(G)$ is a group of exponent at most 2. In particular, if $Z(G) = 1$, then $R_2(G)$ is a subgroup of $G$ of exponent at most 2.

The above theorem is a consequence of results related to the so-called commuting automorphisms of groups. An automorphism $\alpha$ of a group $G$ is called a commuting automorphism if $g\alpha(g) = \alpha(g)g$ for all $g \in G$. The set of all commuting automorphisms of $G$ is denoted here by $\mathcal{A}(G)$ and it is clear that set $\mathcal{A}(G)$ contains the group $Aut_c(G)$ of central automorphisms of $G$. The converse inclusion does not hold in general; in fact, see [2], there exist finite nonabelian 2-groups $G$ such that $\mathcal{A}(G)$ is not a subgroup of $Aut(G)$. Previous results suggest, however, the following:

**Conjecture.** If $G$ is a group and if $Aut_c(G) = \{id_G\}$, then $\mathcal{A}(G) = \{id_G\}$.

This conjecture was shown to be true in the following particular cases: $G$ is a simple nonabelian group (I. N. Herstein [4]), or $G$ has no nontrivial abelian normal subgroups (T. J. Laffey [6]), or $G = G'$ and $Z(G) = 1$ (M. Pettet [8]), or $G$ is finite (Deaconescu et al. [2]).

The second main result of this paper shows that the above conjecture is false:

**Theorem 1.2.** There exist infinite groups $G$ such that $Z(G) = 1$ and $\mathcal{A}(G) \neq \{id_G\}$.

More information about the groups whose existence is ensured by Theorem 1.2 is contained in

**Theorem 1.3.** Let $G$ be a group such that $Z(G) = 1$ and $\mathcal{A}(G) \neq \{id_G\}$.

(i) $\mathcal{A}(G)$ is a subgroup of $Aut(G)$ and $exp(\mathcal{A}(G)) = 2$.

(ii) $1 < R_2(G) \leq C_G(G')$ and $exp(R_2(G)) = 2$. 
(iii) If $\alpha \in \mathcal{A}(G)$, then $\alpha$ fixes $G^2R_2(G)$, where $G^2$ is the subgroup of $G$ generated by the set of squares of elements in $G$.

The above result has an immediate consequence which shows that if a group $G$ has no nontrivial central automorphisms, then the structure of the set $\mathcal{A}(G)$ is very simple:

**Corollary 1.4.** If $G$ is a group such that $\text{Aut}_c(G) = \{id_G\}$, then $\mathcal{A}(G)$ is a subgroup of $\text{Aut}(G)$ of exponent at most 2.

It is well known that if $G$ is a group with trivial center, then the center of $\text{Aut}(G)$ is trivial too. Our last result shows that the same property holds for $R_2(G)$:

**Theorem 1.5.** If $G$ is a group such that $R_2(G)$ is trivial, then $R_2(\text{Aut}(G))$ is trivial.

### PRELIMINARY LEMMAS

The first lemma is due to T. J. Laffey [6] and is very useful in computations:

**Lemma 2.1.** Let $\alpha \in \mathcal{A}(G)$. Then $[\alpha(x), y] = [x, \alpha(y)]$ for all $x, y \in G$.

The next result bridges together the right 2-Engel elements and the commuting automorphisms of a group:

**Lemma 2.2.** (i) If $\alpha \in \mathcal{A}(G)$ and if $x \in G$, then $x^{-1}\alpha(x) \in R_2(G)$.

(ii) Let $g \in G$ and denote by $T_g$ the inner automorphism induced by $g$. Then $T_g \in \mathcal{A}(G)$ if and only if $g \in R_2(G)$. In particular, $\text{Inn}(G) \cap \mathcal{A}(G)$ is isomorphic to $R_2(G)/Z(G)$.

**Proof.** The proofs of both assertions are elementary and can be found in [2].

The following properties of commuting automorphisms are essential for what follows:

**Lemma 2.3.** Let $\alpha \in \mathcal{A}(G)$.

(i) If $x \in G'$, then $x^{-1}\alpha(x) \in Z(G)$.

(ii) $G' \subseteq C_G(\alpha)$ if and only if $\alpha^2 \in \text{Aut}_c(G)$.

**Proof.** See [2].
The following result provides a criterion for \( \mathcal{A}(G) \) to be trivial:

**Lemma 2.4.** Let \( G \) be a group with no nontrivial central automorphisms. Then \( \mathcal{A}(G) = \{ \text{id}_G \} \) if and only if \( R_2(G) = Z(G) \). In particular, if \( G = G' \), then \( R_2(G) = Z(G) \).

**Proof.** If \( \mathcal{A}(G) = \{ \text{id}_G \} \), then \( R_2(G) = Z(G) \) by Lemma 2.2(ii). If \( R_2(G) = Z(G) \), it follows from Lemma 2.2(i) that \( \mathcal{A}(G) \subseteq \text{Aut}_c(G) \), whence \( \mathcal{A}(G) = \{ \text{id}_G \} \).

To prove the last assertion assume that \( G = G' \). Then \( \text{Aut}_c(G) \) is trivial since central automorphisms of \( G \) fix \( G' \) elementwise. Lemma 2.3(i) implies that \( \mathcal{A}(G) \) is trivial. Thus \( R_2(G) = Z(G) \) by the above paragraph.

**Proofs of the Main Results**

**Proof of Theorem 1.2.** We will construct an infinite group \( G \) with trivial center and such that \( R_2(G) \) is nontrivial. Since \( Z(G) = 1 \), it follows at once that \( \text{Aut}_c(G) \) is trivial; the fact that \( R_2(G) \) is nontrivial implies, via Lemma 2.4, that \( \mathcal{A}(G) \) is not trivial.

Let \( H \) denote the direct sum (restricted direct product) of copies of the cyclic group of order two, indexed over the set of positive integers. Let \( G \) denote the standard restricted wreath product of \( Z \) by \( H \). That is, \( G \) is the semidirect product of the base group \( B \) which is the direct sum of \( |H| \) copies of \( Z \) by \( H \) by using the regular representation of \( H \) to define the action of \( H \) on \( B \). In this representation each nontrivial element of \( H \) acts as a formal product of disjoint 2-cycles which fixes no point in \( B \). Also, \( B \) has a \( Z \)-basis (corresponding to the elements of \( H \)) on which \( H \) acts transitively.

It is known that any standard wreath product of an abelian group by an infinite group is centerless, thus \( Z(G) \) is trivial.

Let now \( x = (1, 0, 0, \ldots) \in B \) and let \( y = bh \) be an arbitrary element of \( G \) with \( b \in B \) and \( h \in H \). Assuming that \( h \) contains the 2-cycle \( (1, i) \), then \( x^y = x^h = (0, 0, \ldots, 1, 0, 0, \ldots) \) because 1 and 0 are interchanged and all other interchanges are between zeros. Then \( [x, y] = (1, 0, 0, 0, \ldots, 1, 0, 0, \ldots) \) and \( y \) commutes with \([x, y]\) since \( b \) does and since \( h \) will interchange the two 1's and a set of zeros. It follows that \((1, 0, 0, \ldots)\) is a right 2-Engel element, as are all tuples with only one nonzero component. But these elements generate \( B \). Thus \( B \) is contained in \( R_2(G) \) and since \( R_2(G) \) is abelian it follows that \( B = R_2(G) \). Thus here \( R_2(G) \) is an infinite group of exponent 2 and this completes the proof of Theorem 2.1.

**Remark.** The above considerations show that all the elements with exactly two 1's are in \( G' \). These elements generate the subgroup of \( B \) of elements which have an even number of 1's. That is clearly a normal
subgroup of $G$ with abelian quotient. So $G'$ is this subgroup and therefore $G'$ centralizes $B = R_3(G)$. This last property of our example is not accidental; it will turn out that $R_3(G)$ centralizes $G'$ in every group $G$ with trivial center.

**Proof of Theorem 1.3.** Since $Z(G) = 1$, it follows at once that $Aut(G)$ is trivial. By hypothesis, $A(G)$ is nontrivial, so by Lemma 2.4, $R_3(G) \neq 1$.

Fix a nontrivial automorphism $\alpha \in A(G)$. Since $Z(G) = 1$, Lemma 2.3$i$ implies that $\alpha$ fixes $G'$ and by Lemma 2.3$ii$ one concludes that $\alpha^2 = id_G$. This shows that $exp(A(G)) = 2$.

For $x \in G$, we have that $x^{-1} \alpha(x) \in R_3(G)$ by Lemma 2.2(i). Now since $T_{x^{-1} \alpha(x)} \in A(G)$ by Lemma 2.2$(ii)$, one obtains that the order of this inner automorphism is at most 2. Hence $(x^{-1} \alpha(x))^2 = 1$ and since $x \alpha(x) = \alpha(x)x$ it follows that $\alpha(x^2) = x^2$. Thus the subgroup $G^2 = \langle \{g^2 \mid g \in G \} \rangle$ is fixed by $\alpha$.

Take now $r \in R_3(G)$, so that $T_r \in A(G)$ by Lemma 2.2$(ii)$ and $T_r^2 = id_G$. Then $r^2 \in Z(G) = 1$ and we obtain that $exp(R_3(G)) = 2$. Moreover, Lemma 2.3$(ii)$ implies that $T_r$ fixes $G'$, whence $r \in C_G(G')$. Hence $R_3(G) \leq C_G(G')$ and the proof of part $(ii)$ of the theorem is complete.

To finish the proof of part $(i)$, note that $A(G)$ is closed with respect to taking inverses. Let $\alpha, \beta \in A(G)$ and let $x \in G$. Then, since $x^{-1} \alpha(x), x^{-1} \beta(x) \in R_3(G)$ and since $R_3(G)$ is abelian, we derive, using Lemma 2.1, that $1 = [x^{-1} \alpha(x), x^{-1} \beta(x)] = [\alpha(x), \beta(x)] = [x, \alpha \beta(x)]$, which shows that $\alpha \beta \in A(G)$. Thus $A(G)$ is a subgroup of $Aut(G)$, which proves $(i)$.

To complete the proof of part $(iii)$, we only need to show that every $\alpha \in A(G)$ fixes $R_3(G)$. Take $r \in R_3(G)$, so that the inner automorphism $T_r \in A(G)$. Since by part $(i)$, $A(G)$ is abelian, it follows that $[T_r, \alpha] = id_G$. Since $Z(G) = 1$, a short calculation shows that $\alpha(r) = r$.

**Proof of Theorem 1.1.** Let $r \in R_3(G) \cap C_G(G')$. Then the inner automorphism $T_r$ is a commuting automorphism of $G$ which fixes $G'$, so by Lemma 2.3$(ii)$, $T_r^2 \in Aut(G) \cap Inn(G)$. But then, since $Aut(G) \cap Inn(G) = Z(Inn(G)) \cong Z_3(G)/Z(G)$, it follows at once that $r^2 \in Z_3(G)$.

This proves the first assertion of Theorem 1.1.

If $Z(G) = 1$, then $Z_3(G) = 1$ and by Theorem 1.3$(ii)$, $R_3(G) \leq C_G(G')$.

Thus $R_3(G) \cong R_3(G) \cap C_G(G')/Z(G)$ is a group of exponent at most 2 by the first part of the theorem. The proof is now complete.

**Proof of Corollary 1.4.** If $G$ has no nontrivial central automorphisms, then necessarily $Z_3(G) = Z(G)$. Set $\overline{G} = G/Z(G)$ and observe that every $\alpha \in A(G)$ induces an automorphism $\overline{\alpha} \in A(\overline{G})$. But since $\overline{G}$ has trivial center, it follows from Theorem 1.3$(i)$ that $\overline{\alpha}^2 = id_{\overline{G}}$. Now the function $\varphi : Aut(G) \rightarrow Aut(\overline{G})$, defined by $\varphi(\alpha) = \overline{\alpha}$, is one-by-one since $\text{Ker} \varphi = Aut(G) = \{id_G\}$. Thus $A(G)$ is embedded into $A(\overline{G})$ and the result follows from Theorem 1.3$(i)$.
Proof of Theorem 1.5. We claim first that if $G$ is a group with $Z(G) = 1$, then $R_2(Aut(G)) \subseteq \mathcal{A}(G)$. Indeed, let $\alpha \in R_2(Aut(G))$. Then $[\alpha, T, T] = id_G$ for every $T \in Inn(G)$. Since $Z(G) = 1$, a short computation shows that $[\alpha, x, x] = 1$ for all $x \in G$. This means that $\alpha \in \mathcal{A}(G)$ and proves the claim.

To prove the theorem, note that if $R_2(G) = 1$ then $\mathcal{A}(G) = \{id_G\}$ by Lemma 2.4. But then $R_2(Aut(G)) = \{id_G\}$ by the above claim since obviously $Z(G) = 1$.

CONCLUDING REMARKS

(1) If $G$ is a group, if $\alpha$ is a nontrivial automorphism in $\mathcal{A}(G)$, and if we form the semidirect product $H = [G] \langle \alpha \rangle$, then it can be shown that $\alpha \in R_2(H)$.

(2) It is easy to prove that if $G$ is a finite group and if $\mathcal{A}(G)$ is trivial, then $\mathcal{A}(Aut(G))$ is also trivial. Indeed, since $\mathcal{A}(G)$ is trivial, it follows that $Aut(G)$ is trivial and since $Aut(G)$ is the centralizer of $Inn(G)$ in $Aut(G)$ one obtains that $Z(Aut(G))$ is trivial. By a result in [2] which applies to finite groups, one infers that $\mathcal{A}(Aut(G))$ is trivial. We were not able to prove the corresponding result for arbitrary groups $G$.

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REFERENCES

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