Periodic solutions for equation \( \dot{x} = A(t)x^m + B(t)x^n + C(t)x^l \) with \( A(t) \) and \( B(t) \) changing signs

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**Abstract**

In this paper, we investigate the differential equation \( \dot{x} = S(x,t) = A(t)x^m + B(t)x^n + C(t)x^l \), where \( A, B, C \in C^\infty([0,1]) \), \( m > n > l \) and \( m, n, l \in \mathbb{Z}^+ \). A solution \( x(t) \) with \( x(1) = x(0) \) is called a periodic solution. Under some hypotheses which admit \( A(t) \) and \( B(t) \) without fixed sign, we obtain the upper bound (sometimes sharp) for the number of isolated periodic solutions of the equation. Applying these results for the Abel equation (i.e. \( m = 3, n = 2, l = 1 \)), we get that if there exists \( \lambda \neq 0 \) such that \( S(\lambda, t) \cdot C(t) \cdot \lambda < 0 \) (resp. \( S(\lambda, t) \cdot (A(t)\lambda + B(t)) < 0 \)), then the equation has at most 2 (resp. 4) non-zero isolated periodic solutions. Furthermore, suppose that \( \gamma = (a(t), t) \) is a smooth curve which lies in \((\mathbb{R} \setminus \{0\}) \times [0,1]\) with \( a(0) = a(1) \). We obtain that if vector fields \( (S(x, t), 1) \) (resp. \( \dot{\gamma} \)) and \( (C(t)x, 1) \) are transverse to \( \dot{\gamma} \) (resp. \( S(x, t), 1) \) on \( \gamma \) in opposite directions, then the number of non-zero isolated periodic solutions of this Abel equation is still no more than 2 (resp. 4). These conclusions generalize the known criteria about the Abel equation which only refer to the cases with either \( A(t) \) or \( B(t) \) keeping sign. Finally, as an application we study a kind of trigonometrical Abel equation.

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1. Introduction and statements of main results

Consider a non-autonomous differential equation

\[ \dot{x} = \frac{dx}{dt} = S(x, t), \]  

(1)

where \( S(x, t) \in C^\infty(\mathbb{R} \times [0, 1]) \). A solution \( x(t) \) of (1) which is defined in \([0, 1]\) with \( x(0) = x(1) \), is called a **periodic solution**. Observe that if \( S(x, t) \) is 1-periodic in \( t \), then (1) is an equation defined on a cylinder, and the **periodic solutions** are actual periodic solutions of this equation.

One of the most important problems for (1) is to control its number of isolated periodic solutions, because the equation can be used to analyze the limit cycles of planar differential systems (see Devlin et al. [7], Lins-Neto [12] and Lloyd [15]). Unfortunately, even the simplest case

\[ S(x, t) = \sum_{i=0}^{m} a_i(t)x^i, \]

is not completely understood yet. This case is firstly studied and motivated by Lins-Neto [12] and Lloyd [13–15]. Both authors prove that (1) has at most one (resp. two) isolated periodic solutions if \( m = 1 \) (resp. \( m = 2 \)). However, an unexpected result shown in [12] is that the number of isolated periodic solutions is not bounded for (1) with \( m = 3 \) (see also Panov [17]). Furthermore, such result is easily extended for the equation with \( m > 3 \) (see Gasull and Guillamon [9]). Therefore, people's attention is naturally focused on finding the condition of coefficients \( a_i(t) \), which can bound the number of isolated periodic solutions.

When \( m = 3 \), (1) is called Abel equation. It is well known that if \( a_3(t) \) does not change sign, then (1) has at most three isolated periodic solutions (see Gasull and Llibre [8], Lins-Neto [12], Lloyd [15] and Pliss [18]). Also for the case that \( a_0(t) \equiv 0 \) and \( a_2(t) \) keeps the sign, the authors in paper [8] prove that the number of isolated periodic solutions of (1) is still no more than three. Another notable result is due to Ilyashenko [10]. When \( m > 3 \) and \( a_m(t) \equiv 1 \), he give an upper bound for the number of isolated periodic solutions in terms of the bounds of \( |a_i(t)|, i = 0, \ldots, m - 1 \). Some generalized Abel equations with either \( a_m(t) \) or \( a_n(t) \) keeping sign are also investigated by several authors, where \( a_i(t) \equiv 0 \) for \( n < i < m \) (see Gasull and Guillamon [9], Panov [16]).

The results presented above are mainly based on the fixed sign hypothesis for one of the first two non-zero coefficients. Recently, a motivated work on Abel equation is given by Álvarez, Gasull and Giacomini [1]. They prove that if \( S(x, t) = a_3(t)x^3 + a_2(t)x^2 \) with \( a \cdot a_3(t) + b \cdot a_2(t) \neq 0 \) (i.e. \( a \cdot a_3(t) + b \cdot a_2(t) \) does not change sign), \( a, b \in \mathbb{R} \), then (1) has at most one non-zero isolated periodic solution. This is a new criterion of getting the upper bound for the number of isolated periodic solutions of Abel equation. Bravo, Fernández and Gasull in [3] study a generalized Abel equation

\[ \dot{x} = S(x, t) = a_m(t)x^m + a_n(t)x^n + a_1(t)x, \]

where \( a_m(t), a_n(t) \) and \( a_1(t) \) can change signs. Under some symmetric hypotheses they also obtain the upper bound.

For more works on the periodic solution of Abel equation, see [2,4–6] and [11], etc.

The purpose of this paper is to investigate the following equation:
\[ \dot{x} = \frac{dx}{dt} = S(x, t) = A(t)x^m + B(t)x^n + C(t)x^l, \]  

where \( A(t), B(t), C(t) \in C^\infty([0, 1]), m > n > l \) and \( m, n, l \in \mathbb{Z}^+ \) (we write \( A, B, C \) instead of \( a_m, a_n, a_l \) defined above for convenience). Clearly, (2) is a kind of generalized Abel equation. And the equations studied in [1] and [8] are the particular cases of (2).

Our theorems are mainly extend the results of Álvarez, Gasull and Giacomini [1]. More precisely, under the following hypotheses for (2), we give some criterions to control the upper bound for the number of periodic solutions.

(H.1) There exists \( \lambda \neq 0 \) such that \( A(t)x^{m-n} + B(t) \neq 0 \), and \( C(t) \geq 0 (\leq 0) \).

(H.2) There exists \( \lambda \neq 0 \) such that \( S((\pm 1)^{m-n+1}\lambda, t) \neq 0 \).

**Remark 1.1.** Since

\[
\frac{S((\pm 1)^{m-n+1}\lambda, t)}{(\pm 1)^{(m-n+1)(n-l)}} = (\pm 1)^{(m-n+1)(n-l)}(A(t)x^{m-n} + B(t))^{λ-n-l} + C(t),
\]

(H.2) is equivalent to (H.1) when \( C(t) \equiv 0 \), and it is the main condition used in [1].

**Theorem 1.2.** Suppose that (2) satisfies Hypothesis (H.1). The following statements hold.

(i) Assume that both \( m - l \) and \( m - n \) are even. If

\[
|\left(A(t)x^{m-n} + B(t)\right)^{\lambda-n-l}| > |C(t)|, \quad \left(A(t)x^{m-n} + B(t)\right) \cdot C(t) \leq 0,
\]

then (2) has at most 4 non-zero isolated periodic solutions. This upper bound is sharp.

(ii) Assume that \( m - l \) is even and \( m - n \) is odd. If

\[
|\left(A(t)x^{m-n} + B(t)\right)^{\lambda-n-l}| > |C(t)|, \quad \left(A(t)x^{m-n} + B(t)\right) \cdot C(t) \cdot \lambda \leq 0,
\]

then (2) has at most 2 non-zero isolated periodic solutions. This upper bound is sharp.

(iii) Assume that \( m - l \) is odd and \( m - n \) is even. If

\[
\left|\left(A(t)x^{m-n} + B(t)\right)^{\lambda-n-l}\right| > \left|C(t)\right|, \quad \left(A(t)x^{m-n} + B(t)\right) \cdot \int_0^1 B(t) \, dt \geq 0,
\]

then (2) has at most 3 non-zero isolated periodic solutions. This upper bound is sharp.

(iv) Assume that \( m - l \) and \( m - n \) are odd. If

\[
\left(A(t)x^{m-n} + B(t)\right) \cdot C(t) \geq 0, \quad \left(A(t)x^{m-n} + B(t)\right) \cdot \lambda \cdot \int_0^1 A(t) \, dt \leq 0,
\]

then (2) has at most 1 non-zero isolated periodic solution. This upper bound is sharp.

Note that the assumption in each statement of Theorem 1.2 actually implies Hypothesis (H.2). In particular, when \( l = 1 \) in (2), the number of periodic solutions can be bounded by some weaker conditions. See the theorem below.
Theorem 1.3. Suppose that (2) satisfies Hypotheses (H.1) and (H.2) for the same \( \lambda \), and \( l = 1 \). The following statements hold.

(i) Assume that both \( m - 1 \) and \( m - n \) are even.
   (i.a) (2) has at most 8 non-zero isolated periodic solutions when \( |(A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| < |C(t)| \) and \( (A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot C(t) < 0 \).
   (i.b) (2) has at most 4 non-zero isolated periodic solutions except the case in (i.a). This upper bound is sharp.

(ii) Assume that \( m - 1 \) is even and \( m - n \) is odd.
   (ii.a) (2) has at most 4 non-zero isolated periodic solutions when \( (A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot C(t) \leq 0 \).
   (ii.b) (2) has at most 2 non-zero isolated periodic solutions when \( (A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot C(t) \geq 0 \) and \( (A(t)\lambda^{m-n} + B(t)) \int_0^1 B(t) \, dt \leq 0 \). This upper bound is sharp.

(iii) Assume that \( m - 1 \) is odd and \( m - n \) is even.
   (iii.a) (2) has at most 3 non-zero isolated periodic solutions when \( |(A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| > |C(t)| \). This upper bound is sharp.

(iv) Assume that both \( m - 1 \) and \( m - n \) are odd.
   (iv.a) (2) has at most 3 non-zero isolated periodic solutions when \( (A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot C(t) \geq 0 \). This upper bound is sharp.
   (iv.b) (2) has at most 2 non-zero isolated periodic solutions when \( (A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot C(t) \leq 0 \) and \( (A(t)\lambda^{m-n} + B(t)) \int_0^1 B(t) \, dt \leq 0 \).

When \( l = 1 \) and \( m - n \) is even, Theorem 1.3 in fact tells us that (2) has a bounded number of isolated periodic solutions under Hypotheses (H.1) and (H.2).

Clearly, Abel equation is of the type that \( m - 1 \) is even and \( m - n \) is odd. Hence statement (ii) in Theorem 1.2 and statement (ii) in Theorem 1.3 are results for estimating the number of the periodic solutions of Abel equation. A direct conclusion is given below.

Corollary 1.4. Suppose that (2) is an Abel equation (i.e. \( m = 3, n = 2, l = 1 \)).

(i) If there exists \( \lambda \neq 0 \) such that \( S(\lambda, t) \cdot C(t) \cdot \lambda < 0 \) (i.e. \( (A(t)\lambda^2 + B(t)\lambda + \lambda C(t)) \cdot C(t) < 0 \)), then (2) has at most 2 non-zero isolated periodic solutions. This upper bound is sharp.

(ii) If there exists \( \lambda \neq 0 \) such that \( S(\lambda, t) \cdot (A(t)\lambda + B(t)) < 0 \) (i.e. \( A(t)\lambda^2 + B(t)\lambda + C(t)) \cdot (A(t)\lambda^2 + B(t)\lambda) < 0 \)), then (2) has at most 4 non-zero isolated periodic solutions.

Furthermore, in Section 4 we consider the vector field \((S(x, t), 1)\) induced by (2), and show that Hypotheses (H.1) and (H.2) imply some transversality conditions. These lead us to get the next corollary.

Corollary 1.5. Suppose that (2) is an Abel equation, and \( \gamma = (a(t), t) \) is a smooth curve which lies in \( (\mathbb{R} \setminus \{0\}) \times [0, 1] \) with \( a(0) = a(1) \).

(i) If vector fields \((S(x, t), 1)\) and \((C(t)x, 1)\) are transverse to \( \gamma \) on \( \gamma \) in opposite directions, then (6) has at most 2 non-zero isolated periodic solutions. This upper bound is sharp.

(ii) If \( \gamma \) and \((C(t)x, 1)\) are transverse to \((S(x, t), 1)\) on \( \gamma \) in opposite directions, then (6) has at most 4 non-zero isolated periodic solutions.

Observe that in Corollary 1.5, all the coefficients of (2) are allowed to change signs.

It is notable that the cases with unbounded numbers of periodic solutions given in [12] and [17] are trigonometrical Abel equations. Those numbers of periodic solutions increase with respect to the degree of the trigonometrical coefficients. For this reason, Lins [12] and Ilyashenko [11] propose to find the bound for Abel equation in terms of the degrees of the coefficients. As an application of Corollary 1.5 we study a simple case of trigonometrical Abel equation.
Example 1.6. Consider differential equation
\[
\frac{dx}{dt} = \cos(2\pi t)x^3 + (b_0 + b_1 \cos(2\pi t) + b_2 \sin(2\pi t))x^2 + c_0 x,
\]
where \( t \in [0, 1] \) and \( b_0, b_1, b_2, b_3, b_4, c_0 \in \mathbb{R} \).

(i) If \( c_0 b_0 b_1 > 0, b_1^2 > b_2^2 + b_4^2 \) and
\[
c_0^2 > 4\pi^2 \frac{b_2^2 + b_4^2}{b_1^2 - b_2^2 - b_4^2}, \quad |b_0| - |b_2| > 2 \frac{|c_0|}{|b_1| - \sqrt{b_2^2 + b_4^2}},
\]
then (3) has at most 2 non-zero isolated periodic solutions.

(ii) If \( c_0 b_0 b_1 > 0, b_1^2 > b_3^2 + b_4^2 \) and
\[
|c_0| > 2\pi \sqrt{\frac{b_3^2 + b_4^2}{b_1^2 - b_3^2 - b_4^2}} + (|b_0| + |b_2|) \left( |b_1| + \sqrt{b_3^2 + b_4^2} \right), \quad |b_0| > |b_2|,
\]
then (3) has at most 4 non-zero isolated periodic solutions.

The assertions in Example 1.6 cannot be obtained directly by Theorem 1.3, see Remark 4.2 for details.

The rest of this paper is organized as follows: In Section 2 we give several preliminary results. In Section 3 we prove Theorem 1.2 and Theorem 1.3. Corollary 1.4, Corollary 1.5 and the statements of Example 1.6 are obtained in Section 4.

2. Preliminaries

In this section we mainly give two lemmas and three properties, which are useful for the proofs of the theorems.

Suppose that Hypothesis (H.2) holds for (2). Then together with \( S(0, t) = 0 \), the non-zero periodic solutions of (2) are located in \( \mathbb{R} \setminus [0, (\pm 1)^{m-n+1} \lambda] \), i.e. located in
\[
V_1 = \{x \mid \lambda > |x| > |\lambda|\}, \quad V_2 = \{x \mid 0 < x < |\lambda|\},
\]
\[
V_3 = \{x \mid -|\lambda| < x < 0\}, \quad V_4 = \{x \mid x < -|\lambda|\}, \quad (4)
\]
if \( m - n \) is even, or
\[
U_1 = \{x \mid x \cdot \lambda > 0, \ |x| > |\lambda|\}, \quad U_2 = \{x \mid x \cdot \lambda > 0, \ |x| < |\lambda|\}, \quad U_3 = \{x \mid x \cdot \lambda < 0\}, \quad (5)
\]
if \( m - n \) is odd. Therefore, firstly we have the following lemma.

Lemma 2.1. If (2) satisfies Hypothesis (H.2), then for arbitrary non-zero periodic solution \( x(t) \) of (2),
\[
\int_0^1 \frac{\partial S}{\partial x}(x(t), t) \, dt = \int_0^1 \frac{x^{\lambda-1}(t)}{\lambda^{m-n} - x^{m-n}(t)} \cdot I_5(x(t), t) \, dt,
\]
(6)
where

\[ I_S(x, t) \overset{\text{def}}{=} (m - n)(A(t)\lambda^{m-n} + B(t))x^{n-1} + (m - l)C(t)x^{m-n} - (n - l)C(t)\lambda^{m-n}. \tag{7} \]

**Proof.** Denote \( d = m - n \). Since \( x(t) \) is a periodic solution, it follows from Hypothesis (H.2) and the above argument that

\[ \left| (\lambda^d - x^d(t)) x(t) \right| > 0, \quad t \in [0, 1]. \]

Combining Eq. (2), we have

\[
\begin{align*}
\frac{\partial S}{\partial x}(x(t), t) &= -m \frac{\dot{x}(t)}{x(t)} + d \frac{\dot{x}(t)}{(\lambda^d - x^d(t))x(t)} \\
&= -dB(t)x^{n-1}(t) - (m - l)C(t)x^d(t) \\
&\quad + d \frac{\lambda^d}{\dot{\lambda}^d - x^d(t)} \left[ (A(t)x^{m-1}(t) + B(t)x^{n-1}(t) + C(t)x^d(t)) \\
&\quad - (dB(t)x^n(t) + (m - l)C(t)\lambda^d(t)) \right] \\
&= \frac{x^{d-1}(t)}{\lambda^d - x^d(t)} \left[ d(A(t)\lambda^d + B(t))x^{m-1}(t) + (m - l)C(t)x^d(t) - (n - l)C(t)\lambda^d \right] \\
&= \frac{x^{d-1}(t)}{\lambda^d - x^d(t)} \cdot I_S(x(t), t). \tag{8} \end{align*}
\]

Observe that

\[
\begin{align*}
\int_0^1 \frac{\dot{x}(t)}{x(t)} dt &= \int_0^1 \frac{x^{(1)}}{x} dt = 0, \\
\int_0^1 \frac{\dot{x}(t)}{(\lambda^d - x^d(t))x(t)} dt &= \int_0^1 \frac{x^{(1)}}{(\lambda^d - x^d(t))} dt = 0,
\end{align*}
\]

we obtain

\[
\int_0^1 \frac{\partial S}{\partial x}(x(t), t) dt = \int_0^1 \frac{x^{d-1}(t)}{\lambda^d - x^d(t)} \cdot I_S(x(t), t) dt,
\]

i.e. Eq. (6) holds. \( \square \)

Let \( x(t, x_0) \) be the solution of (2) with \( x(0, x_0) = x_0 \). We consider the return map

\[ H(x_0) = x(1, x_0). \tag{9} \]

It is well known that (see Lloyd [15] for instance)
Thus the proposition below is obtained.

**Proposition 2.2.** Suppose that (2) satisfies Hypothesis (H.2), \((a, b) \subset \mathbb{R} \setminus [0, (\pm 1)^{m-n+1} \lambda]\) and \(U = (a, b) \times [0, 1]\). If \(I_5(x, t)\) (resp. \(\partial(x^{-1} I_5(x, t)/(',d' - x^d))/\partial x)\) does not change sign in \(U\), then (2) has at most 1 (resp. 2) periodic solution in \((a, b)\).

**Proof.** Define a function as

\[
g(x_0) = \int_0^1 \frac{x^{-1}(t, x_0)}{\lambda^d - x^d(t, x_0)} \cdot I_5(x(t, x_0), t) \, dt, \quad x_0 \in \mathbb{R} \setminus (0, (\pm 1)^{m-n+1} \lambda].
\]

We know by Lemma 2.1 that \(\hat{H}(x_0) = \exp g(x_0)\) if \(x(t, x_0)\) is a non-zero periodic solution.

Let \((a_1, b_1)\) be the interval such that \(x(t, x_0) \in (a, b)\) for \((x_0, t) \in (a_1, b_1) \times [0, 1]\). If \(I_5(x, t)\) does not change sign in \(U\), then \(g(x_0)\) keeps the sign in \((a_1, b_1)\), i.e. all periodic solutions of (2) in \((a, b)\) have the same stability. Thus (2) has at most 1 periodic solution in \(U\).

For the case that \(\partial(x^{-1} I_5(x, t)/(',d' - x^d))/\partial x)\) does not change sign in \(U\), we get that \(g(x_0)\) is a strictly monotonic function, where \(x_0 \in (a_1, b_1)\). Using the fact that two consecutive hyperbolic periodic solutions have different stability, (2) has at most 3 periodic solutions in \((a, b)\).

Now assume that there exist three consecutive periodic solutions \(x(t, x_1), x(t, x_2)\) and \(x(t, x_3)\) appearing in \((a_2, b_2) \subset (a, b)\) with \(a < a_2 < b_2 < b\) and \(b_2 - a_2 < +\infty\). We have \(g(x_1) \cdot g(x_3) < 0\) and \(g(x_2) = 0\), i.e. \(x(t, x_1)\) and \(x(t, x_3)\) are two hyperbolic periodic solutions with different stability, and \(x(t, x_2)\) is semi-stable. Without loss of generality, suppose that \(0 < a_2 < b_2\) and \(g(x_1) < g(x_2) = 0 < g(x_3)\). Then \(x(t, x_2)\) is unstable from below and stable from above. Consider the following equation

\[
\frac{dx}{dt} = S_\varepsilon (x, t) = A(t)x^m + B(t)x^n + (C(t) + \varepsilon)x^l, \tag{10}
\]

which is also of the form (2). For \((x, t) \in [a_2, b_2] \times [0, 1]\), \(S_\varepsilon (x, t)\) is monotonically increasing with respect to \(\varepsilon\). Hence when \(\varepsilon > 0\) small enough, (10) has at least four periodic solutions in \((a_2, b_2)\). However, for small \(\varepsilon\), (10) still satisfies Hypothesis (H.2), and the function \(\partial(x^{-1} I_5(x, t)/(',d' - x^d))/\partial x)\) does not change sign in \([a_2, b_2] \times [0, 1]\). According to the same argument, (10) has at most 3 periodic solutions in \((a_2, b_2)\). This shows a contradiction. As a result, (2) has at most 2 periodic solutions in \((a, b)\). \(\square\)

**Proposition 2.3.** Suppose that (2) satisfies Hypotheses (H.1) and (H.2) for the same \(\lambda\). The following statements hold.

(i) Assume that \(n - l\) is odd. Then for each \(t_0 \in [0, 1]\), \(\partial I_5(x, t_0)/\partial x = 0\) has one real non-zero solution

\[
x = \left(\frac{-C(t_0)}{A(t_0)\lambda^{m-n} + B(t_0)}\right)^{1/(n-l)}
\]

when \(C(t_0) \neq 0\), and has no real non-zero solution when \(C(t_0) = 0\).

(ii) Assume that \(n - l\) is even. Then for each \(t_0 \in [0, 1]\), \(\partial I_5(x, t_0)/\partial x = 0\) has two real non-zero solutions

\[
x = \pm \left(\frac{-C(t_0)}{A(t_0)\lambda^{m-n} + B(t_0)}\right)^{1/(n-l)}
\]
As a result, statement (iii) is obtained.

Proof. It is trivial from a direct computation.

If \(ab \geq 0\) and \(a \neq 0\), then \(\text{sgn}(a + b) = \text{sgn}(a)\).

Lemma 2.4. Assume that \(a, b \in \mathbb{R}\). The following statements hold.

(i) \(\text{sgn}(ab) = \text{sgn}(a) \cdot \text{sgn}(b)\).
(ii) If \(|a| > |b|\), then \(\text{sgn}(a + b) = \text{sgn}(a)\).
(iii) If \(ab \geq 0\) and \(a \neq 0\), then \(\text{sgn}(a + b) = \text{sgn}(a)\).

Proof. It is trivial from a direct computation.
Proposition 2.5. Suppose that (2) satisfies Hypothesis (H.1).

(i) The following statements hold for $I_S(x, t)$.

   (i.a) \[
   \text{sgn}(I_S(0, t)) = -\text{sgn}(\lambda^{m-n}) \cdot \text{sgn}(C(t)),
   \]
   \[
   \lim_{x \to +\infty} \text{sgn}(I_S(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)),
   \]
   \[
   \lim_{x \to -\infty} \text{sgn}(I_S(x, t)) = (-1)^{m-l} \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)).
   \]

   (i.b) If $m - n$ is odd, then
   \[
   \text{sgn}(I_S(\lambda, t)) = \text{sgn}(\lambda) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t))\lambda^{n-l} + C(t)).
   \]

   If $m - n$ is even, then
   \[
   \text{sgn}(I_S(\pm|\lambda|, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)) + C(t)).
   \]

   (i.c) If $n - l$ is odd, then
   \[
   \text{sgn}\left(I_S \left(\left(\frac{-C(t)}{A(t)\lambda^{m-n} + B(t)}\right)^{1/(n-l)}, t\right)\right)
   \]
   \[
   = \text{sgn}(C(t)) \cdot \text{sgn}\left((\pm 1)^{n-m}\left(\frac{-C(t)}{A(t)\lambda^{m-n} + B(t)}\right)^{(n-m)/l} - \lambda^{m-n}\right).
   \]

   If $n - l$ is even and $(A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0$, then
   \[
   \text{sgn}\left(I_S \left(\left(\frac{-C(t)}{A(t)\lambda^{m-n} + B(t)}\right)^{1/(n-l)}, t\right)\right)
   \]
   \[
   = \text{sgn}(C(t)) \cdot \text{sgn}\left((\pm 1)^{m-n}\left(\frac{-C(t)}{A(t)\lambda^{m-n} + B(t)}\right)^{(m-n)/l} - \lambda^{m-n}\right).
   \]

(ii) When $l = 1$, the following statements hold for $P_S(x, t)$.

   (ii.a) \[
   \text{sgn}(P_S(0, t)) = -\text{sgn}(\lambda^{m-n}) \cdot \text{sgn}(C(t)),
   \]
   \[
   \lim_{x \to +\infty} \text{sgn}(P_S(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)),
   \]
   \[
   \lim_{x \to -\infty} \text{sgn}(P_S(x, t)) = (-1)^{m-1} \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)).
   \]

   (ii.b) If $m - n$ is odd, then
   \[
   \text{sgn}(P_S(\lambda, t)) = -\text{sgn}(\lambda) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t))\lambda^{n-1} + C(t)).
   \]

   If $m - n$ is even, then
   \[
   \text{sgn}(P_S(\pm|\lambda|, t)) = -\text{sgn}(A(t)\lambda^{m-n} + B(t)) + C(t)).
   \]
Proof. Following statement (i) of Lemma 2.4 and a direct calculation, the assertions are valid. □

Proposition 2.6. Let $F(x)$ be a polynomial and $g \in C^\infty([0, 1])$, respectively. If $x(t)$ is a solution of differential equation

$$\frac{dx}{dt} = g(t)F(x), \quad (14)$$

then

$$\text{sgn}(x(t) - x(0)) = \text{sgn}\left(\int_0^t g(s) \, ds\right) \cdot \text{sgn}(F(x(0))). \quad (15)$$

Furthermore, assume that $q(x, t) \in C^\infty(\mathbb{R} \times [0, 1])$ and $|q(x, t)| > 0$ in $(a, b) \times [0, 1]$, where either $F(a) = 0$ or $F(b) = 0$. If

$$\text{sgn}\left(\int_0^1 g(s) \, ds\right) \cdot \text{sgn}(F(x)) \cdot \text{sgn}(q(x, t)) \geq 0, \quad (x, t) \in (a, b) \times [0, 1], \quad (16)$$

then the differential equation

$$\frac{dx}{dt} = g(t)F(x) + q(x, t) \quad (17)$$

has no periodic solution in $(a, b)$.

Proof. Clearly if $F(x(0)) = 0$, then $x(t) \equiv x(0)$. Eq. (15) holds. If $F(x(0)) \neq 0$, then $F(x(t)) \neq 0$, i.e. $F(x(t))$ does not change sign. As a result, we have

$$\int_{x(0)}^{x(t)} \frac{dx}{F(x)} = \int_0^t g(s) \, ds,$$

which also implies Eq. (15).

To prove the second part of the Proposition we suppose $F(a) = 0$ and $q(x, t) > 0$ in $(a, b) \times [0, 1]$, without loss of generality. Assume that (17) has a periodic solution $\tilde{x}(t)$ in $(a, b)$. Comparing (17) with (14), we know that if $x(t)$ is a solution of (14) with $x(0) = \tilde{x}(0) = \tilde{x}(1)$, then $x(t)$ is defined in $[0, 1]$ and $\tilde{x}(t) > x(t) > a$. Thus, $\text{sgn}(x(1) - x(0)) = -1$. However, it follows from (16) that

$$\text{sgn}\left(\int_0^1 g(s) \, ds\right) \cdot \text{sgn}(F(x)) \geq 0, \quad (x, t) \in (a, b) \times [0, 1].$$

Combining (15) we obtain $\text{sgn}(x(1) - x(0)) \geq 0$. This shows a contradiction. As a result, (17) has no periodic solution in $(a, b)$. □
3. Proofs of Theorem 1.2 and Theorem 1.3

Before starting the proofs of our main results, we give two equations as follows.

\[ \frac{dx}{dt} = A(t)x^{p}(x^{m-n} - \lambda^{m-n}), \]  \hspace{1cm} (18)

and

\[ \frac{dx}{dt} = \frac{B(t)}{\lambda^{m-n}} x^{q}(\lambda^{m-n} - x^{m-n}). \]  \hspace{1cm} (19)

Noting that (2) can be rewritten as

\[ \frac{dx}{dt} = S(x, t) = A(t)x^{p}(x^{m-n} - \lambda^{m-n}) + \left( A(t)\lambda^{m-n} + B(t) \right)x^{q} + C(t) \]

\[ = \frac{B(t)}{\lambda^{m-n}} x^{q}(\lambda^{m-n} - x^{m-n}) + \frac{C(t)}{\lambda^{m-n}} \left[ (A(t)\lambda^{m-n} + B(t))x^{m-n} + C(t)\lambda^{m-n} \right], \]  \hspace{1cm} (20)

these two equations will be compared with (2) in our proofs repeatedly.

We begin to prove Theorem 1.2.

**Proof of Theorem 1.2.** Under Hypothesis (H.1), it is easy to verify that the first inequality in each statement of this theorem implies Hypothesis (H.2), respectively. Thus the periodic solutions of (2) only appear in \( \mathbb{R} \setminus (0, (\pm 1)^{m-n+1}\lambda) \), and Proposition 2.2 is usable.

Now we prove the statements one by one.

(i) By assumption, \( n-l \) is even. Since \( |(A(t)\lambda^{m-n} + B(t))x^{n-l}| > |C(t)| \) and \( (A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0 \), we get

\[ 0 \leq \left( \frac{-C(t)}{A(t)\lambda^{m-n} + B(t)} \right)^{1/(n-l)} < |\lambda|. \]

Combining statement (i) of Proposition 2.5 and statement (ii) of Lemma 2.4, we obtain

\[ \text{sgn}(I_{\delta}(0, t)) = -\text{sgn}(C(t)), \]

\[ \lim_{x \to +\infty} \text{sgn}(I_{\delta}(x, t)) = \lim_{x \to -\infty} \text{sgn}(I_{\delta}(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)), \]

and

\[ \text{sgn}\left( I_{\delta}\left( \pm \left( \frac{-C(t)}{A(t)\lambda^{m-n} + B(t)} \right)^{1/(n-l)} , t \right) \right) = \text{sgn}(C(t)) \cdot \text{sgn}(\lambda^{m-n}) = -\text{sgn}(C(t)). \]

Hence, from statement (ii) of Proposition 2.3, \( I_{\delta}(x, t) \leq 0 \) (resp. \( I_{\delta}(x, t) \geq 0 \)) in \( \mathbb{R} \times [0, 1] \) if \( C(t) \geq 0 \) (resp. \( C(t) \leq 0 \)), and \( I_{\delta}(x, t) = 0 \) if and only if \( x = C(t) = 0 \). As a result, \( I_{\delta}(x, t) \) does not change sign in \( (\mathbb{R} \setminus (0, \pm \lambda)) \times [0, 1] = \bigcup_{i=1}^{d}(V_{i} \times [0, 1]) \), where \( V_{i} \) are defined in (4). Using Proposition 2.2, Eq. (2) has at most 4 non-zero periodic solutions.
Furthermore, taking
\[ S(x, t) = A(t)x^m + B(t)x^n + C(t)x^l = -x^m + \frac{4^{m-l} - 1}{4^{n-l} - 1} x^n - \frac{4^{m-l} - 4^{n-l}}{4^{n-l} - 1} x^l, \quad \lambda = 2, \] (21)
we can verify that \( C(t) < 0, \)
\[ (A(t)^2m-n + B(t))2^{n-l} = -2^{m-l} + 2^{n-l} \frac{2^{2(m-l)} - 1}{2^{2(n-l)} - 1} \]
\[ = \frac{2^{2(m-l)+n-l} - 2^{m-l+2(n-l)} + 2^{m-l} - 2^{n-l}}{2^{2(n-l)} - 1} > 0, \]
and
\[ (A(t)^2m-n + B(t))^{2n-l} + C(t) = \frac{2^{2(m-l)}(2^{n-l} - (m-n) - 1) + 2^{2(n-l)} + 2^{m-l} - 2^{n-l}}{2^{2(n-l)} - 1} > 0. \]
Thus \(|(A(t)^2m-n + B(t))^{2n-l}| > |C(t)|\) and \((A(t)^2m-n + B(t)) \cdot C(t) \leq 0.\) Clearly \(S(\pm 1, t) = S(\pm 4, t) = 0,\) which implies that \((2)\) has 4 periodic solutions \(x = \pm 1, x = \pm 4.\) Therefore, we know that the upper bound is sharp.

(ii) Similar to above, it follows from assumption that \(n - l\) is odd and
\[ \left| \left( \frac{-C(t)}{A(t)^m-n + B(t)} \right)^{1/(n-l)} \right| < |\lambda|. \]
Since \(m - l\) is even and \(m - n\) is odd, statement (i) of Proposition 2.5 tells us that
\[ \text{sgn}(I_5(0, t)) = -\text{sgn}(C(t)) \cdot \text{sgn}(\lambda), \]
\[ \lim_{x \to +\infty} \text{sgn}(I_5(x, t)) = \lim_{x \to -\infty} \text{sgn}(I_5(x, t)) = \text{sgn}(A(t)^m-n + B(t)), \]
and
\[ \text{sgn}\left( I_5\left( \left( \frac{-C(t)}{A(t)^m-n + B(t)} \right)^{1/(n-l)}, t \right) \right) = -\text{sgn}(C(t)) \cdot \text{sgn}(\lambda). \]
Applying \((A(t)^m-n + B(t)) \cdot C(t) \cdot \lambda \leq 0\) in assumption and statement (i) of Proposition 2.3, \(I_5(x, t)\) does not change sign in \((\mathbb{R} \setminus [0, \lambda]) \times [0, 1] = \bigcup_{i=1}^{3} U_i \times [0, 1],\) where \(U_i\) are defined in (5). Hence, by Proposition 2.2 each \(U_i\) has at most 1 periodic solution of \((2), \) \(i = 1, 2, 3.\)

Now we rewrite \((2)\) in (20), and compare it with (18). Since \(|(A(t)^m-n + B(t))\lambda^{n-l}| > |C(t)|,\) for \(x \in U_1\) we get
\[ |(A(t)^m-n + B(t))x^n| > |(A(t)^m-n + B(t))\lambda^{n-l}x^l| > |C(t)x^l|. \] (22)
Together with statement (ii) of Lemma 2.4,
\[ \text{sgn}\left( (A(t)^m-n + B(t))x^n + C(t)x^l \right) = \text{sgn}(x^n) \cdot \text{sgn}(A(t)^m-n + B(t)) \neq 0, \quad x \in U_1. \]
In addition, for \( x \in U_3 \) we have by assumption that \((A(t)\lambda^{m-n} + B(t)) \cdot x \cdot C(t) \geq 0\). Recalling that \( n - l \) is odd, \((A(t)\lambda^{m-n} + B(t)) \cdot x^{m-l} \cdot C(t) \geq 0\) for \( x \in U_3 \). Therefore, from statement (iii) of Lemma 2.4 we still get

\[
\text{sgn}\left( (A(t)\lambda^{m-n} + B(t))x^n + C(t)x^l \right) = \text{sgn}(x^n) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)) \neq 0, \quad x \in U_3.
\]

As a result,

\[
\text{sgn}\left( \int_0^1 A(t) \, dt \right) \cdot \text{sgn}(x^n(\lambda^{m-n} - \lambda^{m-n})) \cdot \text{sgn}\left( (A(t)\lambda^{m-n} + B(t))x^n + C(t)x^l \right)
\]

\[
= \begin{cases} 
\text{sgn}\left( \int_0^1 A(t) \, dt \right) \cdot \text{sgn}(\lambda) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)), & x \in U_1, \\
\text{sgn}\left( \int_0^1 A(t) \, dt \right) \cdot \text{sgn}(-\lambda) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)), & x \in U_3.
\end{cases}
\]

This implies that if for \((x, t) \in U_1 \times [0, 1],\)

\[
\text{sgn}\left( \int_0^1 A(t) \, dt \right) \cdot \text{sgn}(x^n(\lambda^{m-n} - \lambda^{m-n})) \cdot \text{sgn}\left( (A(t)\lambda^{m-n} + B(t))x^n + C(t)x^l \right)
\]

\[
= \text{sgn}\left( \int_0^1 A(t) \, dt \right) \cdot \text{sgn}(\lambda) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t))
\]

\[
\leq 0,
\]

then for \((x, t) \in U_3 \times [0, 1],\)

\[
\text{sgn}\left( \int_0^1 A(t) \, dt \right) \cdot \text{sgn}(x^n(\lambda^{m-n} - \lambda^{m-n})) \cdot \text{sgn}\left( (A(t)\lambda^{m-n} + B(t))x^n + C(t)x^l \right)
\]

\[
= -\text{sgn}\left( \int_0^1 A(t) \, dt \right) \cdot \text{sgn}(\lambda) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t))
\]

\[
\geq 0
\]

(it is similar for the opposite case). Hence, for either \((x, t) \in U_1 \times [0, 1] \) or \((x, t) \in U_3 \times [0, 1],\)

\[
\text{sgn}\left( \int_0^1 A(t) \, dt \right) \cdot \text{sgn}(x^n(\lambda^{m-n} - \lambda^{m-n})) \cdot \text{sgn}\left( (A(t)\lambda^{m-n} + B(t))x^n + C(t)x^l \right) \geq 0.
\]

According to Proposition 2.6, (2) has no periodic solution in \( U_1 \) or \( U_3 \). Based on the above, (2) has at most 2 non-zero periodic solutions.

We still use the example (21) with \( m - l \) being even and \( m - n \) being odd. Thus \(|(A(t)2^{m-n} + B(t))2^{m-l}| > |C(t)|\), (2) has \( (A(t)2^{m-n} + B(t)) \cdot 2 \cdot C(t) \leq 0 \) and \( S(1, t) = S(4, t) = 0 \). Therefore, (2) has 2 periodic solutions \( x = 1, x = 4 \). The upper bound is sharp.

(iii) By assumption \( n - l \) is odd. Since \(|(A(t)\lambda^{m-n} + B(t))\lambda^{n-l}| > |C(t)|\), we get
Based on the above, (2) has at most 3 non-zero periodic solutions.

Following statement (i) of Proposition 2.5 and Lemma 2.4, we obtain

\[
\lim_{x \to +\infty} \text{sgn}(I_5(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]

\[
\lim_{x \to -\infty} \text{sgn}(I_5(x, t)) = -\text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]

\[
\text{sgn}(I_5(\pm|\lambda|, t)) = \text{sgn}((\pm|\lambda|)^{n-l}(A(t)\lambda^{m-n} + B(t))) = \pm \text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]

and

\[
\text{sgn}\left(I_5\left(\left(\frac{-C(t)}{A(t)\lambda^{m-n} + B(t)}\right)^{1/(n-l)}, t\right)\right) = \text{sgn}(I_5(0, t)) = -\text{sgn}(C(t)).
\]

Together with statement (i) of Proposition 2.3, \(I_5(x, t)\) does not change sign in \(V_1 \times [0, 1]\), \(V_2 \times [0, 1]\) and \(V_4 \times [0, 1]\), respectively. As a result, it follows from Proposition 2.2 that (2) has at most 3 periodic solutions in \(V_1 \cup V_2 \cup V_4\).

Now we study the periodic solutions in \(V_3\). Rewrite (2) in (20), and compare it with (19). Since \(m - n\) is even, it follows from (23) that

\[
(A(t)\lambda^{m-n} + B(t))\lambda^{n-l} \cdot C(t)\lambda^{m-n} = (A(t)\lambda^{m-n} + B(t)) \cdot \lambda^{n-l} \cdot C(t) \cdot (\lambda\lambda)^{m-n} \geq 0,
\]

where \((x, t) \in V_3 \times [0, 1]\). Thus, using statement (iii) of Lemma 2.4, the equation below holds for \((x, t) \in V_3 \times [0, 1]\),

\[
\text{sgn}\left(\frac{\lambda^l}{\lambda^{m-n}}\left[(A(t)\lambda^{m-n} + B(t))\lambda^{n-l} + C(t)\lambda^{m-n}\right]\right) = \text{sgn}((A(t)\lambda^{m-n} + B(t))\lambda^n) \neq 0.
\]

Noting that \((A(t)\lambda^{m-n} + B(t)) \cdot \int_0^1 B(t) \text{dt} \geq 0\) by assumption,

\[
\text{sgn}\left(\int_0^1 \frac{B(t)}{\lambda^{m-n}} \text{dt} \cdot x^n(\lambda^{m-n} - x^{m-n})\right) \cdot \text{sgn}\left(\frac{x^l(A(t)\lambda^{m-n} + B(t))\lambda^{n-l} + C(t)\lambda^{m-n}}{\lambda^{m-n}}\right)
\]

\[
= \text{sgn}\left(\int_0^1 B(t) \text{dt} \cdot x^n\right) \cdot \text{sgn}((A(t)\lambda^{m-n} + B(t))\lambda^n)
\]

\[
\geq 0,
\]

where \((x, t) \in V_3 \times [0, 1]\). As a result, Proposition 2.6 tells us that (2) has no periodic solutions in \(V_3\). Based on the above, (2) has at most 3 non-zero periodic solutions.

To show the example with sharp upper bound, we again use the (21) with \(m - l\) being odd and \(m - n\) being even. So \(|(A(t)2^{m-n} + B(t))2^{n-l}| > |C(t)|\), and it is easy to check that \((A(t)2^{m-n} + B(t)) \cdot \ldots\)
Lemma 2.4 that (2) has a non-zero periodic solution. This implies that the upper bound is sharp.

(iv) Since both \( m - l \) and \( m - n \) are odd, we know \( n - l \) is even. Combining assumption \( (A(t)\lambda^{m-n} + B(t))^n \cdot C(t) \geq 0 \) and statement (ii) Proposition 2.3, \( I_S(x, t) \) is monotonic with respect to \( x \) in \((-\infty, 0) \times [0, 1] \) and \((0, +\infty) \times [0, 1] \), respectively. In addition, we get from statement (i) of Proposition 2.5 and statement (iii) of Lemma 2.4 that

\[
\text{sgn}(I_S(\lambda, t)) = \text{sgn}(\lambda) \cdot \text{sgn}(\lambda^{n-l} (A(t)\lambda^{m-n} + B(t)) + C(t)) = \text{sgn}(\lambda) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)).
\]

and

\[
\lim_{x \to +\infty} \text{sgn}(I_S(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]

\[
\lim_{x \to -\infty} \text{sgn}(I_S(x, t)) = -\text{sgn}(A(t)\lambda^{m-n} + B(t)).
\]

Hence, \( I_S(x, t) \) does not change sign in \( U_1 \times [0, 1] \). Following Proposition 2.2, (2) has at most 1 periodic solution in \( U_1 \).

Now rewrite (2) in (20) and compare it with (18). For \((x, t) \in (U_2 \cup U_3) \times [0, 1]\) we obtain by Lemma 2.4 that

\[
\text{sgn}
\left(
\int_0^1 A(t) \, dt \right) \cdot \text{sgn}(x^n (x^{m-n} - \lambda^{m-n})) = -\text{sgn}
\left(
\int_0^1 A(t) \, dt \right) \cdot \text{sgn}(x^n \lambda),
\]

and

\[
\text{sgn}( (A(t)\lambda^{m-n} + B(t))x^n + C(t)x^l ) = \text{sgn}(x^n) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)) \neq 0, \quad x \neq 0.
\]

Observe that \((A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot \int_0^1 A(t) \, dt \leq 0\) from assumption, we get

\[
\text{sgn}
\left(
\int_0^1 A(t) \, dt \right) \cdot \text{sgn}(x^n (x^{m-n} - \lambda^{m-n})) \cdot \text{sgn}( (A(t)\lambda^{m-n} + B(t))x^n + C(t)x^l )
\]

\[
= -\text{sgn}
\left(
\int_0^1 A(t) \, dt \right) \cdot \text{sgn}(\lambda) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t))
\]

\[
\leq 0,
\]

where \((x, t) \in (U_2 \cup U_3) \times [0, 1]\). As a result, (2) has no periodic solution in \( U_2 \) and \( U_3 \) from Proposition 2.6. Based on the above, (2) has at most 1 non-zero periodic solution.

Let

\[
S(x, t) = A(t)x^n + B(t)x^n + C(t)x^l = x^m - 2 \cdot 3^{m-n-1}x^n - 3^{m-l-1}x^l, \quad \lambda = 1.
\]

It is easy to verify that \( A(t) + B(t) < 0, C(t) < 0 \) and \((A(t) + B(t)) \cdot \int_0^1 A(t) \, dt < 0\). Noting \( S(3, t) = 0 \), (2) has a non-zero periodic solution \( x = 3 \), which means the upper bound is sharp. \( \Box \)
Now we shall prove Theorem 1.3.

**Proof of statement (i) of Theorem 1.3.** According to Hypothesis (H.2), the non-zero periodic solutions of (2) only appear in $\mathbb{R} \setminus \{ 0, \pm \lambda \} = \bigcup_{i=1}^{4} V_{i}$. Hence in what follows we calculate the number in each $V_{i}$.

(i.a) Firstly we know $n - 1$ is even. Since $|A(t)\lambda^{m-n} + B(t)| \lambda^{n-1} < |C(t)|$ and $(A(t)\lambda^{m-n} + B(t)) \cdot C(t) < 0$, statement (i) of Proposition 2.5 and Lemma 2.4 tell us that
\[
\text{sgn}(I_{S}(0, t)) = \lim_{x \to +\infty} \text{sgn}(I_{S}(x, t)) = \lim_{x \to -\infty} \text{sgn}(I_{S}(x, t)) = -\text{sgn}(C(t)),
\]
\[
\text{sgn}(I_{S}(\pm \lambda, t)) = \text{sgn}((\pm \lambda)^{n-1}(A(t)\lambda^{m-n} + B(t)) + C(t)) = \text{sgn}(C(t)).
\]
Thus $I_{S}(x, t)$ changes sign in each $V_{i}$, $i = 1, \ldots, 4$. In order to use Proposition 2.2 we need to consider the sign of $P_{S}(x, t)$.

From statement (iii) of Proposition 2.3, $P_{S}(x, t)$ is monotonic with respect to $x$ in $V_{i} \times [0, 1]$, respectively. In addition, it follows from statement (ii) of Proposition 2.5 that
\[
\text{sgn}(P_{S}(0, t)) = \text{sgn}(P_{S}(\pm \lambda, t)) = -\text{sgn}(C(t)),
\]
\[
\lim_{x \to +\infty} \text{sgn}(P_{S}(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)) = -\text{sgn}(C(t)).
\]

Therefore, $P_{S}(x, t)$ does not change sign in $(\mathbb{R} \setminus \{ 0, \pm \lambda \}) \times [0, 1] = \bigcup_{i=1}^{4} (V_{i} \times [0, 1])$. Following (11) and Proposition 2.2, (2) has at most 8 non-zero periodic solutions.

(i.b) From Hypotheses (H.1) and (H.2), if (2) does not satisfy the condition in statement (i.a), then either $(A(t)\lambda^{m-n} + B(t))\lambda^{n-1} + C(t) \neq 0$ and $(A(t)\lambda^{m-n} + B(t)) \cdot C(t) \geq 0$, or $|A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| > |C(t)|$ and $(A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0$. In the following we prove these two cases, respectively.

Case 1. $(A(t)\lambda^{m-n} + B(t))\lambda^{n-1} + C(t) \neq 0$ and $(A(t)\lambda^{m-n} + B(t)) \cdot C(t) \geq 0$.

According to statement (ii) of Proposition 2.3, $I_{S}(x, t)$ is monotonic with respect to $x$ in $(-\infty, 0) \times [0, 1]$ and $(0, +\infty) \times [0, 1]$, respectively. In addition, statement (i) of Proposition 2.5 tells us that
\[
\lim_{x \to -\infty} \text{sgn}(I_{S}(x, t)) = \lim_{x \to -\infty} \text{sgn}(I_{S}(x, t)) = \text{sgn}(I_{S}(\pm \lambda, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)).
\]

Therefore, $I_{S}(x, t)$ does not change sign in $V_{1} \times [0, 1]$ and $V_{4} \times [0, 1]$, which implies that (2) has at most 2 periodic solutions in $V_{1}$ and $V_{4}$, respectively.

Since $\text{sgn}(I_{S}(0, t)) \cdot \text{sgn}(I_{S}(\pm \lambda, t)) = -\text{sgn}(C(t)) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)) \leq 0$ is also obtained, the sign of $I_{S}(x, t)$ in $V_{2} \times [0, 1]$ and $V_{3} \times [0, 1]$ is unknown. We need to consider the sign of $P_{S}(x, t)$. Similar to the argument in (i.a), we can verify that $P_{S}(x, t)$ keeps the sign in $V_{2} \times [0, 1]$ and $V_{3} \times [0, 1]$. Hence, (2) has at most 2 periodic solutions in $V_{2}$ and $V_{3}$ from (11) and Proposition 2.2, respectively.

Now we rewrite (2) in (20)\(\|_{l=1}\) and compare it with (18). Since $(A(t)\lambda^{m-n} + B(t)) \cdot C(t) \geq 0$ and $n - 1$ is even, we obtain by Lemma 2.4 that
\[
\text{sgn}\left(\int_{0}^{1} A(t) dt \right) \cdot \text{sgn}(x^{n} - \lambda^{m-n}) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)) \cdot \text{sgn}(x^{n} + C(t)x) \neq 0,
\]
and
\[
\text{sgn}\left(\int_{0}^{1} A(t) dt \right) \cdot \text{sgn}(x^{n} - \lambda^{m-n}) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)) \cdot \text{sgn}(x^{n} + C(t)x) = \text{sgn}\left(\int_{0}^{1} A(t) dt \right) \cdot \text{sgn}(x^{n} - \lambda^{m-n}) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)).
\]
Together with Proposition 2.6, the statements below hold.

(a) If \( \int_0^1 A(t) \, dt = 0 \), then (2) has no periodic solutions in \( \bigcup_{i=1}^4 V_i \). Therefore it has no non-zero periodic solutions.

(b) If \( \int_0^1 A(t) \, dt \cdot (A(t)\lambda^{m-n} + B(t)) < 0 \), then (2) has no periodic solutions in \( V_2 \cup V_3 \). Therefore it has at most 2 non-zero periodic solutions.

(c) If \( \int_0^1 A(t) \, dt \cdot (A(t)\lambda^{m-n} + B(t)) > 0 \), then (2) has no periodic solutions in \( V_1 \cup V_4 \). Therefore it has at most 4 non-zero periodic solutions.

As a result, (2) has at most 4 non-zero periodic solutions.

Case 2. \(|A(t)\lambda^{m-n} + B(t)| > |C(t)| \) and \((A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0 \). This is a particular case \((l = 1)\) in statement (i) of Theorem 1.2. Thus (2) has at most 4 non-zero periodic solutions, and the upper bound is sharp.

Based on the above, statement (i.a) and statement (i.b) of Theorem 1.3 are obtained. \( \square \)

**Proof of statement (ii) of Theorem 1.3.** Firstly Hypothesis (H.2) tells us that the non-zero periodic solutions of (2) are located in \( \mathbb{R} \setminus \{0, \lambda\} = \bigcup_{i=1}^4 U_i \).

(ii.a) By assumption, \( n - 1 \) is odd and \((A(t)\lambda^{m-n} + B(t))\lambda^{n-1} + C(t) \neq 0 \). Since \((A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot C(t) \leq 0 \), we know \(|A(t)\lambda^{m-n} + B(t)|\lambda^{n-1} - |C(t)| \neq 0 \). Therefore if \(|A(t)\lambda^{m-n} + B(t)|\lambda^{n-1} > |C(t)|\), then statement (ii) of Theorem 1.2 tells us that (2) has at most 2 non-zero periodic solutions.

For the case that \(|A(t)\lambda^{m-n} + B(t)|\lambda^{n-1} < |C(t)|\), we have

\[
|C(t)| > 0, \quad \left( \frac{-C(t)}{A(t)\lambda^{m-n} + B(t)} \right)^{1/(n-1)} \cdot \lambda > 0.
\]

Combining statement (i) of Proposition 2.3, \( I_5(x, t) \) is monotonic with respect to \( x \) in \( U_3 \times [0, 1] \). Furthermore, it follows from Proposition 2.5 that

\[
\text{sgn}(I_5(0, t)) = \lim_{x \to +\infty} \text{sgn}(I_5(x, t)) = \lim_{x \to -\infty} \text{sgn}(I_5(x, t)) = -\text{sgn}(\lambda) \cdot \text{sgn}(C(t)),
\]

\[
\text{sgn}(I_5(\lambda, t)) = \text{sgn}(\lambda) \cdot \text{sgn}(C(t)).
\]

Thus, \( I_5(x, t) \) has fixed sign in \( U_3 \times [0, 1] \), and change sign in both \( U_1 \times [0, 1] \) and \( U_2 \times [0, 1] \). This means that (2) has at most 1 periodic solution in \( U_3 \), and we need to consider the sign of \( P_5(x, t) \) in \( U_1 \times [0, 1] \) and \( U_2 \times [0, 1] \).

According to statement (iii) of Proposition 2.3, \( P_5(x, t) \) is monotonic with respect to \( x \) in \( U_1 \times [0, 1] \) and \( U_2 \times [0, 1] \). Following Proposition 2.5 and a calculation,

\[
\text{sgn}(P_5(0, t)) = \lim_{x \to +\infty} \text{sgn}(P_5(x, t)) = \lim_{x \to -\infty} \text{sgn}(P_5(x, t)) = -\text{sgn}(\lambda) \cdot \text{sgn}(C(t)),
\]

\[
\text{sgn}(P_5(\lambda, t)) = -\text{sgn}(\lambda) \cdot \text{sgn}((A(t)\lambda^{m-n} + B(t))\lambda^{n-1} + C(t)) = -\text{sgn}(\lambda) \cdot \text{sgn}(C(t)).
\]

Therefore, \( P(x, t) \) does not change sign in \( U_1 \times [0, 1] \) and \( U_2 \times [0, 1] \). Applying (11) and Proposition 2.2, (2) has at most 2 periodic solutions in \( U_1 \) and \( U_2 \), respectively.

Observe that \(|C(t)| > |A(t)\lambda^{m-n} + B(t)|\lambda^{n-1}| > |(A(t)\lambda^{m-n} + B(t))x^{n-1}| \) for \( x \in U_2 \), and \((A(t)\lambda^{m-n} + B(t)) \cdot \chi \cdot C(t) > 0 \) for \( x \in U_3 \), we obtain by Lemma 2.4 that

\[
\text{sgn}((A(t)\lambda^{m-n} + B(t))x^n + C(t)x) = \begin{cases} 
\text{sgn}(\lambda) \cdot \text{sgn}(C(t)), & x \in U_2, \\
-\text{sgn}(\lambda) \cdot \text{sgn}(C(t)), & x \in U_3.
\end{cases}
\]

If we compare (2) (i.e. \((20)_{x=1}\) with \((18)_{x=1}\), then combining Proposition 2.6,
(a) (2) has no periodic solution in \( U_2 \cup U_3 \) if \( \int_0^1 A(t) \, dt = 0 \).
(b) (2) has no periodic solution in either \( U_2 \) or \( U_3 \) if \( \int_0^1 A(t) \, dt \neq 0 \).

As a result, (2) has at most 4 non-zero periodic solutions, statement (ii.a) is valid.

(ii.b) First compare (2) (i.e. (20)\(\lambda=1\)) with (19)\(\lambda=1\). Using \((A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot C(t) \geq 0\) and Lemma 2.4, we get

\[
\begin{align*}
\text{sgn} \left( \frac{x}{\lambda^{m-n}} \left[ (A(t)\lambda^{m-n} + B(t))x^{m-1} + C(t)\lambda^{m-n} \right] \right) \\
= \begin{cases} 
\text{sgn}(A(t)\lambda^{m-n} + B(t)), & x \cdot \lambda > 0, \\
-\text{sgn}(A(t)\lambda^{m-n} + B(t)), & x \cdot \lambda < 0.
\end{cases}
\end{align*}
\]

Hence together with \((A(t)\lambda^{m-n} + B(t)) \int_0^1 B(t) \, dt \leq 0\) and Proposition 2.6, Eq. (2) has no periodic solution in \( U_1 \) and \( U_3 \).

Second observe that \( n - 1 \) is odd, we have by assumption and Lemma 2.4 that

\[
\text{sgn}((A(t)\lambda^{m-n} + B(t))\lambda^{n-1} + C(t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)).
\]  

(24)

According to Proposition 2.5,

\[
\text{sgn}(I_5(\lambda, t)) \cdot \text{sgn}(I_5(0, t)) = -\text{sgn}(C(t)) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)) \leq 0.
\]

Thus the sign of \( I_5(x, t) \) in \( U_2 \times [0, 1] \) is unknown, and the sign of \( P_5(x, t) \) is needed.

Recall that \( P_5(x, t) \) is monotonic with respect to \( x \) in \( U_2 \times [0, 1] \). Following Proposition 2.5 and (24),

\[
\text{sgn}(P_5(0, t)) \cdot \text{sgn}(P_5(\lambda, t)) = \text{sgn}(\lambda) \cdot \text{sgn}(C(t)) \cdot \text{sgn}(A\lambda^{m-n} + B) \geq 0.
\]

This implies that \( P(x, t) \) has fixed sign in \( U_2 \times [0, 1] \). Applying (11) and Proposition 2.2, (2) has at most 2 periodic solutions in \( U_2 \). Based on the above, (2) has at most 2 non-zero periodic solutions.

To show the example with sharp upper bound, let

\[
S(x, t) = A(t)x^m + B(t)x^n + C(t)x = -x^m + \frac{2^{m-1}-1}{2^{n-1}-1} x^n - \frac{2^{m-1}-2^{n-1}}{2^{n-1}-1} x, \quad \lambda = 4.
\]  

(25)

We can verify that \( B(t) > 0, C(t) < 0 \) and

\[
(A(t)^4m-n + B(t))^4n-1 = -4^{n-1} + 4^n \frac{2^{(m-1)}-1}{2^{(n-1)}-1}
\]

\[
= \frac{2^{2(n-1)+m-1}(-2^{(m-n)}+2^{(m-n)-(n-1)}+1)-2^{2(n-1)}}{2^{(n-1)}-1} < 0.
\]

Therefore, \( S(4, t) = (A(t)^4m-n + B(t))^4 + 4C(t) \leq 0, (A(t)^4m-n + B(t)) \cdot 4 \cdot C(t) > 0 \) and \((A(t)^4m-n + B(t)) \int_0^1 B(t) \, dt < 0\). Observe that \( S(1, t) = S(2, t) = 0 \), (2) has 2 periodic solutions \( x = 1, x = 2 \), and the upper bound is reached.

Statement (ii.b) is obtained. \( \Box \)

**Proof of statement (iii) of Theorem 1.3.** By assumption the periodic solutions of (2) only appear in \( \mathbb{R} \setminus \{0, \pm \lambda\} = \bigcup_i V_i \). Moreover, Hypotheses (H.1) and (H.2) tell us that \((A(t)\lambda^{m-n} + B(t)) \cdot C(t) \geq 0(\leq 0)\)
and \(|(A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| - |C(t)| \neq 0\). Noting that \(m\) and \(n\) are even, the transformation \((x, t) \mapsto (-x, 1-t)\) changes (2) into

\[
\frac{dx}{dt} = \tilde{A}(t)x^m + \tilde{B}(t)x^n + \tilde{C}(t)x = A(1-t)x^m + B(1-t)x^n - C(1-t)x.
\]

Clearly,

\[
(\tilde{A}(t)\lambda^{m-n} + \tilde{B}(t)) \cdot \tilde{C}(t) = -(A(1-t)\lambda^{m-n} + B(1-t)) \cdot C(1-t),
\]

\[
|(\tilde{A}(t)\lambda^{m-n} + \tilde{B}(t))\lambda^{n-1}| - |\tilde{C}(t)| = |(A(1-t)\lambda^{m-n} + B(1-t))\lambda^{n-1}| - |C(1-t)|.
\]

Hence in the following proof we suppose

\[
(A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0,
\]

without loss of generality.

(iii.a) Firstly since \(|(A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| > |C(t)|\), we have

\[
0 \leq \left( \frac{-C(t)}{A(t)\lambda^{m-n} + B(t)} \right)^{1/(n-1)} < |\lambda|.
\]

Together with Proposition 2.5, we obtain

\[
\text{sgn}(I_5(|\lambda|, t)) = \lim_{x \to +\infty} \text{sgn}(I_5(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]

\[
\text{sgn}(I_5(-|\lambda|, t)) = \lim_{x \to -\infty} \text{sgn}(I_5(x, t)) = -\text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]

\[
\text{sgn}(I_5(0, t)) = \text{sgn}(I_5\left(\frac{-C(t)}{A(t)\lambda^{m-n} + B(t)}\right)^{1/(n-1)}, t)) = -\text{sgn}(C(t)).
\]

Thus, by statement (i) of Proposition 2.3, the sign of \(I_5(x, t)\) is fixed in \(V_i \times [0, 1]\) with \(i = 1, 2, 4,\) and is unknown in \(V_3 \times [0, 1]\). This implies that (2) has at most 1 periodic solution in \(V_1, V_2,\) and \(V_4,\) respectively, and we need the sign of \(P_5(x, t)\) in \(V_3 \times [0, 1]\). Since a calculation shows that \(\text{sgn}(P_5(0, t)) = -\text{sgn}(C(t))\) and \(\text{sgn}(P_5(-|\lambda|, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)),\) it follows from statement (iii) of Proposition 2.3 that \(P_5(x, t)\) does not change sign in \(V_3 \times [0, 1].\) As a result, (2) has at most 2 periodic solutions in \(V_3\) from (11) and Proposition 2.2.

Now we compare (2) (i.e. (20)) with (18) in \((V_1 \cup V_4) \times [0, 1]\) and with (19) in \(V_3 \times [0, 1],\) respectively. Noting that \(n-1\) is even and

\[
\text{sgn}(A(t)\lambda^{m-n} + B(t))x^n + C(t)x = \text{sgn}(x^n) \cdot \text{sgn}(A(t)\lambda^{m-n} + B(t)), \quad x \in V_1 \cup V_4,
\]

\[
\text{sgn}\left(\frac{x}{\lambda^{m-n}}[(A(t)\lambda^{m-n} + B(t))\lambda^{n-1} + C(t)\lambda^{m-n}]ight) = \text{sgn}(A(t)\lambda^{m-n} + B(t)), \quad x \in V_3,
\]

the statements below hold by Proposition 2.6.

(a) (2) has no periodic solution in \(V_1 \cup V_4\) if \((A(t)\lambda^{m-n} + B(t)) \int_0^1 A(t) dt \geq 0.\)

(b) (2) has no periodic solution in \(V_3\) if \((A(t)\lambda^{m-n} + B(t)) \int_0^1 B(t) dt \geq 0.\)
On the other hand, since \(|A(t)\lambda^{m-n} + B(t)| > 0\), we obtain

\[
\left( A(t)\lambda^{m-n} + B(t) \right) \int_{0}^{1} \left( A(t)\lambda^{m-n} + B(t) \right) dt > 0.
\]

Observe that \(m - n\) is even and \(\lambda^{m-n} > 0\), either statement (a) or statement (b) holds. Based on the above, (2) has at most 3 non-zero periodic solutions.

Furthermore, statement (iii) of Theorem 1.2 with \(l = 1\) in (2) is a particular case in this statement (iii.a), which implies that the upper bound is sharp.

(iii.b) From the above argument, \(|(A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| < |C(t)|\) holds if (2) does not satisfy the condition in statement (iii.a). So firstly we have

\[
|C(t)| > 0, \quad \left( \frac{-C(t)}{A(t)\lambda^{m-n} + B(t)} \right)^{1/(n-l)} > |\lambda|.
\]

Combining statement (i) of Proposition 2.5, we get

\[
\text{sgn}(I_5(0, t)) = \lim_{x \to +\infty} \text{sgn}(I_5(x, t)) = -\text{sgn}(C(t)),
\]

\[
\text{sgn}(I_5(\pm |\lambda|, t)) = \lim_{x \to -\infty} \text{sgn}(I_5(x, t)) = \text{sgn}(C(t)).
\]

Therefore, statement (i) of Proposition 2.3 tells us that \(I_5(x, t)\) has fixed sign in \(V_4 \times [0, 1]\), but changes signs in \(V_2 \times [0, 1], \ i = 1, 2, 3\). This implies that (2) has at most 1 periodic solution in \(V_4\), and the sign of \(P_5(x, t)\) in \(V_i \times [0, 1]\) is needed, \(i = 1, 2, 3\). Following Proposition 2.5,

\[
\text{sgn}(P_5(0, t)) = \text{sgn}(P_5(\pm |\lambda|, t)) = \lim_{x \to +\infty} \text{sgn}(P_5(x, t)) = -\text{sgn}(C(t)).
\]

Hence, \(P_5(x, t)\) keeps the sign in each \(V_i \times [0, 1]\) from statement (iii) of Proposition 2.3, \(i = 1, 2, 3\). As a result, (2) has at most 2 periodic solutions in \(V_1, V_2\) and \(V_3\), respectively.

Now we compare (2) (i.e. (20)|\(l=1\)) with (18)\(|l=1\) in \((V_2 \cup V_4) \times [0, 1]\) and with (19)\(|l=1\) in \(V_3 \times [0, 1]\), respectively. Noting that \(n - 1\) is odd, from Lemma 2.4 we have

\[
\text{sgn}\left(\left( A(t)\lambda^{m-n} + B(t) \right)x^n + C(t)x \right) = \text{sgn}(x) \cdot \text{sgn}(C(t)) \neq 0, \quad x \in V_2 \cup V_4,
\]

\[
\text{sgn}\left( \frac{x}{\lambda^{m-n}} \left[ (A(t)\lambda^{m-n} + B(t))x^{n-1} + C(t)\lambda^{m-n} \right] \right) = -\text{sgn}(C(t)) \neq 0, \quad x \in V_3.
\]

Therefore the statements below hold by Proposition 2.6.

(c) (2) has no periodic solution in \(V_2 \cup V_4\) if \(C(t) \int_{0}^{1} A(t) dt \leq 0\).
(d) (2) has no periodic solution in \(V_3\) if \(C(t) \int_{0}^{1} B(t) dt \leq 0\).

On the other hand, since \(|A(t)\lambda^{m-n} + B(t)| > 0\) and \((A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0\), it follows that

\[
C(t) \int_{0}^{1} \left( A(t)\lambda^{m-n} + B(t) \right) dt \leq 0.
\]
Recalling that \( m - n \) is even and \( \lambda^{m-n} > 0 \), either statement (c) or statement (d) holds. Based on the above, (2) has at most 5 non-zero periodic solutions.

The proof of statement (iii) of Theorem 1.3 is finished. □

**Proof of statement (iv) of Theorem 1.3.** (iv.a) First we know \( n - 1 \) is even. According to assumption and statement (ii) of Proposition 2.3, \( I_5(x,t) \) is monotonic with respect to \( x \) in \((-\infty,0) \times [0,1] \) and \((0, +\infty) \times [0,1] \). Using Proposition 2.5 we have

\[
\begin{align*}
\sgn(I_5(0,t)) &= -\sgn(\lambda) \cdot \sgn(C(t)), \\
\sgn(I_5(\lambda,t)) &= \sgn(\lambda) \cdot \sgn(A(t)\lambda^{m-n} + B(t)), \\
\lim_{x \to +\infty} \sgn(I_5(x,t)) &= \sgn(A(t)\lambda^{m-n} + B(t)), \\
\lim_{x \to -\infty} \sgn(I_5(x,t)) &= -\sgn(A(t)\lambda^{m-n} + B(t)).
\end{align*}
\]

As a result, the sign of \( I_5(x,t) \) is fixed in \( U_1 \times [0,1] \) and \( U_3 \times [0,1] \), but is not known in \( U_2 \times [0,1] \). This implies that (2) has at most 1 periodic solution in \( U_1 \) and \( U_3 \) from Proposition 2.2, respectively. Furthermore, by Proposition 2.5 and Lemma 2.4 we have

\[
\sgn(P_5(0,t)) \cdot \sgn(P_5(\lambda,t)) = \sgn(C(t)) \cdot \sgn((A(t)\lambda^{m-n} + B(t))\lambda^{n-1}) \geq 0.
\]

Thus statement (iii) of Proposition 2.3 tells us that \( P_5(x,t) \) keeps the sign in \( U_2 \times [0,1] \). Applying (11) and Proposition 2.2, Eq. (2) has at most 2 periodic solutions in \( U_2 \).

Now we compare (2) (i.e. (20)) with (18). Observe that \( n - 1 \) is even, we have

\[
\sgn((A(t)\lambda^{m-n} + B(t))x^n + C(t)x) = \sgn(x) \cdot \sgn(A(t)\lambda^{m-n} + B(t)) \neq 0, \quad x \neq 0.
\]

Hence the statements below hold by Proposition 2.6.

1. (2) has no periodic solution in \( U_1 \) if \( (A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot \int_0^1 A(t) \, dt > 0 \).
2. (2) has no periodic solution in \( U_2 \cup U_3 \) if \( (A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot \int_0^1 A(t) \, dt < 0 \).
3. (2) has no periodic solution in \( U_1 \cup U_2 \cup U_3 \) if \( (A(t)\lambda^{m-n} + B(t)) \cdot \lambda \cdot \int_0^1 A(t) \, dt = 0 \).

As a result, (2) has at most 3 non-zero periodic solutions. Noting that statement (iv) of Theorem 1.2 with \( l = 1 \) in (2) is a particular case in this statement (iv.a), the upper bound is sharp.

Statement (iv.b) is obtained.

(iv.b) Since \( (A(t)\lambda^{m-n} + B(t))\lambda^{n-1} + C(t) \neq 0 \) from Hypothesis (H.2) and \( (A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0 \), we obtain \( |(A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| > |C(t)| \neq 0 \). In what follows we prove in two cases.

Case 1. \( |(A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| > |C(t)| \).

First for \( x \in U_1 \), we get

\[
|\left((A(t)\lambda^{m-n} + B(t))x^{n-1}\right)| > |\left((A(t)\lambda^{m-n} + B(t))\lambda^{m-n}\right)| > |C(t)\lambda^{m-n}|.
\]

Recalling that \( (A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0 \) and both \( m - 1 \) and \( m - n \) are odd, for \( x \in U_3 \) we have

\[
((A(t)\lambda^{m-n} + B(t))x^{n-1}) \cdot (C(t)\lambda^{m-n}) = \left((A(t)\lambda^{m-n} + B(t))C(t)\right) \cdot (x^{m-1}\lambda^{m-n}) \geq 0.
\]

Therefore, it follows from Lemma 2.4, (26) and (27) that
\[
\sgn((A(t)\lambda^{m-n} + B(t))x^{m-1} + C(t)\lambda^{m-n}) = \sgn((A(t)\lambda^{m-n} + B(t))x^{m-1}), \quad x \in U_1 \cup U_3.
\]

This implies that

\[
\sgn\left(\frac{x}{\lambda^{m-n}}\left[(A(t)\lambda^{m-n} + B(t))x^{m-1} + C(t)\lambda^{m-n}\right]\right)
= \sgn\left(\frac{x}{\lambda^{m-n}}\right) \cdot \sgn((A(t)\lambda^{m-n} + B(t))x^{m-1})
= \sgn\left(\frac{x^m}{\lambda^{m-n}}\right) \cdot \sgn(A(t)\lambda^{m-n} + B(t))
= \sgn(\lambda) \cdot \sgn(A(t)\lambda^{m-n} + B(t)), \quad x \in U_1 \cup U_3.
\]

As a result, if we compare (2) (i.e. (20)) with (19), then together with \( (A(t)\lambda^{m-n} + B(t)) \int_0^1 B(t) dt \leq 0 \) and Proposition 2.6, Eq. (2) has no periodic solution in \( U_1 \cup U_3 \).

Second according to assumption and Proposition 2.5,

\[
\sgn(I_5(0, t)) = \sgn\left(I_5\left(\pm \left(\frac{-C(t)}{A(t)\lambda^{m-n} + B(t)}\right)^{1/(n-1)}, t\right)\right) = -\sgn(\lambda) \cdot \sgn(C(t)),
\]

\[
\sgn(I_5(\lambda, t)) = \sgn(\lambda) \cdot \sgn(A(t)\lambda^{m-n} + B(t)).
\]

Using Proposition 2.3, \( I_5(x, t) \) does not change sign in \( U_2 \times [0, 1] \). Therefore Proposition 2.2 tells us that (2) has at most 1 periodic solution in \( U_2 \). As a result, (2) has at most 1 non-zero periodic solution.

Case 2. \(|(A(t)\lambda^{m-n} + B(t))\lambda^{n-1}| < |C(t)|\).

Similar to case 1, for \( x \in U_2 \) we have

\[
0 < |(A(t)\lambda^{m-n} + B(t))x^{m-1}| < |(A(t)\lambda^{m-n} + B(t))\lambda^{m-1}| < |C(t)\lambda^{m-n}|.
\]

Combining Lemma 2.4 and \( (A(t)\lambda^{m-n} + B(t)) \cdot C(t) \leq 0 \) in assumption,

\[
\sgn((A(t)\lambda^{m-n} + B(t))x^{m-1} + C(t)\lambda^{m-n})
= \sgn(C(t)\lambda^{m-n})
= -\sgn((A(t)\lambda^{m-n} + B(t))x^{m-1})
= \sgn(-\lambda \lambda) \cdot \sgn((A(t)\lambda^{m-n} + B(t))x^{m-1}), \quad x \in U_2.
\]

On the other hand, observe that (27) still holds for \( x \in U_3 \), we get

\[
\sgn((A(t)\lambda^{m-n} + B(t))x^{m-1} + C(t)\lambda^{m-n})
= \sgn((A(t)\lambda^{m-n} + B(t))x^{m-1})
= \sgn(-\lambda \lambda) \cdot \sgn((A(t)\lambda^{m-n} + B(t))x^{m-1}), \quad x \in U_3.
\]

Thus,
\[
\text{sgn}\left(\frac{x}{\lambda^{m-n}}\left(\left(A(t)\lambda^{m-n} + B(t)\right)x^{m-1} + C(t)\lambda^{m-n}\right)\right)
\]
\[
= \text{sgn}\left(\frac{x}{\lambda^{m-n}}\right) \cdot \text{sgn}(-x\lambda) \cdot \text{sgn}\left(\left(A(t)\lambda^{m-n} + B(t)\right)x^{m-1}\right)
\]
\[
= \text{sgn}\left(\frac{\lambda^{m-n}}{\lambda^{m-n-1}}\right) \cdot \text{sgn}(-x) \cdot \text{sgn}\left(A(t)\lambda^{m-n} + B(t)\right)
\]
\[
= \text{sgn}(-x) \cdot \text{sgn}\left(A(t)\lambda^{m-n} + B(t)\right), \quad x \in U_2 \cup U_3.
\]

If we compare (2) (i.e. (20)\(\text{for } i = 1\)) with (19)\(\text{for } i = 1\), then (2) has no periodic solution in \(U_2 \cup U_3\) from \(\int_0^1 B(t) dt \leq 0\) and Proposition 2.6.

Since \(A(t)\lambda^{m-n} + B(t) \cdot C(t) \leq 0\) and Proposition 2.5 tells us that
\[
\text{sgn}(I_5(\lambda, t)) = \text{sgn}(\lambda) \cdot \text{sgn}(C(t)),
\]
\[
\lim_{x \to +\infty} \text{sgn}(I_5(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]
\[
\lim_{x \to -\infty} \text{sgn}(I_5(x, t)) = -\text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]
the sign of \(I_5(x, t)\) in \(U_1 \times [0, 1]\) is not known. So we consider the sign of \(P_5(x, t)\). Observe that \(P_5(x, t)\) is monotonic with respect to \(x\) in \(U_1 \times [0, 1]\), applying Proposition 2.5 we get
\[
\text{sgn}(P_5(0, t)) = -\text{sgn}(\lambda) \cdot \text{sgn}(C(t)),
\]
\[
\lim_{x \to +\infty} \text{sgn}(P_5(x, t)) = \text{sgn}(A(t)\lambda^{m-n} + B(t)),
\]
\[
\lim_{x \to -\infty} \text{sgn}(I_5(x, t)) = -\text{sgn}(A(t)\lambda^{m-n} + B(t)).
\]

Hence, \(P_5(x, t)\) keeps the sign in \(U_1 \times [0, 1]\), which means that (2) has at most 2 periodic solutions in \(U_1\) by (11) and Proposition 2.2. As a result, (2) has at most 2 non-zero periodic solution.

Based on the above, the assertion of (iv,b) holds, and the proof of (iv) of Theorem 1.3 is finished. \(\square\)

4. Application on Abel equations

In this section we suppose that (2) is an Abel equation, and use the results above to study it. Firstly the proof of Corollary 1.4 is given below.

Proof of Corollary 1.4. (i) Since \((A(t)\lambda^2 + B(t)\lambda + C(t)) \cdot C(t) < 0\), we have \(C(t) \neq 0\). Without loss of generality, suppose that \(C(t) > 0\). Then \((A(t)\lambda + B(t))\lambda < -C(t) < 0\), which means \(|(A(t)\lambda + B(t))\lambda| > |C(t)| > 0\) and \((A(t)\lambda + B(t)) \cdot \lambda \cdot C(t) < 0\). According to statement (ii) of Theorem 1.2, statement (i) holds.

(ii) Similarly we assume \((A(t)\lambda^2 + B(t)\lambda) > 0\), without loss of generality. Then \(C(t) < -(A(t)\lambda + B(t))\lambda < 0\) from assumption, which implies \(|C(t)| > |(A(t)\lambda + B(t))\lambda| > 0\) and \((A(t)\lambda + B(t)) \cdot \lambda \cdot C(t) < 0\). Using statement (ii.a) of Theorem 1.3, we obtain statement (ii). \(\square\)

Secondly assume that \(\gamma = (a(t), t)\) is a smooth curve which lies in \((\mathbb{R}\setminus\{0\}) \times [0, 1]\) with \(a(0) = a(1)\). We know that (2) has an invariant form under the transformation \(y = x/a(t)\). This is trivial because the transformation changes (2) into
\[
\frac{dy}{dt} = \hat{S}(x, t)
\]
\[
= \hat{A}(t)y^3 + \hat{B}(t)y^2 + \hat{C}(t)y
\]
\[
= A(t)a^2(t)y^3 + B(t)a(t)y^2 + \left(C(t) - \frac{\dot{a}(t)}{a(t)}\right)y. \quad (28)
\]

Now consider the vector fields \(v_S = (S(x, t), 1)\) and \(v_C = (C(t)x, 1)\). One can check that
\[
(v_S \land v_C)|_y = a(t)(\hat{A}(t) + \hat{B}(t)), \quad (v_S \land \dot{y})|_y = a(t)\hat{S}(1, t),
\]
\[
(v_C \land \dot{y})|_y = a(t)\hat{C}(t), \quad (29)
\]
where \(\land\) is the outer product. As a result, Abel equation (28) satisfies Hypotheses (H.1) and (H.2) for \(\lambda = 1\), if and only if (2) satisfies the following hypotheses.

(H.1') \(v_S\) is transverse to \(v_C\) on \(y\), and \((v_C \land \dot{y})|_y \geq 0(\leq 0)\).

(H.2') \(v_S\) is transverse to \(\dot{y}\) on \(y\).

**Proof of Corollary 1.5.** According to the discussion above, we transform (2) into (28).

(i) From Eq. (29), the assumption implies
\[
(v_S \land v_C)|_y \cdot (v_C \land \dot{y})|_y = a^2(t) \cdot \hat{S}(1, t) \cdot \hat{C}(t) < 0,
\]
i.e. \(\hat{S}(1, t) \cdot \hat{C}(t) \cdot 1 < 0\). Thus, by using statement (i) of Corollary 1.4, (28) has at most 2 non-zero periodic solutions, and the upper bound is sharp. Noting that \(a(0) = a(1)\), Eqs. (2) and (28) have the same number of non-zero periodic solutions. Statement (i) is valid.

(ii) It is the same argument to (i). By assumption we get
\[
(v_S \land \dot{y})|_y \cdot (v_S \land v_C)|_y = a^2(t) \cdot \hat{S}(1, t) \cdot (\hat{A}(t) + \hat{B}(t)) < 0,
\]
i.e. \(\hat{S}(1, t) \cdot (\hat{A}(t) + \hat{B}(t)) < 0\). Hence, statement (ii) of Corollary 1.4 tells us that (28) and (2) have at most 4 non-zero periodic solutions. Statement (ii) holds. \(\Box\)

**Remark 4.1.** Clearly, Hypotheses (H.1') and (H.2') are much weaker than Hypotheses (H.1) and (H.2), and they do not refer to the signs of the coefficients \(A(t), B(t)\) and \(C(t)\). Noting that for general equation \((2)\) with \(l = 1\), transformation \(y = x/a(t)\) also changes it into a new equation which has the same form. Therefore, using a similar discussion above, Hypotheses (H.1) and (H.2) can be reduced to some weaker transversality conditions, and Theorem 1.3 is extended. In this point of view, it seems that the number of isolated periodic solutions of (2) is able to be controlled by enough transversality conditions.

**Proof of the statements of Example 1.6.** Let
\[
a(t) = -(b_1 + b_3 \cos(2\pi t) + b_4 \sin(2\pi t)),
\]
\[
S(x, t) = \cos(2\pi t)x^3 + (b_0 + b_1 \cos(2\pi t) + b_2 \sin(2\pi t) + b_3 \cos^2(2\pi t) + b_4 \cos(2\pi t) \sin(2\pi t))x^2 + c_0x.
\]

Firstly we have \(a(t) \neq 0\) by assumption, and \(a(0) = a(1)\). In addition, a direct calculation shows that
\[ S(a(t), t) - \dot{a}(t) = \left[ (b_0 + b_2 \sin(2\pi t)) \cdot a(t) + c_0 - \frac{\dot{a}(t)}{a(t)} \right] \cdot a(t), \]
\[ S(a(t), t) - c_0 a(t) = (b_0 + b_2 \sin(2\pi t)) \cdot a^2(t). \]  
(30)

(i) Since
\[ \max_{t \in [0, 1]} \left| \frac{\dot{a}(t)}{a(t)} \right| = \max_{t \in [0, 1]} \left| 2\pi \frac{-b_3 \sin(2\pi t) + b_4 \cos(2\pi t)}{b_1 + b_3 \cos(2\pi t) + b_4 \sin(2\pi t)} \right| = 2\pi \sqrt{\frac{b_3^2 + b_4^2}{b_1^2 - b_3^2 - b_4^2}}. \]  
(31)

we obtain from assumption that
\[ |c_0| > \left| \frac{\dot{a}(t)}{a(t)} \right|, \quad \text{sgn} \left( c_0 - \frac{\dot{a}(t)}{a(t)} \right) = \text{sgn}(c_0), \quad 2|c_0| > \left| c_0 - \frac{\dot{a}(t)}{a(t)} \right|. \]

Furthermore,
\[ |(b_0 + b_2 \sin(2\pi t)) \cdot a(t)| > (|b_0| - |b_2|) \cdot |b_1 + b_3 \cos(2\pi t) + b_4 \sin(2\pi t)| \]
\[ > (|b_0| - |b_2|) \cdot (|b_1| - \sqrt{b_3^2 + b_4^2}) \]
\[ > 2|c_0| \]
\[ > \left| c_0 - \frac{\dot{a}(t)}{a(t)} \right|. \]

Using \( b_1^2 > b_3^2 + b_4^2 \) and Lemma 2.4 we get
\[ \text{sgn} \left( S(a(t), t) - \dot{a}(t) \right) = \text{sgn} \left( (b_0 + b_2 \sin(2\pi t)) \cdot a(t) + c_0 - \frac{\dot{a}(t)}{a(t)} \right) \]
\[ = \text{sgn} \left( (b_0 + b_2 \sin(2\pi t)) \cdot a(t) \right) \]
\[ = \text{sgn} \left( b_0 + b_2 \sin(2\pi t) \right) \cdot \text{sgn}(a(t)) \]
\[ = \text{sgn}(b_0) \cdot \text{sgn}(-b_1), \]

which implies
\[ \text{sgn} \left( S(a(t), t) - \dot{a}(t) \right) \cdot \text{sgn}(c_0 a(t) - \dot{a}(t)) = -1. \]

As result, \((S(x, t), t) \) and \((c_0 x, t) \) are transverse to \((\dot{a}(t), 1) \) on curve \((a(t), t) \) in opposite directions. According to statement (i) of Corollary 1.5, (3) has at most 2 non-zero periodic solutions. Statement (i) is obtained.

(ii) Following assumption and (31),
\[ |(b_0 + b_2 \sin(2\pi t)) \cdot a(t) - \frac{\dot{a}(t)}{a(t)}| < (|b_0| + |b_2|) \cdot (|b_1| + \sqrt{b_3^2 + b_4^2}) + 2\pi \sqrt{\frac{b_3^2 + b_4^2}{b_1^2 - b_3^2 - b_4^2}} \]
\[ < |c_0|. \]

Combining \( b_1^2 > b_3^2 + b_4^2 \) and (30) we have
\[
\text{sgn}(S(a(t), t) - \dot{a}(t)) = \text{sgn}(c_0) \cdot \text{sgn}(-b_1), \quad \text{for } a(t) \neq 0
\]
\[
\text{sgn}(S(a(t), t) - c_0 a(t)) = \text{sgn}(b_0).
\]
which implies
\[
\text{sgn}(S(a(t), t) - \dot{a}(t)) \cdot \text{sgn}(S(a(t), t) - c_0 a(t)) = -1.
\]

Hence, \((\dot{a}(t), 1)\) and \((c_0 x, t)\) are transverse to \((S(x, t), t)\) on curve \((a(t), t)\) in opposite directions. According to statement (ii) of Corollary 1.5, (3) has at most 4 non-zero periodic solutions. Statement (ii) holds. \(\square\)

**Remark 4.2.** Example 1.6 cannot be obtained in a straightforward way from Theorem 1.2 and Theorem 1.3. In fact, consider a particular case
\[
\frac{dx}{dt} = A(t)x^2 + B(t)x^2 + C(t)x = \cos(2\pi t)x^3 + (1 - 10\cos(2\pi t) - 2\cos(2\pi t))x^2 - 2x.
\]
Then for arbitrary \(\lambda \in \mathbb{R}\),
\[
A(0)\lambda + B(0) = \lambda - 11, \quad A(1/4)\lambda + B(1/4) = 1, \quad A(1/2)\lambda + B(1/2) = -\lambda + 9.
\]
Noting that \(A(0)\lambda + B(0) + A(1/2)\lambda + B(1/2) = -2 < 0\), either \(A(0)\lambda + B(0)\) or \(A(1/2)\lambda + B(1/2)\) is negative. Therefore, \(A(t)\lambda + B(t)\) changes sign for arbitrary fixed \(\lambda \in \mathbb{R}\), which means that Hypothesis (H.1) does not hold. However, one can check that this equation satisfies the condition of statement (i) of Corollary 1.5. Hence its number of non-zero periodic solutions is bounded and less than 2.

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**References**


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