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Research Progress on the Kelly Game

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Abstract

The growth-optimal portfolio optimization strategy has been investigated in many ways since firstly pioneered by Kelly. This paper firstly introduces the research progress of this so-called Kelly game. Based on the original Kelly game the optimality is shortly proofed. Especially generalized research is introduced such as the relation between M-V approach and the Kelly approach, the question of diversification, the influence of transaction fees and limited information, etc. Then the application of Kelly strategy is discussed with some conclusions.

Keywords: Kelly game; Long run investing; Portfolio optimization; Application

1. Introduction

Portfolio optimization is one of the main topics in finance. It aims at maximizing the return of the investment while simultaneously minimizing the risk which caused by unfavorable events. Modern portfolio theory(see[1-2] for a review) is mostly based on Mean-Variance approach which was first introduced by Markowitz in his paper "portfolio selection" [3]. This M-V approach minimizes the variance of the portfolio under the constraint of a fixed expected return value. In this paper we focus on a different approach put forward by Kelly[4] from a long run prospect.

Kelly first proposed "investment for the long run", which was developed by Markowitz [5], Latane[6-7], and Breiman[8-9]. It is concerned with an investor who reinvests his portfolio each turn to maximum growth of wealth over the indefinitely long run, and never draws nor deposits new cash to the portfolio in this process. As Markowitz said in his paper[10]" In the long run, a penny invested at 6.01% is better-eventually becomes and stays greater than a million dollars invested at 6%". Based on this hypothesis, the approach focused in this paper mainly considers this question: in gambling games the fundamental problem for a gambler is to find positive expectation betting opportunities, while the problem in investing for an investor is analogous and just more complex. After finding investments with excess risk-adjusted expected returns, the investor (or we can call him a gambler) must decide how much of his capital to bet, to make his wealth have an optimal growth.

This question was first discussed by Daniel Bernoulli[11] in about 1730 in connection with the St. Petersburg game. Later it was studied by many economists and any others, giving some different ideas. One approach is to choose a goal, such as minimize the probability of total loss within a specified number of trials. Another example would be to maximize the probability of reaching a fixed goal on or before N trials[12]. A different approach much studied by many economists is to use a utility function as a criteria to value money. Once a utility function is specified, the object is to maximize the expected value of the utility of wealth. Daniel Bernoulli used the utility function logx to "solve" the St. Petersburg Paradox much earlier. Being similar to this utility function approach, Kelly proposed another function i.e. the average exponential growth rate $E\ln W$ to be the criterion. However, the quantity $E\ln W$ here is not a logarithmic utility function. Once been put forward, this so-called growth-optimal or Kelly portfolio has been shown to be optimal according to various criteria[9, 13] and generalized in different ways[10, 14-17] . In this paper, we will turn to study the research progress of the Kelly game, to summarize some interesting research findings and to make it a starting point of the future work. The paper is organized as follows. In next chapter we will introduce the basic original model which proposed by Kelly and give a short proof about its optimality. In chapter 3 some new investigations and generalized research will be introduced. Then the application of Kelly approach will be discussed with some conclusion. Finally it is the conclusion and some discussion about the future work.

2. Short summary of the origin Kelly game

For simplification a gambling game will be modeled for introducing the Kelly approach which is also called "geometric mean maximizing portfolio strategy" ," the growth- optimal strategy"," the capital growth criterion", etc. by economists and financial

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theorists. In financial market the case the investor faces is almost the same as the one of a gambler but more complicated. Consider a situation where an investor (gambler) with an initial wealth W_n is allowed to repeatedly invest into a risky assert. In each turn with the probability p his capital is doubled and with the complementary probability $1 - p$ his capital is lost. Besides the winning probability p is constant and known by the investor. Our purpose here is to illustrate the essential framework through simplest examples thus risk-free interest rate (set 0 here), asset's dividends, and transaction costs are ignored. If we define W_i as the invested wealth and W_r as the resulting wealth, the game return would be $R= W_r\text{-}W_i/W_i$. Apparently in this game the return per turn is +1 with probability p and -1 with probability $1\text{-}p$.

In the long run the investor must decide how much of his wealth to bet to get the best growth of his wealth. Suppose the fraction is f_k which means he bets f_k of his total wealth on the k-th turn. Since properties of the risky game do not change in time, the investor bets the same fraction f of the actual wealth in each turn, i.e. $f_k = f$. Then the question is how can we choose an appropriate f. Straightforward maximization of $\langle W_N \rangle = W_0 \langle 1+fR_i \rangle^N = W_0[1+(2p\text{-}1)f]^N$ can be used to optimize the investment, however, we can easily justify this expected return is not a good criterion. ($\langle\cdot\rangle$ here is labeled as the average)

Since for $p<1/2$, $\langle W_N \rangle$ is a decreasing function of f, the optimal strategy is to refrain from investing, $f^* = 0$. By contrast, for $p>1/2$, the optimal strategy is $f^* = 1$ as $\langle W_N \rangle$ increases with f. At this time if we win all the time, the wealth will be $2W_o$, $4W_o$, $8W_o$…, but once we lose one turn we will get ruined. The probability of getting ruined is $1 \text{-}P^N$ after N turns and when $N\rightarrow\infty$ we can see the investor bankrupts inevitably. Likewise, if we play to minimize the probability of eventual ruin, the well-known gambler's ruin formula in[18] shows that we minimize ruin by making a minimum bet on each trial, but this unfortunately also minimizes the expected gain. Thus "timid" betting is also unattractive. This suggests an intermediate strategy which is somewhere between maximizing $\langle W_N \rangle$ and minimizing the probability of ruin. An asymptotically optimal strategy was firstly proposed by Kelly[4] where he proposed to use maximizing the long-term growth rate G of the investor's capital which was defined by $W_N = W_0\exp[GN]$, thus its limit value has the form：

$$G = \lim_{n\to\infty} \frac{1}{N} \log_2 \frac{W_N}{W_0} \tag{1}$$

as a criterion for a long run investment, where W_0 and W_N represent the initial wealth and the wealth after N terns respectively. And

$$W_N = W_0 \prod_{i=1}^{N} (1 + fR_i) = W_0 (1+f)^S (1-f)^F \tag{2}$$

where $S+F=N$, S represents the number of winning turns and F represents the number of losing turns. Without affecting the results, in our analysis we use natural logarithms. Due to the multiplicative character of W_N and the law of large numbers, G can be rearranged as

$$G = \lim_{n\to\infty} \frac{1}{N} \sum_{i=1}^{N} \ln(1+fR_i) = \langle \ln(1+fR) \rangle = \langle \ln W_1 \rangle \tag{3}$$

Notice that while we investigate repeated investments, wealth W_1 after turn step plays a prominent role in the optimization.

The long-term profitability of the risky asset can be measured by the average return per time step, $\Re$. By definition $W_N = W_0(1+f\ \Re_N)^N$, its limitation $\Re = lim_{N\to\infty}\Re_N$, from Eq.(1), it can be expressed in terms of G simply as:

$$\Re = \exp(G) - 1 \tag{4}$$

From this point we can see that maximizing G and maximizing the average return per time step $\Re$ is not different actually.

In this origin Kelly game, it is easy to figure out $G = \langle \ln W_1 \rangle = p\ln(1+f)+(1-p)\ln(1-f)$. To get the maximum of G, note that $G'(f) = 0$, we can see the optimal fraction is:

$$f*(p) = 2p - 1 \tag{5}$$

While $p < 0.5$, $f^* < 0$, a short selling is suggested and while $p > 1$ it means the investor should borrow additional money. The non-trivial investment which means $0 < f^* < 1$ occurs while $0.5 < p < 1$ and at this time using $\Re = \exp(G) - 1$ the maximum of $\Re$ can be written as

$$\Re(p) = 2p^p(1-p)^{1-p} - 1 \tag{6}$$

We can see when $p = 1/2$, $\Re = 0$; when $p \rightarrow 1$, $\Re = 1$. Also

$$G''(f) = -\frac{p}{(1+p)^2} - \frac{1-p}{(1-f)^2} < 0 \tag{7}$$

So G' is monotone strictly decreasing on [0,1), and $G'(0) = 2p - 1$, $\lim_{f\to 1} G'(f) = -\infty$. Therefore by the continuity of G', $G(f)$ has a unique maximum at $f = f^*$, where $G(f^*) = p\ln p+(1-p)\ln(1-p)+\ln 2>0$. Moreover, $G(0) = 0$ and $\lim_{f\to 1} G(f) = -\infty$, so there is a unique number $f = f_c$, where $0< f^*< f_c<1$,such that $G(f_c) = 0$. When $f>f_c$, G would be negative.

As this Kelly strategy has been shown optimal according to various criteria, it will be shown a shortly proof of the optimality here by a theorem[19]:

i). Given a strategy Φ^* which maximizes $E\log X_N$, and any other "essentially different" strategy Φ, then $\lim_{N\to\infty} X_N(\Phi^*)/X_N(\Phi)=\infty$ almost surely.

ii). The expected time for the current X_N to reach any fixed preassigned goal C is, asymptotically, least with a strategy which maximizes $E\log X_N$.

The theorem shows that the Kelly strategy of maximizing $E\log X_N$, is asymptotically optimal by two important criteria and the "essentially different" strategy is one such that the difference $E\log X_N^* - E\log X_N$, between the Kelly strategy and the other strategy grows faster than the standard deviation of $\ln X_N^* - \ln X_N$, ensuring $P(\ln X_N^* - \ln X_N)>0\to 1$.The proof of the theorem can be seen in Breiman[9].

3. Generalized research

This Kelly portfolio has been investigated in details over a few different aspects. For examples, the relation between M-V approach and the Kelly approach[16], the question of diversification[20-21], the influence of transaction fees[22] and limited information[21, 23], etc. In this chapter we will try to summarize some work about the generalized research.

3.1The relation between M-V and Kelly approach

As mentioned in chapter 1, modern portfolio theory is mostly based on the M-V approach which minimizes the variance of the portfolio under the constraint of a fixed expected return value. The Kelly strategy investigates the portfolio from a different way for the long run. However, research by [16] found that there are some connections between the two approaches.

In paper[16], the author assumed a simply model which led to log-normally distributed returns, i.e. the assets price undergo uncorrelated geometric Brownian motions:

$$p_i(t) = e^{\eta_i(t)} p_i(t-1) \tag{8}$$

where $p_i(t)$ represents the price of asset i within N assents. Then they analyzed the optimal portfolio with *M-V* approach and Kelly strategy respectively, and proved when returns and volatilities of the assets are small and borrowing is forbidden, the Kelly-optimal portfolio lies on Markowitz Efficient Frontier. The results is shown in Fig. 1, the author plotted the Efficient Frontier together with the constrained Kelly portfolio for the same three assets. μ_p represents the average return, σ_p represents the volatility. The additional constraint is no short selling-SS (i.e. investment fractions are non-negative). While the original EF is not bounded (for any fix μ_p we can find an appropriate σ_p), the EF with the additional constraint starts at the point corresponding to the full investment in the least profitable asset and ends at the point corresponding to the most profitable asset. The two lines coincide on a wide range of μ_p. From the figure, we can see the Kelly portfolio lies close to the EF and the same holds in the case with the additional constraint.

Markowitz's Efficient Frontier is the line where efficient portfolios are supposed to lie in the Mean-Variance picture. In this paper using typical instead of average quantities to follow the M-V procedure such as modifying the M-V approach by replacing the averages $\langle W\rangle$ and $\langle W^2\rangle$ with the logarithm related quantities $\langle \ln W\rangle$ and $\langle(\ln W)^2\rangle$ was also considered. These are less affected by rare events and allow capturing the typical behavior of the system. As a matter of fact, they found the difference between the traditional M-V approach and the modification proposed here ("Logarithmic Efficient Frontier") is very small and does not justify the additional complexity thus induced.

Another connection of applying the M-V analysis of correlation to observe the impact of that on Kelly's optimal criterion was also investigated in[24] which will be introduced in section 3.2.

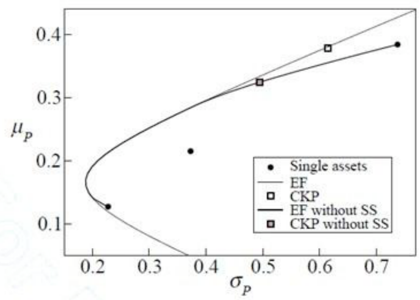


FIG 1. is from[16]: The Efficient Frontier (EF) and the constrained Kelly portfolio (CKP) in a particular case of three assets: m_1=0.1, D_1=0.04, m_2=0.15, D_2=0.09, m_3=0.2, D_3=0.25.

3.2 Diversification

Consider the origin model mentioned in chapter 2, suppose the winning probability p=0.6>0.5, it is a profitable game and the expected return is $\langle R \rangle$ =2p-1=20%. According to Eq. (5), the optimal investment fraction is f_K=2p-1=0.2, therefore from Eq.(6) the average return per time step is $\Re(p)=2p^p(1-p)^{1-p}-1$=2%. We see that a wise investor gets in the long run much less than the illusive return 20% of the given game, a naive investor gets even less. In this section we will discuss diversification and try to investigate how diversification can improve this performance.

Most portfolio strategy lead to diversification, a few research investigate diversification in Kelly approach such as [23], the question of diversification and constant rebalancing among a certain number of uncorrelated stocks was investigated. Based on the investigation of [21], we will shortly summarize the influence of diversification both with uncorrelated stocks and correlated stocks.

We first generalize the origin Kelly game to a new game with M independent assets, and the M assets are simultaneous. In other words the investor will play M independent games simultaneous. Assuming the M games are identical and the properties are fixed, therefore the fraction invested in game i is $f_i=f$ all the time. Then the investment optimization is simplified to a one-variable problem. Under this condition there is the probability $(1\text{-}p)^M$ that in one turn all the M games are losing. In consequence, for all p<1 the optimal investment fraction is smaller than $1/M$, thus $M f^*$<1, otherwise the gambler risks getting bankrupted and the chance that this happens approaches one in the long run. Assuming there are w winning games in one turn and M-w games losing, the one turn return would be (2w-M)f and the investor's wealth is multiplied by the factor (2w-M)f. Consequently, it can be seen the exponential rate is:

$$G=\langle \ln W_1 \rangle = \sum_{w=0}^{M} p(w;M,p)\ln[1+(2w-M)f] \tag{9}$$

where $P(w/L, p)=C_w^M$-$p^w(1-p)^{L-w}$ is a binomial distribution. Using dG/df=0 one can get the optimal fraction f^* , and rewrite 2w-M = [f(2w-M)+1-1]/f and use the normalization of $P(w;M,p)$, we can simplify the resulting equation to:

$$\sum_{w=0}^{M} \frac{p(w;M,p)}{f(2w-M)+1} = 1 \tag{10}$$

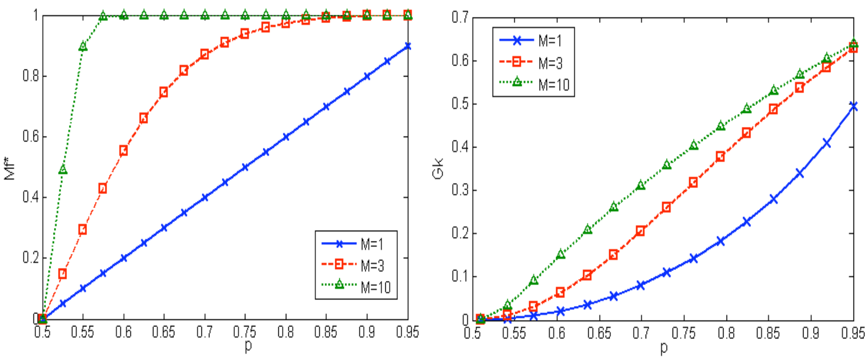
while M=1, we can get the well-known result f^* =2p-1,M=2,3,4 the solution also can be get. However, when M>5, Eq.(10) has no close solution. Using the approximations in[21] we can get approximate analysis solutions:

$$f^* = \frac{2p-1}{M(2p-1)^2+4p(1-p)} \tag{11}$$

when $p \to 1/2$ means an unsaturated portfolio and

$$f^* = \frac{2p-1}{M(2p-1)^2+4p(1-p)} \tag{12}$$

When $p \to 1$ means a saturated portfolio. The two solutions can be continuously joined if $p \in [1/2, p_c]$ the first one and $p \in (p_c, 1]$ the second one is used: the boundary value p_c is determined by the intersection of these two results. The numerical solution can be obtained by using the algorithms of[25], while a comparison of the derived approximate results with numerical solutions of Eq.(9) is shown in [21] and displayed a good agreement. Fig. 2 (a) shows the total investment fraction is not simply M times of a single game, and Fig. 2 (b) shows that diversification significantly improves the return.



(a) (b)

FIG 2. a: total investment fraction. b: critical value G of diversification

The paper also proved the superiority of diversification from a different perspective. In a noisy game where winning probability fluctuates between p-Δ and p+Δ, an informed inside investor knows the exact winning probability of one game in every turn, while an outsider only knows the time average winning probability p but he could invest on multiple games for compensation. By comparing the long-term exponential growth of both investors, the critical situation where they perform equally well is found and shown in Fig. 3. This shows the advantage of inside information (represented by delta) can be eliminated by diversification to some extent. From this perspective, the performance of diversified portfolio arising from original Kelly game could be estimated more visualized and well-founded.

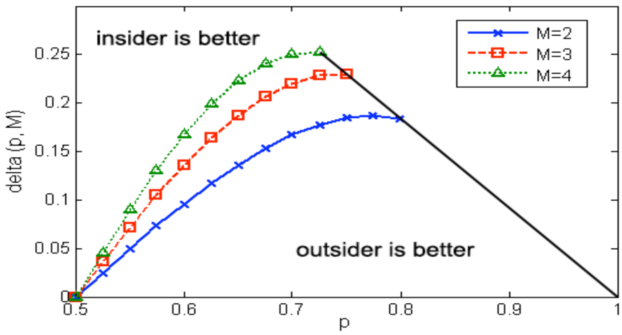


FIG 3. Critical value of the competition between diversification and information.

If assets are correlated rather than independent, the return of diversified investment will decrease. In another paper[24], the concept of “effective” portfolio size was introduced to quantify the influence of correlation.

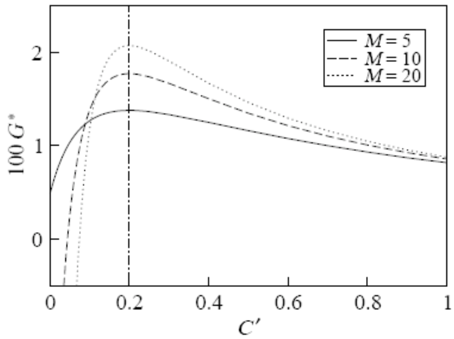


FIG4. is from [24]: G^* for an investor assuming a wrong magnitude of the asset correlations.

The above figure is from[24], which applies the M-V analysis of correlation to observe the impact of that on Kelly’s optimal criterion – the average growth rate in after each time step. From this figure, we can see “underestimation of correlations can lead to a significant reduction of investment performance”. In this work, a new quality was introduced – the effective portfolio size – to measure the influence of correlations, which showed that, the size of effective portfolio can be much smaller than that of the actual portfolio, in other words, correlation could eliminate the performance of diversification to some extent.

3.3 Transaction fees

Kelly's optimization scheme is based on the long-term prospects of the investor and requires continual rebalancing of the portfolio so that the investment fraction is kept constant. This rebalancing represents the key advantage of the Kelly portfolio over the simple buy-and-hold strategy. On the other hand, when non-zero transaction costs are imposed, investment performance may deteriorate considerably. A new paper[22] focuses on this influence of transaction fees. In this section, we intend to study the effect of non-zero transaction costs on the Kelly portfolio.

Different from the origin model, the game is modified a little here: assume that the asset price x(t) undergoes a multiplicative stochastic process：

$$x(t)=\begin{cases} x(t)(1+r) & \text{with probality } \dfrac{1}{2}+p \\ x(t)(1-r) & \text{with probality } \dfrac{1}{2}-p \end{cases} \tag{13}$$

Here r is a positive parameter ($0<r<1$) representing the rate of return or loss of the investment, $1/2+P$ is the “winning" probability and $P\in(0,1/2)$ (when $P<0$, the asset is not profitable and no investment takes place). The other parameters are the same as in chapter 2. The requirement of keeping the investment fraction f constant implies that the investor needs to constantly rebalance the portfolio: after a “winning" turn, some part of wealth has to be moved from the asset to cash and after a “loosing" turn, some additional wealth has to be invested in the asset. This constant portfolio rebalancing may require payment of substantial transaction fees. Assume that for any wealth W_T transferred from or to the risky asset, a transaction fee $\alpha/W_T/$ must be paid ($\alpha > 0$, the absolute value reflects the fact that transaction fees are paid regardless of the direction of the transfer). In this game the resulting wealth after N turns would be obtained:

$$W_N = W_0[1 + fr_1 - \frac{arf(1-f)}{1-\alpha f}]^W [1 - fr_1 - \frac{arf(1-f)}{1-\alpha(1-f)}]^{N-w} \tag{14}$$

It is straightforward to use Eq. (14) to obtain the exponential growth rate $G(f)$ and maximize it to get the optimal investment fraction. Using the approximation in[22], assuming p and r are sufficiently small, we can obtain the optimal investment fraction is:

$$f^* = \frac{2p-\alpha}{r-2\alpha} \tag{15}$$

Fig. 5 illustrates the dependency of this result on both p and α. Naturally in the limit $\alpha \to 0$ we recover the fee-free result $f = 2p/r$. Interestingly, transaction fees may both decrease and increase the optimal investment fraction (in comparison with the value corresponding to $\alpha=0$). On the other hand, the average return is always reduced by transaction fees. Solving the equation $f^*(\alpha)=0$, we can obtain a lower bound for p at which the asset becomes profitable, $p^{low}=\alpha/2$, similarly, we can solve the equation $f^*(\alpha)=1$ to obtain the upper bound for p at which the investor is advised to invest all wealth in the asset, $p^{up}=(r-\alpha)/2$. As we can see, transaction fees narrow the region where non-trivial optimal investment fractions ($0 < f < 1$) realize (this effect is well visible in Fig. 5).Transaction fees are in this sense similar to friction in mechanics which also both attenuates motion and leads to dissipation of energy (in the case of transaction fees we face dissipation of wealth).

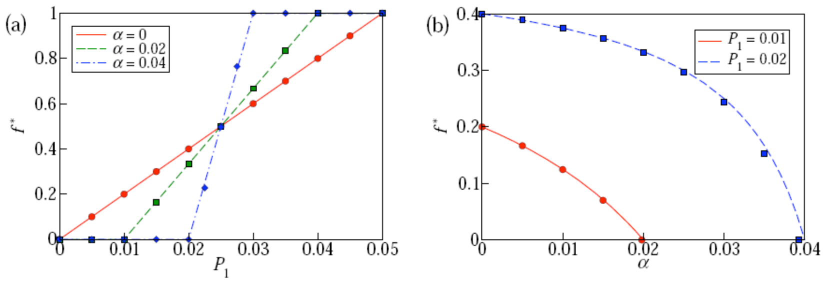


FIG 5. is from [22]: The influence of transaction fees on the optimal investment fraction: the dependency on P, for r, α fixed (a) and the dependency on α, for r, P fixed (b); r = 10% in both cases. Analytical and numerical results are shown as lines and symbols, respectively.

While in the original Kelly game the investor should rebalance the portfolio as often as possible (i.e., after each time step), in the presence of transaction fees it may be profitable to rebalance the portfolio less often. In the same paper above, the authors investigated the intermittent portfolio rebalancing for a further research, and tried to answer what rebalancing period T_{opt} maximizes the exponential growth rate per turn. As this question cannot be answered analytically, some numerical result would be obtained. On the below figure, we can see: T_{opt} decreases with both P and r, notably T_{opt} seems to grow with α roughly linearly, while f_{opt} decreases with α exponentially.

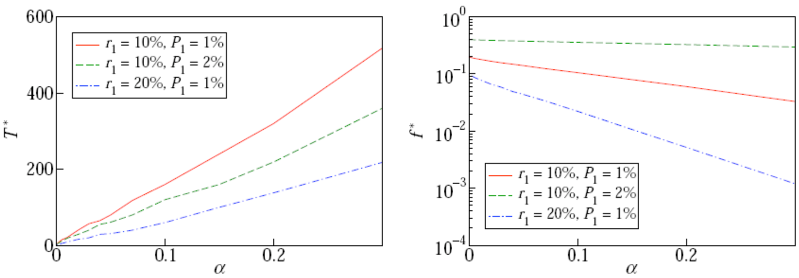


FIG 6. is from[22]: The optimal rebalancing period T_{opt} and the optimal investment fraction f_{opt} vs. α for different values of r and P.

Assets' properties are in real life generally non-stationary. Considering this situation by a simple model where the price of the asset undergoes a stochastic binary process on two distinct time scales was proposed in [22] as well. Albeit principally simple, the described situation is out of scope of analytical optimization tools and hence the authors only gave a numerical analysis. The below Fig. 7 is from [22]. It is shown that irregularities corresponding to the longer time scale are visible on both $f^*(T)$ and $G^*(T)$, with the conclusion that the presence of multiple time scales is important only if portfolio rebalancing occurs in time intervals comparable to the longest time scale of asset's returns.

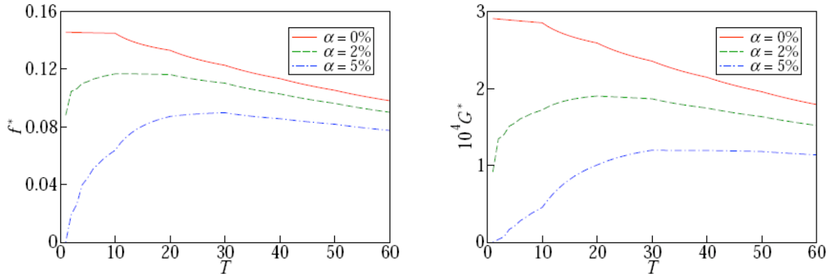


FIG 7. is from[22]:Optimal investment fraction and average return of the Kelly portfolio for an asset with price change on two time scales with p_1=-0.01, r_1 =0.05, p_2=0.05,r_2=0.5.

3.4 Limited information

Since in reality we seldom know the precise probabilities and payoffs, when investing in games without specified levels of risk and reward, the Kelly criterion can be merged with a Bayesian statistical learning as in, for example, [21, 23], yielding generalized results for the optimal investment fractions.

When the past game outcomes represent the only source of information, the author of [21]introduced a simple analytical formula for the optimal investment. Labeling the number of winning games in last L turns as w (w = 0, . . . ,L), the resulting knowledge about p can be quantified by the Bayes theorem[26] as：

$$\rho(p\,|\,w,L)=\frac{\pi(p)P(w\,|\,p,L)}{\int_0^1 \pi(p)P(w\,|\,L,p)dp} \tag{16}$$

here $\pi(p)$ is the prior probability distribution of p, and $P(w|p,L)$ is the probability distribution of w given the values p and L., and $P(w|L, p)=(C_w^M)p^w(1-p)^{L-w}$ due to the mutual independence of consecutive outcomes. With the maximum prior ignorance of $\pi(p)$=1 for $0 \leqslant p \leqslant 1$, Eq.(16) can be simplified to

$$\rho(p\,|\,w,L)=\frac{(L+1)!}{w!(L-w)!}p^w(1-p)^{L-w} \tag{17}$$

This is the investor's information about p after observing w wins in the last L turns. Since in the original Kelly game, the optimal solution is $f^*(p)=2p-1$ it is not difficult to prove that[21], when the probability distribution $\rho(p)$ is known, maximization of $G=\langle \ln W_1 \rangle$ results in $f^*=2\langle p \rangle -1$. From Eq.(17) follows $\langle p \rangle =(w+1)/(L+2)$, consequently

$$f^*(w,L)=\frac{2w-L}{L+2} \tag{18}$$

for $w \geqslant L/2$(Since in the paper they do not consider the possibility of short selling),and the exponential growth rate of an investor with the memory length L is:

$$G(p,L)=\sum_{w=0}^{L}P(w\,|\,p,L)[pIn(1+f^*(w,L))+(1-p)In(1-f^*(w,L))] \tag{19}$$

Then we can compare the limited information with the original Kelly model which p is known and fixed. In-depth analysis can be seen in[21] as shown in Fig. 8: $\Re$ (p,L) and $\Re$ (p) represent the compounded return on the condition of limited information and original Kelly model respectively as well as $G(p,L)$ and $G(p)$. As L increases, the investor's information about p improves and $\xi \rightarrow 1$. When p is small, a very long memory is needed to make a profitable investment. This agrees with the experience of finance practitioners—according to them the Kelly portfolio is sensitive to a wrong examination of the investment profitability.

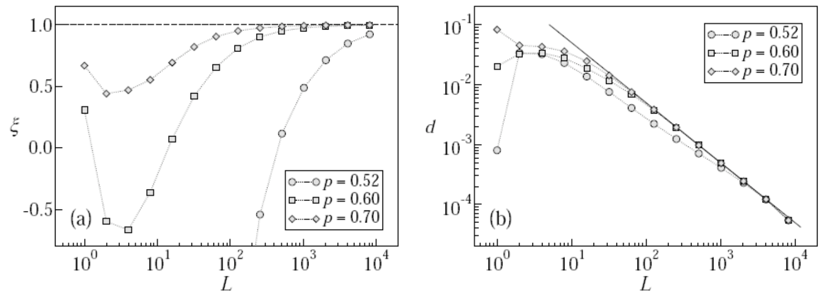


FIG 8. is from[21]:The ratioξ as a function of the memory length L. b) the difference d=$G(p)$-$G(p, L)$ is shown as a function of L.

4. Application of Kelly strategy

Here are just some conclusions, for more analysis one can refer to[19, 27]. As this so-called growth-optimal or Kelly portfolio has been shown to be optimal according to various criteria, it's instructive to investigate the application in real financial market. In[19], Thorp introduced how to use the Kelly strategy in Blackjack, Sports betting i.e., and gave a case study about the Kelly strategy. However, in[27] application of Kelly's optimization process to real stock prices (based on the time evolution of the New York Stock Exchange composite index)was studied with conclusions: The optimal investment ratio fluctuates very rapidly in time; it depends strongly on the time, when the investment strategy started to be applied, a non-trivial investment (i.e., investing only a part of one's wealth, which means $0<f^*<1$) occurs rarely. This is related to the general notion that Kelly's portfolio is very aggressive and investment outcomes are sensitive to errors in estimates of assets' properties. Any more characteristic of real market such as tax, transaction cost etc. will affect this uncertainty as well. Modifications such as fractional Kelly strategies[28] and controlled drawdowns[29] have been consequently proposed to make the resulting portfolios more secure.

Since the Kelly approach is for the long run, it is important to understand "the long run", i.e., the time it takes for f^* to dominate a specified neighbor by a specified probability, can vary without limit. Each application requires a separate analysis[30-31]. Another one of the most important ideas in Kelly is that betting more than the Kelly amount decreases the probability of very good results, while still increasing the probability of very bad results. Therefore since in reality the precise probabilities and payoffs are seldom known, and since over-betting is worse than under-betting, it makes sense to err on the side of caution and bet less than the Kelly amount.

5. Conclusion

Since its present, the Kelly strategy has been investigated in details in many works by a few researchers. In this paper we try to review these significant works such as, the optimality of Kelly strategy. The relation between Mean-variance approach and Kelly strategy, display the connection between them. The case of diversification improves the performance of the investor and lead to a further discussion about the correlation among different assets. The influence of transaction fees significantly change the origin Kelly strategy and call for a new intermittent portfolio rebalancing. As in real financial market the precise quality is not known at moments, the Kelly criterion can be merged with a Bayesian statistical learning, yielding generalized results for the optimal investment fractions. After these discussions, finally, a case of application of Kelly strategy is mentioned with some conclusion. Besides the work reviewed here there are also much future work need to be discussed such as some variations about the original Kelly game for more consideration on real market, modifications about the original Kelly strategy and more investigations on the application of Kelly strategy and so on.

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