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On the finitistic dimension conjecture I: related to representation-finite algebras

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Abstract

We use the class of representation-finite algebras to investigate the finitistic dimension conjecture. In this way we obtain a large class of algebras for which the finitistic dimension conjecture holds. The main results in this paper are: (1) Let A be an artin algebra and let $I_j, 1 \leqslant j \leqslant n$ be a family of ideals in A with $I_1 I_2 \cdots I_n = 0$, such that $\text{proj.dim}({}_A I_j) < \infty$ and $\text{proj.dim}(I_j)_A = 0$ for all $j \geqslant 3$. If A/I_1 and A/I_2 are representation-finite and if A/I_j has finite finitistic dimension for $j \geqslant 3$, then the finitistic dimension of A is finite. In particular, the finitistic dimension conjecture is true for algebras obtained from representation-finite algebras by forming dual extensions, trivially twisted extensions, Hochschild extensions, matrix algebras and tensor products with algebras of radical-square-zero. (2) Let A, B and C be three artin algebras with the same identity such that (i) $C \subseteq B \subseteq A$, and (ii) the Jacobson radical of C is a left ideal of B and the Jacobson radical of B is a left ideal of A. If A is representation-finite, then C has finite finitistic dimension. We also provide a way to construct algebras satisfying all conditions in (2), and this leads to a new reformulation of the finitistic dimension conjecture.

MSC: 16G10; 16P10; 16S20; 18G20

1. Introduction

In the representation theory of algebras and groups, homological invariants of modules and algebras form one of the important topics. Among them is the finitistic dimension, which is defined to be the supremum of projective dimensions of finitely generated

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modules having finite projective dimension. The famous finitistic dimension conjecture says that the finitistic dimension of an arbitrary artin algebra is finite. This conjecture is closely related to the well-known Nakayama conjecture and the generalized Nakayama conjecture. There is a variety of literatures on the studying of finitistic dimensions of special classes of artin algebras (see [5,6,21,13], and others). Recently, it is shown in [8] that if the representation dimension of an artin algebra is upper bounded by 3, then the finitistic dimension of the algebra is finite, where the representation dimension, introduced by Auslander in [1], is by definition the minimum of the global dimensions of algebras of the form $\mathrm{End}({}_AM)$ with M a generator-cogenerator. However, we know that the representation dimension is not always bounded by 3 proved by Rouquier (unpublished), thus the finitistic dimension conjecture is still open. In fact, it is far from being solved.

As we know, the class of representation-finite artin algebras is better understood than other classes of algebras in the representation theory. Of course, the finitistic dimension conjecture holds true for representation-finite artin algebras. From this point of view, in this note we try to use representation-finite algebras to enlarge our knowledge on finitistic dimensions, namely, we study questions of the following type: suppose two artin algebras A and B have certain good relationship. If one of them is representation-finite, what could we say about the finitistic dimension of the other? So our philosophic idea in this note is to approach a homological conjecture, the finitistic dimension conjecture, without imposing homological conditions on algebras, but merely by employing the class of representation-finite artin algebras. In this direction we have already seen some interesting results in [13] and in [4]. These are also the motivation of our philosophy. In this note we shall add the following new results along this direction:

(1) If A is an artin algebra with two ideals I and J such that both A/I and A/J are representation-finite, then the finitistic dimension of A/IJ is finite. In particular, the finitistic dimension conjecture is true for algebras obtained from representation-finite algebras by forming

- dual extensions,
- trivially twisted extensions,
- Hochschild extensions,
- matrix algebras,
- tensor products with algebras of radical-square-zero.

Thus statement (1) describes the finitistic dimensions of extension algebras, while the following result describes the finitistic dimensions of subalgebras.

(2) Let A, B and C be three artin algebras with the same identity such that (i) $C \subseteq B \subseteq A$, and (ii) the Jacobson radical of C is a left ideal of B and the Jacobson radical of B is a left ideal of A. If A is representation-finite, then C has finite finitistic dimension.

In particular, we have the following consequence.

(3) Let B be a subalgebra of an artin algebra A with the same identity such that the Jacobson radical of B is a left ideal in A. If A is representation-finite, then the finitistic dimension of B is finite. Particularly, if A and B have the same Jacobson radical and if A is representation-finite, then B has finite finitistic dimension.

Note that the last statement in (3) was proved in [4], but we re-prove it by a more direct manner. Since there are plenty of examples of subalgebras such that their radicals are only left ideals in the overalgebras, our result (3) is a proper generalization of the result on finitistic dimensions in [4]. As a consequence of (3) together with the splitting method in [4] we re-obtain the result that the finitistic dimension conjecture is true for string algebras. Note also that the proofs in [4] do not extend to our cases of (3) and (2).

This note is detailed as follows: after we list in Section 2 some basic results needed for our proofs, we start with Section 3 the proofs of (1) and (3), in this section we shall also construct algebras of representation dimension 3 by trivially twisted extensions in [17]. In Section 4 we prove (2) and also give a construction of algebras satisfying all conditions of (2) by the idealizer method. In the last section some questions on the finitistic dimension and the representation dimension related to the results in this note are mentioned.

2. Preliminaries

In this section we recall some basic definitions and results needed in the paper.

Let A be an artin algebra, that is, A is a finitely generated module over its center which is assumed to be a commutative artin ring. We denote by A-mod the category of all finitely generated left A-modules and by $\mathrm{rad}(A)$ the Jacobson radical of A. Given an A-module M, we denote by $\mathrm{proj.dim}(M)$ the projective dimension of M. Let $K(A)$ be the quotient of the free abelian group generated by the isomorphism classes $[M]$ of modules M in A-mod modulo the relations (i) $[Y] = [X] + [Z]$ if $Y \simeq X \oplus Z$; and (ii) $[P] = 0$ if P is projective. Thus $K(A)$ is a free abelian group with the basis of non-isomorphism classes of non-projective indecomposable A-modules. Igusa and Todorov define a function Ψ on this abelian group, which depends on the algebra A and takes values of non-negative integers.

The following result is due to Igusa and Todorov [8].

Lemma 2.1. *For any artin algebra A there is a function Ψ defined on the objects of A-mod such that*

(1) *$\Psi(M) = \mathrm{proj.dim}(M)$ if M has finite projective dimension. Moreover, if M is indecomposable and $\mathrm{proj.dim}(M) = \infty$, then $\Psi(M) = 0$.*

(2) *For any natural number n, $\Psi(\bigoplus_{j=1}^{n} M) = \Psi(M)$.*

(3) *For any A-modules X and Y, $\Psi(X) \leqslant \Psi(X \oplus Y)$.*

(4) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in A-mod with $\mathrm{proj.dim}(Z) < \infty$, then $\mathrm{proj.dim}(Z) \leqslant \Psi(X \oplus Y) + 1$.*

(5) *If $0 \to X \to Y \to Z \to 0$ is an exact sequence in A-mod with Z indecomposable, then $\Psi(Z) \leqslant \Psi(X \oplus Y) + 1$.*

Note that given an exact sequence $0 \to X \to Y \to Z \to 0$ in A-mod, there are two relevant exact sequences $0 \to \Omega(Y) \to \Omega(Z) \oplus P \to X \to 0$ and $0 \to \Omega^2(Z) \to \Omega(X) \oplus P' \to \Omega(Y) \to 0$, where Ω^i is the ith syzygy operator, and P, P' are projective modules. So the following result is a consequence of 2.1 (see also [13]).

Lemma 2.2. *If* $0 \to X \to Y \to Z \to 0$ *is an exact sequence in A-mod, then*
(1) *proj.dim*$(Y) \leqslant \Psi(\Omega(X) \oplus \Omega^2(Z)) + 2$ *in case proj.dim*$(Y) < \infty$,
(2) *proj.dim*$(X) \leqslant \Psi(\Omega(Y \oplus Z)) + 1$ *in case proj.dim*$(X) < \infty$.

Given an artin algebra A, the **finitistic dimension** of A, denoted by fin.dim(A), is defined as

$$\text{fin.dim}(A) = \sup\{\text{proj.dim}(_AM) \mid M \in A\text{-mod and proj.dim}(_AM) < \infty\}.$$

Note that fin.dim(A) may be different from fin.dim(A^{op}), where A^{op} is the opposite algebra. Finally, recall that A is called *representation-finite* if in A-mod there are only finitely many non-isomorphic indecomposable modules.

3. Results and proofs

In this section we shall show how the representation-finite algebras can be used to control the finitistic dimension in the question mentioned in the introduction. Let us first prove the following result which generalizes properly a result in [4]. At the end of this section we provide an example of a pair $B \subseteq A$ such that rad(B) is just a left ideal of A, but not a two-sided ideal in A.

Theorem 3.1. *Let B be a subalgebra of an artin algebra A with the same identity such that the Jacobson radical rad*(B) *of B is a left ideal in A. If A is representation-finite, then the finitistic dimension of B is finite.*

Proof. Since A is representation-finite, we may assume that $M_1, M_2, \ldots, M_t$ are a complete list of non-isomorphic indecomposable A-modules. Since B is a subalgebra of A, each A-module can be considered as a B-module just by restriction of the scalars of A to B. Let X be a B-module with finite projective dimension. We take a minimal projective cover $f: P_B(X) \to X$, thus the top of X and the top of $P_B(X)$ are isomorphic. If we denote by rad(X) the radical of the B-module X, then we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \Omega(X) \arrow[r, equal] \arrow[d] & \Omega(X) \arrow[d] & & \\
0 \arrow[r] & \mathrm{rad}(P_B(X)) \arrow[r] \arrow[d, "f'"'] & P_B(X) \arrow[r] \arrow[d, "f"'] & \mathrm{top}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{rad}(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \mathrm{top}(X) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where f' is the restriction of f. Since rad(B) is a left ideal in A and since rad$(_BM) =$ rad$(B)M$ for all B-modules $_BM$, we know that rad$(P_B(X))$ and rad(X) are A-modules and that f' is in fact an A-module homomorphism, thus the kernel $\Omega(X)$ of f' is also an A-module. So we may write this A-module as $\Omega(X) = \bigoplus_{j=1}^{t} M_j^{s_j}$, where s_j is a non-negative integer for each j. Note that this is also a B-module decomposition. Now we use 2.1 to bound the projective dimension of ${}_BX$:

$$\begin{aligned}\text{proj.dim}_B X &\leqslant \text{proj.dim}\,\Omega({}_BX) + 1\\ &= \Psi(\Omega({}_BX)) + 1\\ &= \Psi(\oplus M_j^{s_j}) + 1\\ &\leqslant \Psi\left(\bigoplus_j M_j\right) + 1.\end{aligned}$$

Thus the finitistic dimension of B is upper bounded by $\Psi(\bigoplus_j M_j) + 1$. This finishes the proof. □

Now we turn to the proof of the following result.

Theorem 3.2. *If A is an artin algebra with two ideals I and J such that $IJ = 0$ and both A/I and A/J are representation-finite, then the finitistic dimension of A is finite.*

Proof. By assumption we suppose that $\{M_1, M_2, \ldots, M_s\}$ is a complete list of non-isomorphic indecomposable A/I-modules and that $\{N_1, N_2, \ldots, N_t\}$ is a complete list of non-isomorphic indecomposable A/J-modules. Now let X be an A-module with finite projective dimension. We consider the exact sequence $0 \to JX \to X \to X/JX \to 0$. Since $IJ = 0$, the module JX is also an A/I-module, thus $JX = \bigoplus_{j=1}^{s} M_j^{s_j}$ for some non-negative integers s_j. Clearly, X/JX is an A/J-module and therefore $X/JX = \bigoplus_{j=1}^{t} N_j^{t_j}$ for some non-negative integers t_j. By 2.2, we have

$$\begin{aligned}\text{proj.dim}_A X = \Psi({}_AX) &\leqslant \Psi\left(\Omega\left(\bigoplus_{j=1}^{s} M_j^{s_j}\right) \oplus \Omega^2\left(\bigoplus_{j=1}^{t} N_j^{t_j}\right)\right) + 2\\ &= \Psi\left(\bigoplus_{j=1}^{s} \Omega(M_j)^{s_j} \oplus \bigoplus_{j=1}^{t} \Omega^2(N_j)^{t_j}\right) + 2\\ &\leqslant \Psi\left(\bigoplus_j \Omega(M_j) \oplus \bigoplus_i \Omega^2(N_i)\right) + 2.\end{aligned}$$

Thus the projective dimension of X is bounded by $\Psi(\bigoplus_j \Omega(M_j) \oplus \bigoplus_i \Omega^2(N_i)) + 2$, and Theorem 3.2 follows. □

Let us remark that this result seems to have the following generalization: if I_j, $1 \leqslant j \leqslant n$, are a family of ideals in A such that $I_1 \cdots I_n = 0$ and that all A/I_j are representation-finite, then A has finite finitistic dimension. It would be interesting to have a proof of this generalization.

The following result is a partial answer in this direction.

Theorem 3.3. *Let I_j, $1 \leqslant j \leqslant n \geqslant 2$, be a family of ideals in an artin algebra A such that $I_1 \cdots I_n = 0$ and that A/I_j are representation-finite for $j = 1, 2$, and that A/I_j has finite finitistic dimension for $j \geqslant 3$. If the projective dimension of ${}_AI_j$ is finite for all $j \geqslant 3$ and if I_j is projective as a right A-module for all $j \geqslant 3$, then A has finite finitistic dimension.*

To prove the result, we need the following lemma in [12, Lemma 7.3.9, p. 240].

Lemma 3.4. *Let A be an artin algebra, I an ideal in A and ${}_AM$ an A-module. Then: if I_A is projective and ${}_AM$ is a submodule of a projective module, then $proj.dim_A IM \leqslant proj.dim_A M + proj.dim_A I$.*

Proof of Theorem 3.3. Note that given an exact sequence $0 \to X \to Y \to Z \to 0$ in A-mod, if two of the modules have finite projective dimension then the third has also finite projective dimension, and in this case $\text{proj.dim}_A Y \leqslant \max\{\text{proj.dim}_A X, \text{proj.dim}_A Z\}$.

Suppose that Y is an A-module of finite projective dimension. Then $X := \Omega_A(Y)$ is a submodule of a projective A-module. Since I_j is a projective right A-module and $\text{proj.dim}_A I_j < \infty$ for $j \geqslant 3$, we know that $I_jI_{j+1} \cdots I_nX$ has finite projective dimension by Lemma 3.4. Thus $\text{proj.dim}_A I_{j+1} \cdots I_nX/I_jI_{j+1} \cdots I_nX < \infty$ for $j \geqslant 3$. If $\{M_1, M_2, \ldots, M_s\}$ is a complete list of non-isomorphic indecomposable A/I_1-modules and if $\{N_1, N_2, \ldots, N_t\}$ is a complete list of non-isomorphic indecomposable A/I_2-modules, then $\text{proj.dim}_A I_3 \cdots I_nX \leqslant \Psi_A(\bigoplus_j \Omega(M_j) \oplus \bigoplus_i \Omega^2(N_i)) + 2$ by 3.2. Let us denote by $\text{fin.dim}(A)$ the finitistic dimension of A. So we have

$$\begin{aligned}
\text{proj.dim}_A X &\leqslant \max\{\text{proj.dim}_A I_nX, \text{proj.dim}_A X/I_nX\} \\
&\leqslant \max\{\text{proj.dim}_A I_nX, \text{fin.dim}(A/I_n)\} \\
&\leqslant \max\{\text{proj.dim}_A I_{n-1}I_nX, \text{fin.dim}(A/I_{n-1}), \text{fin.dim}(A/I_n)\} \\
&\leqslant \ldots\ldots \\
&\leqslant \max\{\text{proj.dim}_A I_3I_4 \cdots I_nX, \text{fin.dim}(A/I_3), \ldots, \text{fin.dim}(A/I_n)\} \\
&= \max\{\Psi_A(I_3I_4 \cdots I_nX), \text{fin.dim}(A/I_3), \ldots, \text{fin.dim}(A/I_n)\} \\
&\leqslant \max\left\{\Psi_A\left(\bigoplus_j \Omega(M_j) \oplus \bigoplus_i \Omega^2(N_i)\right) + 2, \text{fin.dim}(A/I_3),\right. \\
&\qquad \left. \ldots, \text{fin.dim}(A/I_n)\right\}.
\end{aligned}$$

This shows that $\text{proj.dim}_A Y$ is upper bounded by $\max\{\Psi_A(\bigoplus_j \Omega(M_j) \oplus \bigoplus_i \Omega^2(N_i)) + 2, \text{fin.dim}(A/I_3), \ldots, \text{fin.dim}(A/I_n)\} + 1$. The proof is completed. $\square$

The next result is a dual statement of 3.1 in some sense.

Proposition 3.5. *Let A and B be two artin algebras such that $A/soc(A) \simeq B/soc(B)$, and suppose there is a surjective homomorphism $f : A \to B$. If B is representation-finite, then A has finite finitistic dimension.*

The proof of Proposition 3.5 follows from the following observation.

Lemma 3.6. *Let A and B be two artin algebras, and let $f : A \to B$ be an algebra homomorphism such that the kernel of f is contained in the socle of A. If $A/\ker(f)$ is representation-finite, then A has finite finitistic dimension.*

Proof. Let I be the kernel of f and J the radical of A. Then $J\ker(f) = 0$. Since A/I and A/J are representation-finite, the result follows from 3.2. $\square$

Similarly, we have the following result which generalizes the main result in [13] and also re-proves that the finitistic dimension conjecture is true for algebras with radical-cube-zero.

Proposition 3.7. *Let A be an artin algebra with an ideal I such that $I^n rad(A) = 0$ for a natural number $n \geqslant 2$. If A/I^{n-1} is representation-finite, then A has finite finitistic dimension.*

Proof. Given an A-module X with finite projective dimension, we consider $\Omega(X)$ instead of X, and then apply 2.2 to the exact sequence $0 \to I^{n-1}\Omega(X) \to \Omega(X) \to \Omega(X)/I^{n-1}\Omega(X) \to 0$ since $\Omega(X)$ has finite projective dimension and since $I^{n-1}\Omega(X)$ is an A/I-module by the fact that $I^n\Omega(X) \subseteq I^n\mathrm{rad}(P(X)) = I^n\mathrm{rad}(A)P(X) = 0$, where $P(X)$ is the projective cover of X. Since A/I^{n-1} is representation-finite, A/I is also representation-finite. By the argument in the proof of Theorem 3.2 we have the proposition. $\square$

Now let us get some other consequences of 3.2. The first case we consider is that $I = J$.

Corollary 3.8. *If A is an artin algebra with an ideal I such that $I^2 = 0$ and A/I is representation-finite, then the finitistic dimension conjecture is true for A.*

A special case of (3.8) is the Hochschild extension of a representation-finite algebra. Let B be an algebra and let M be a B-B-bimodule. For each 2-cocycle $\phi \in H^2(B, M)$, there is an algebra structure on $A_\phi := B \oplus M$ by $(b, m)(b', m') = (bb', \phi(b, b') + bm' + mb')$ for all $m, m' \in M$ and $b, b' \in B$ such that M is an ideal in A_ϕ with $M^2 = 0$ (see [7]). The algebra A_ϕ is called the **Hochschild extension** of B by M via ϕ. It follows from (3.8) that if B is representation-finite then A_ϕ has finite finitistic dimension. Thus the finiteness of finitistic dimension of the Hochschild extension of a representation-finite algebra does not depend on the bimodule M. Note that the algebra A_ϕ may have finite or infinite global dimension (see [11]).

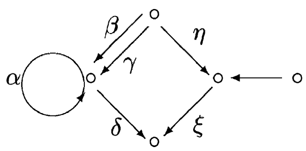
Now let us introduce a new construction which generalizes slightly the notion of dual extensions in [16]. Let C be a finite dimensional algebra over a field given by the quiver $\Gamma = (\Gamma_0, \Gamma_1)$ with relations $\{\sigma_i \mid i \in I_0\}$, and let B be an algebra given by the quiver $\Delta = (\Delta_0, \Delta_1)$ with relations $\{\tau_j \mid j \in J_0\}$. Assume that $S = \{s_1, \ldots, s_m\}$ is a subset contained in $\Gamma_0 \cap \Delta_0$. Now we define a new algebra A, called the **trivially twisted extension** of C and B at S, in the following manner: A is given by the quiver $Q = (Q_0 := \Gamma_0 \dot{\cup} (\Delta_0 \setminus S), Q_1 := \Gamma_1 \dot{\cup} \Delta_1)$, with the relations $\{\sigma_i | i \in I_0\} \cup \{\tau_j | j \in J_0\} \cup \{\alpha\beta | \alpha \in \Gamma_1, \beta \in \Delta_1\}$. Note that if $S = \Delta_0 = \Gamma_0$ and if B is the opposite algebra of C then A is just the **dual extension** of C. Another special case is that $S = \emptyset$. In this case we have that A is the direct sum of B and C. Now let J be the ideal in A generated by $\{\beta | \beta \in \Delta_1\}$ and let I be the ideal in A generated by $\{\alpha | \alpha \in \Gamma_1\}$. Then $IJ = 0$, $A/I \simeq B$ and $A/J \simeq C$. Note that C and B are not only factor algebras but also subalgebras of A.

The following result is an immediate consequence of 3.2.

Corollary 3.9. *If C and B are representation-finite over a field, then the trivially twisted extension of C and B at S has finite finitistic dimension.*

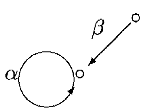
Note that the trivially twisted extension of two representation-finite algebras can be of wild representation type and can also have arbitrary nilpotency index for the radical. For further property of the dual extension we refer the reader to [17]. Now let us illustrate the trivially twisted extension by an example.

Example 1. (1) Let A be an algebra (over a field) given by the following quiver with relations:

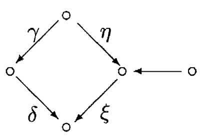


$\eta\xi = \gamma\delta$, $\alpha^3 = \beta\delta = \alpha\delta = 0.$

Let B and C be the algebras given by the following quiver with relations, respectively:

$B:$ 

$\alpha^3 = 0.$

$C:$ 

$\eta\xi = \gamma\delta.$

Suppose both γ and β have the same starting vertex 1 and the same ending vertex 2. Then A is the trivially twisted extension of B and C at the vertex $S = \{1, 2\}$. Since B and C are representation-finite, the algebra A has finite finitistic dimension by 3.9.

(2) Let A be the algebra given by the following quiver

$$1 \circ \begin{array}{c} \xleftarrow{\beta} \\ \xrightarrow[\gamma]{} \\ \xleftarrow[\alpha]{} \end{array} \circ 2$$

with relations $\alpha\gamma = \gamma\alpha = \gamma\beta = 0$. If we take C to be the subalgebra of A generated by the arrows α, γ and the two primitive orthogonal idempotents e_1 and e_2, and B the subalgebra of A generated by $\{e_1, e_2, \beta\}$, then A is the trivially twisted extension of C and B at $S = \{1, 2\}$, and therefore has finite finitistic dimension by 3.9 since C and B are transparently representation-finite.

Let us remark that this famous example, due to Igusa, Smalø and Todorov, is used to show that the subcategory $\mathscr{P}^{\infty}(A)$ of A-mod consisting of all modules with finite projective dimensions is not always contravariantly finite in A-mod. However, if this subcategory is contravariantly finite in A-mod, then the finitistic dimension of A is finite (see [2]). In general, it is not easy to control the category $\mathscr{P}^{\infty}(A)$, for instance, the contravariant finiteness of both $\mathscr{P}^{\infty}(C)$ in C-mod and $\mathscr{P}^{\infty}(B)$ in B-mod even cannot guarantee the contravariant finiteness of $\mathscr{P}^{\infty}(A)$ in A-mod, as the example shows. But our Theorem 3.2 (see also Theorem 4.5 below) provides a chance to avoid the consideration of the contravariant finiteness of $\mathscr{P}^{\infty}(A)$.

The construction of trivially twisted extensions produces also algebras with the representation dimension bounded by 3. This is done in the following manner:

Let A be the trivially twisted extension of C and B at S. If K is the ideal in A generated by $\{\beta\alpha \mid \beta \in \Delta_1, \alpha \in \Gamma_1\}$, then $\mathrm{rad}(A/K) = \mathrm{rad}(C) \oplus \mathrm{rad}(B)$ and the algebra A/K can be embedded in $B \oplus C$, and therefore the representation dimension of A/K is upper bounded by 3 if C and B are representation-finite. This can be seen from the main result in [4]. For further new results on representation dimension we refer to [18,19,20].

As another consequence of 3.8 and 3.2 we have the following results on the finitistic dimension of the tensor product of two algebras. Recall that given two k-algebras A and B over a field k, the **tensor product** of A and B, denoted by $A \bigotimes_k B$, has the multiplication defined by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb', \quad a, a' \in A; b, b' \in B.$$

Proposition 3.10. *If A is a representation-finite k-algebra and if B is a k-algebra with $\mathrm{rad}^2(B) = 0$ such that $B/\mathrm{rad}(B)$ is a split semi-simple k-algebra, then the tensor product $A \bigotimes_k B$ of A and B has finite finitistic dimension.*

Proof. We define $I = A \bigotimes_k \mathrm{rad}(B)$. Then I is an ideal in $A \bigotimes_k B$ with $I^2 = 0$. Since $B/\mathrm{rad}(B)$ is a direct sum of full matrix algebras over k, we see that $A \bigotimes_k (B/\mathrm{rad}(B))$ is Morita equivalent to a direct sum of copies of A. Thus $(A \bigotimes_k B)/I \simeq A \bigotimes_k (B/\mathrm{rad}(B))$ is representation-finite since A is representation-finite by assumption. Now the proposition follows from 3.8 immediately. □

Note that even under the assumption of Proposition 3.10 the radical of $A\bigotimes_k B$ may have arbitrary nilpotency index and the tensor product may not be a monomial algebra in general. So we cannot apply the result in [5].

Proposition 3.11. *Let B and C be two finite dimensional k-algebras over a field k such that $B/rad(B)$ and $C/rad(C)$ are splitting semi-simple k-algebras. If B and C are representation-finite, then the finitistic dimension of $(B\bigotimes_k C)/(rad(B)\bigotimes_k rad(C))$ is finite.*

Proof. We denote by A the tensor product of B and C and by $\bar{A}$ the factor algebra $(B\bigotimes_k C)/(\mathrm{rad}(B)\bigotimes_k \mathrm{rad}(C))$. Let $I=\mathrm{rad}(B)\bigotimes_k C$ and $J=B\bigotimes_k \mathrm{rad}(C)$. The images of I and J under the canonical surjective map $A \to \bar{A}$ are denoted by $\bar{I}$ and $\bar{J}$, respectively. Since $B/\mathrm{rad}(B) \simeq \bigoplus_j M_{n_j}(k)$, where $M_n(k)$ stands for the full matrix algebra over the field k, we have that $\bar{A}/\bar{I} \simeq (B\bigotimes_k C)/(\mathrm{rad}(B)\bigotimes_k C) \simeq (B/\mathrm{rad}(B))\bigotimes_k C \simeq \bigoplus_j M_{n_j}(k)\bigotimes_k C \simeq \bigoplus_j M_{n_j}(C)$. This implies that $\bar{A}/\bar{I}$ is representation-finite. Similarly, we know that $\bar{A}/\bar{J}$ is representation-finite. Clearly, $\bar{I}\bar{J}=0$. Now the proposition follows from Theorem 3.2. □

Remark. If we assume that the field k is a perfect field (for example, a finite field, or a filed of characteristic zero, or an algebraically closed field) then we can drop simply the assumption that $B/\mathrm{rad}(B)$ and $C/\mathrm{rad}(C)$ are splitting semi-simple k-algebras in Propositions 3.10 and 3.11 since if B is representation-finite then $B\bigotimes_k M_n(D)$ is also representation-finite for any finite dimensional division k-algebra D by finding the representation dimension (see [15, Theorem 3.5]).

The next result deals with triangular algebras, here we re-obtain a result in the literature.

Corollary 3.12. *Given two artin algebras A and B, and an A-B-bimodule M, we may form the triangular algebra*

$$\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}.$$

If A and B are representation-finite, then the finitistic dimension of Λ is finite. In particular, if A is representation-finite, then the algebra $T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ has finite finitistic dimension.

More generally, we have the following result which is also a special case of Hochschild extensions.

Corollary 3.13. *Given two artin algebras A and B, an A-B-bimodule M and a B-A-bimodule N, we define a* **matrix algebra** *as follows:*

$$\Lambda = \begin{pmatrix} A & M \\ N & B \end{pmatrix}, \quad \begin{pmatrix} a & m \\ n & b \end{pmatrix}\begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ na' + bn' & bb' \end{pmatrix},$$

where $a, a' \in A$, $b, b' \in B$ and $m, m' \in M, n, n' \in N$. If A and B are representation-finite, then the finitistic dimension of Λ is finite.

Proof. We just take the ideal $\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix}$ of the matrix algebra, and then apply 3.8 since the square of this ideal vanishes. □

In the following we give several examples to show that there do exist algebras which satisfy our more general conditions.

Let us first see an example where the radical of a subalgebra B is a left ideal of the algebra A, but not a right ideal in A.

Example 2. Let A and B be the subalgebras of the 4×4 matrix algebra over a field k defined as follows:

$$B = \left\{ \begin{pmatrix} a & b & c & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ d & e & f & g \end{pmatrix} | a, b, c, d, e, f, g \in k \right\},$$

$$A = \left\{ \begin{pmatrix} a & b & c & 0 \\ 0 & a & x & 0 \\ 0 & 0 & y & 0 \\ d & e & f & g \end{pmatrix} | a, b, c, d, e, f, g, x, y \in k \right\}.$$

One can verify that the radical of B is $\operatorname{rad}(B) = \left\{ \begin{pmatrix} 0 & b & c & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ d & e & f & 0 \end{pmatrix} | b, c, d, e, f \in k \right\}$,

which is a left ideal in A, but not a right ideal in A. Clearly, the radical of A is

$\operatorname{rad}(A) = \left\{ \begin{pmatrix} 0 & b_1 & c & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & 0 \\ d & e & f & 0 \end{pmatrix} | b_1, b_2, c, d, e, f \in k \right\}$. So the radical of B is properly contained in the radical of A. In fact, A is the set of all 4×4 matrices x such that $x \cdot \operatorname{rad}(B) \subseteq \operatorname{rad}(B)$. Since the algebra A is representation-finite by covering technique, we know that B has finite finitistic dimension by 3.1. This follows also from the fact that B is a monomial algebra (see [5,9]).

Example 3. Let us give a very simple example of pair $B \subset A$ for the case $\mathrm{rad}(B) = \mathrm{rad}(A)$.

We take A to be the algebra of 2×2 upper triangular matrices over a field k, and let B be the subalgebra generated by the identity element e and the radical of A. Clearly, A and B have the same radical.

In fact, the general construction of a pair $B \subset A$ with $\mathrm{rad}(B) = \mathrm{rad}(A)$ proceeds in the same way as this example shows: given an algebra A, we fix a decomposition of 1 into orthogonal primitive idempotents, say $1 = \sum_{j=1}^{n} e_j$. To define B, we just fix a partition of the set $I := \{1, 2, \dots, n\}$, say $I = \bigcup_{i=1}^{m} I_i$, and put $f_i = \sum_{j\in I_i} e_j$. Now the algebra B is generated by $f_i, 1 \leqslant i \leqslant m$ together with $\mathrm{rad}(A)$. Clearly, A and B have the same identity and the same radical. Conversely, every such pair $B \subseteq A$ with A an basic algebra appears in this form if the ground field is algebraically closed: choose a maximal semi-simple subalgebra S_0 of B and extend it to a maximal semi-simple subalgebra of A. First we write 1 in S_0 as a sum of primitive orthogonal idempotents of B, say $\sum_j f_j = 1$, and then write each f_j as sum of primitive orthogonal idempotents of A, say $f_j = \sum_{i\in I_j} e_i$. Since A is basic, S is a commutative algebra. Thus S_0 is a product of fields by Wedderburn-theorem, and is generated by f_j's, and also S is generated by e_i's.

In the following we give an example of a pair $B \subseteq A$ of algebras such that B is representation-infinite, A is representation-finite, and $\mathrm{rad}(B)(\neq \mathrm{rad}(A))$ is an ideal in A.

Example 4. Let A and B be the following algebras:

$$B = \left\{ \begin{pmatrix} a & 0 & e & f \\ 0 & b & g & h \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \middle| a, b, c, d, e, f, g \in k \right\},$$

$$A = \left\{ \begin{pmatrix} a & 0 & e & f \\ 0 & b & g & h \\ 0 & 0 & c & i \\ 0 & 0 & 0 & d \end{pmatrix} \middle| a, b, c, d, e, f, g, h, i \in k \right\}.$$

One can easily see that A is a hereditary algebra of Dynkin type, thus representation-finite, but B is a hereditary algebra of affine type, thus representation-infinite. A simple verification shows that $\mathrm{rad}(B)$ is an ideal in A and contained properly in $\mathrm{rad}(A)$.

4. Idealized extensions

In this section we give a construction of the pair $B \subseteq A$ with $\mathrm{rad}(B)$ being a left ideal in A, and prove statement (2) in the introduction.

Let us start with the following lemma which describes some general properties of a pair $B \subseteq A$.

Lemma 4.1. *Let B be a subalgebra of A such that $rad(B)$ is a left ideal in A. Then*
(1) $rad(B) \subseteq rad(B)A = rad({}_BA) \subseteq rad(A)$.
(2) $B \cap rad(A) = rad(B)$, *and hence $B/rad(B)$ is a subalgebra of $A/rad(A)$.*
(3) *If B is a self-injective algebra, then we have an exact sequence of algebra-homomorphisms*:

$$0 \to soc(A_B) \to A \to End(rad(B)_B) \to 0.$$

Proof. (1) Clearly, $\mathrm{rad}(B)A$ is a nilpotent ideal in A, hence $\mathrm{rad}(B)A \subseteq \mathrm{rad}(A)$.
(2) Since $B \cap \mathrm{rad}(A)$ is a nilpotent ideal in B, we have $B \cap \mathrm{rad}(A) \subseteq \mathrm{rad}(B)$. On the other hand, we have $\mathrm{rad}(B) \subseteq B \cap \mathrm{rad}(A)$ by (1), thus (2) follows.
(3) By definition, each element $a \in A$ gives us an endomorphism ϕ_a of the right B-module $\mathrm{rad}(B)$ by the left multiplication. Thus the map $a \mapsto \phi_a$ is an algebra homomorphism from A to $\mathrm{End}(\mathrm{rad}(B)_B)$ with the kernel $\mathrm{soc}(A_B)$. Since B is a self-injective algebra, every endomorphism of $\mathrm{rad}(B)_B$ can be left to an endomorphism of B_B, which is in fact a map by left multiplying of an element in B, thus an element in A. This means that the map ϕ sending a to ϕ_a is surjective. □

The following result is a general categorical property of the pair $B \subseteq A$ with $\mathrm{rad}(B)$ being a left ideal of A. Recall that each A-module can be regarded as a B-module just by the restriction of scalars, this provides us a functor F.

Lemma 4.2. (1) *The restriction functor $F : A\text{-}mod \to B\text{-}mod$ is an exact faithful functor, and has a right adjoint $G = Hom_B({}_BA_A, -) : B\text{-}mod \to A\text{-}mod$ and a left adjoint $E =: A \bigotimes_B - : B\text{-}mod \to A\text{-}mod$. In particular, E preserves projective modules and G preserves injective modules.*
(2) *For any B-module M there is a B-homomorphism $\alpha_M : GM \to M$ such that the induced map $Hom_A(X, GM) \to Hom_B(X, M)$ is an isomorphism for all A-module X.*
(3) *The kernel and the cokernel of α_M are semi-simple B-modules.*
(4) *Each simple A-module is also a semi-simple B-module via restriction.* (*In general, F does not preserve simples.*)
(5) $add(B/rad(B)) = add(F(A/rad(A)))$.
(6) $rad(A) = rad(B)A$ *if and only if* $rad({}_BFX) = F\ rad({}_AX)$ *for all A-module X, and if and only if* $F top_A(X) = top_B(FX)$ *for all A-module X, where $top_A(X)$ stands for the top of the A-module X.*

Proof. Statements (1), (2) and (4) are clear.
(3) Note that the kernel and cokernel of α_M are given by the following exact sequence according to the definition of α_M:

$$0 \to \mathrm{Hom}_B(A/B, {}_BM) \to \mathrm{Hom}_B({}_BA_A, {}_BM) \xrightarrow{\alpha_M} M \to \mathrm{Ext}^1_B(A/B, M).$$

Since the left B-module structure on $\mathrm{Hom}_B(A/B, M)$ is induced from the right B-module structure of $(A/B)_B$ and since $(A/B)\mathrm{rad}(B) = (A\ \ \mathrm{rad}(B) + B)/B \subseteq$

$(\mathrm{rad}(B)+B)/B=0$, we know that $\mathrm{Hom}_B(A/B,M)$ is a semi-simple B-module. Similarly, we have that $\mathrm{Ext}^1_B(A/B,M)$ is a semi-simple B-module.

(5) Clearly, $\mathrm{add}(F(A/\mathrm{rad}(A))) \subseteq \mathrm{add}(B/\mathrm{rad}(B))$ by (4). Since the inclusion $B \subseteq A$ induces an injective B-module homomorphism from $B/\mathrm{rad}(B)$ to the B-module $A/\mathrm{rad}(A)$ by Lemma 4.1, we see that the socle of $B/\mathrm{rad}(B)$ is contained in the socle of $A/\mathrm{rad}(A)$, but both B-modules are semi-simple, thus $\mathrm{add}(B/\mathrm{rad}(B)) \subseteq \mathrm{add}(F(A/\mathrm{rad}(A)))$.

(6) The first statement is obvious, and the second statement follows from the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{rad}(FX) & \longrightarrow & FX & \longrightarrow & \mathrm{top}_B(FX) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & F\mathrm{rad}({}_AX) & \longrightarrow & FX & \longrightarrow & F\mathrm{top}_A(X) & \longrightarrow & 0 \end{array}$$

by the snake lemma. $\square$

The following is a homological property of the pair $B \subseteq A$.

Lemma 4.3. *Let A be an idealized extension of B with $rad(B)A = rad(A)$.*

(1) *If ${}_BX$ is a B-module of positive projective dimension $m < \infty$, then $\Omega^m_B(X)$ is a projective A-module.*

(2) *If ${}_AX$ is an A-module such that FX is a projective B-module, then ${}_AX$ is a projective A-module.*

Proof. (1) It suffices to show that this is true for $m = 1$. In this case, $\Omega_B(X)$ is a projective B-module and also an A-module. Let $f: Q \to \Omega_B(X)$ be a projective cover of the A-module $\Omega_B(X)$. Then there is a B-module homomorphism $f': \Omega_B(X) \to FQ$ such that $f'(Ff) = id$. Note that $\mathrm{top}_B(\Omega_B(X)) = \mathrm{top}_B(F\Omega_B(X)) = F\mathrm{top}_A(\Omega_B(X)) = F\mathrm{top}_A(Q) = \mathrm{top}_B(FQ)$ by Lemma 4.2(6). This implies that f' is surjective by a general homological fact. So the following diagram

$$\begin{array}{ccccccccc} & & & & \Omega_B(X) & = & \Omega_B(X) & \longrightarrow & 0 \\ & & & & {\scriptstyle f'}\downarrow & & \| & & \\ 0 & \longrightarrow & F\Omega_A(\Omega_B(X)) & \longrightarrow & FQ & \xrightarrow{Ff} & \Omega_B(X) & \longrightarrow & 0 \end{array}$$

indicates clearly that $\Omega_A(\Omega_B(X)) = 0$, that is, $\Omega_B(X)$ is a projective A-module.

(2) Let $P \to X$ be a projective cover of the A-module X. Then we have the following exact sequence

$$0 \to F\Omega_A(X) \to FP \to FX \to 0.$$

Since FX is a projective B-module, the sequence splits. On the other hand, the top of FX and the top of FP are isomorphic by 4.2(6). This implies that $F\Omega_A(X) = 0$. Thus (2) follows. $\square$

The following is a way to construct a pair $B \subseteq A$ such that $\mathrm{rad}(B)$ is a left ideal in A.

We start with an algebra B over a field k, and fix a minimal number n such that B is a subalgebra of the $n \times n$ matrix algebra $M_n(k)$ over k, so B and $M_n(k)$ have the same identity. We define A to be the set of all matrices $x \in M_n(k)$ such that $x \cdot \mathrm{rad}(B) \subseteq \mathrm{rad}(B)$. Note that A is the largest subring of $M_n(k)$ containing $\mathrm{rad}(B)$ as a left ideal. We call A the (left) *idealized extension* of B. In the literature the idealizers or subidealizers of right ideals of rings are studied intensively, but most of the authors assume that the right ideals considered are idempotent, this cannot happen in our case. However, our construction appears in the study of orders over a Dedekind domain (for example, see [10] and [14]).

Now we define $A_0 = B$, and $A_1 = A$. For $i \geqslant 1$, we define A_{i+1} is the idealized extension of A_i. Note that all A_i are subalgebras of $M_n(k)$ with the same identity. Thus there is a minimal number s such that $A_s = A_{s+1} = \cdots \subset M_n(k)$. (In practice, we may choose any matrix algebra $M_n(k)$ containing B and do not require the minimality of n.)

Lemma 4.4. (1) *$A_i \neq A_{i+1}$ if and only if $rad(A_{i-1})$ is not a left ideal of A_{i+1}.*

(2) *$A_{i+1}rad(A_{i-1}) \subseteq rad(A_i)$ for all $i \geqslant 1$.*

(3) *A_s is the maximal subalgebra of $M_n(k)$ containing $rad(A_s)$ as a two-sided ideal.*

(4) *If $rad(A_i)A_{i+1} = rad(A_{i+1})$ for all i, then $rad(A_0)A_j = rad(A_j)$ for all j.*

The following result, which is a generalization of Theorem 3.1, shows that our construction can provide algebras of finite finitistic dimension.

Theorem 4.5. *Let A, B and C be three artin algebras with the same identity such that* (i) *$C \subseteq B \subseteq A$, and* (ii) *the Jacobson radical of C is a left ideal of B and the Jacobson radical of B is a left ideal of A. If A is representation-finite, then C has finite finitistic dimension.*

Proof. Suppose that $X_1, \ldots, X_n$ form a complete list of non-isomorphic indecomposable A-modules. Let Y be a C-module of finite projective dimension. Then we know from the proof of 3.1 that the C-syzygy $\Omega_C(Y)$ of Y is a B-module. Let us take a B-projective cover $P_B(\Omega_C(Y))$ of $\Omega_C(Y)$:

$$0 \to \Omega_B(\Omega_C(Y)) \to P_B(\Omega_C(Y)) \to \Omega_C(Y) \to 0.$$

Then $\Omega_B(\Omega_C(Y))$ is an A-module, and thus there are non-negative integers t_i such that $\Omega_B(\Omega_C(Y)) = \bigoplus_i X_i^{t_i}$. Now we consider all these modules as C-modules by restriction and use Lemma 2.1 to bound the projective dimension:

$$\begin{aligned}
\text{proj.dim}({}_CY) &\leqslant \text{proj.dim}\Omega_C(Y) + 1 \\
&= \Psi(\Omega_C(Y)) + 1 \\
&\leqslant \Psi(P_B(\Omega_C(Y)) \oplus \Omega_B(\Omega_C(Y))) + 1 + 1 \\
&= \Psi\left(P_B(\Omega_C(Y)) \oplus \bigoplus_i X_i^{t_i}\right) + 2 \\
&\leqslant \Psi\left({}_CB \oplus \bigoplus_i X_i\right) + 2.
\end{aligned}$$

This shows that proj.dim$({}_CY)$ is bounded by $\Psi({}_CB \oplus \bigoplus_i X_i) + 2$. $\square$

In the following we shall see that the algebra A_s in our construction is in fact a representation-finite algebra. Let us recall some definitions from order theory.

Let R be a discrete valuation ring, and let π be an element of R which generates the unique maximal ideal of R. Let K and k be the fraction field and the residue field of R, respectively. Suppose Σ is a central simple K-algebra. An R-order Λ in Σ is a subring of Σ with the same identity satisfying the following conditions: (i) the center of Λ contains R, (ii) Λ is finitely generated R-module and (iii) $K \cdot \Lambda = \Sigma$. For any R-order Λ in A, we have $\mathrm{rad}(\Lambda) = \pi\Lambda$.

Note that we consider $B = A_0$ as a subalgebra of $M_n(k)$. Now let Γ be the maximal R-order $M_n(R)$ in Σ. Then we have $\Gamma/\mathrm{rad}(\Gamma) = M_n(k)$. Let us denote the canonical projection from Γ to $M_n(k)$ by ϕ. We define Λ to be the pullback:

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \phi \\ A_0 & \longrightarrow & M_n(k). \end{array}$$

Then Λ is an R-order in Σ. We define $\Lambda_0 = \Lambda$ and consider Λ_0 as a subring of Σ. Now let Λ_i be the idealized extension of Λ_{i-1}. Then we obtain a chain of orders which must stop after finite number of steps at an order Λ_s, namely, $\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_s$, all are suborders of Γ satisfying $\mathrm{rad}(\Gamma) \subseteq \mathrm{rad}(\Lambda_i)$ and $\mathrm{rad}(\Lambda_i) \subset \mathrm{rad}(\Lambda_{i+1})$. This chain of R-orders in Σ under the ϕ-projection gives us the chain $A_0 \subset A_1 \subset \cdots \subset A_s$. Since $\mathrm{rad}\Lambda_i = \pi\Lambda_i$, we know that $\mathrm{rad}(\Lambda_i) = \mathrm{rad}(\Lambda_{i-1})\Lambda_i$. Thus $\mathrm{rad}(A_i) = \mathrm{rad}(A_{i-1})A_i$ for all i.

The following lemma shows that the algebra A_s is always representation-finite.

Lemma 4.6. *If k is the residue field of a discrete valuation ring R, then A_s is Morita equivalent to a direct sum of lower triangular matrix algebras $T_m(k)$ over k. In particular, A_s is representation-finite.*

Proof. It follows from the above construction that A_s is a ϕ-projection of a hereditary R-order Λ_s in the central simple algebra Σ and that Λ_s is contained in the maximal R-order $\Gamma := M_n(R)$. (Note that under our assumption, an R-order Λ in Σ is hereditary if and only if Λ is the idealized extension of Λ.) By [3, Theorem 26.28, p. 577], Λ_s has a "block" form, if we factor out from Λ_s the radical of Γ which is contained in the radical of Λ_s, then we get a lower block triangular matrix algebra over k, and thus A_s is Morita-equivalent to a direct sum of copies of some $T_m(k)$ with $m \leqslant n$. $\square$

As a consequence of Lemma 4.6 and Theorem 4.5, we have the following corollary.

Corollary 4.7. *Let k be the residue field of a discrete valuation ring, and let s be defined as above. If $s > 2$, then A_{s-2} and A_{s-1} have finite finitistic dimension.*

Let us look at Example 1 again to demonstrate the above construction.

We take $A_0 := B$ and $A_1 := A$ to be the algebras in Example 1. Then a calculation shows that the idealized extension A_2 of A_1 is

$$A_2 = \left\{ \begin{pmatrix} a & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & f & 0 \\ x & y & z & u \end{pmatrix} | a,b,c,d,e,f,u,x,y,z \in k \right\}.$$

Since the radicals of A_1 and A_2 coincide with each other, we know that the construction stops at $s = 2$. It is easy to see that the algebra A_2 is representation-finite. In fact, a basis change shows that A_2 is isomorphic to a 4×4 upper triangular matrix algebra over k.

Now let us point out that the above consideration gives another formulation of the finitistic dimension conjecture.

Corollary 4.8. *Let k be the residue field of a discrete valuation ring. Then the following are equivalent*:

(1) *The finitistic dimension conjecture is true for all k-algebras*;

(2) *If $B \subseteq A$ is a pair of k-algebras with the same identity such that $rad(B)$ is a left ideal in A and if A has finite finitistic dimension*, *then B has finite finitistic dimension.*

Finally, let us mention the following result concerning the condition $\mathrm{rad}(B)A = \mathrm{rad}(A)$.

Proposition 4.9. *Let B be a subalgebra of an artin algebra A such that $rad(B)$ is a left ideal in A and $rad(A) = rad(B)A$. If the subcategory $\Omega_A(A\text{-}mod) = \{X \in A\text{-}mod|$ there is an A-module Y such that $X \simeq \Omega_A(Y)\}$ is finite, then $fin.dim(B)$ is finite.*

Proof. The condition $\mathrm{rad}(B)A = \mathrm{rad}(A)$ implies that ${}_B\mathrm{top}_A(X) = \mathrm{top}_B({}_BX)$ for all A-modules ${}_AX$ by Lemma 4.2(6).

Now let ${}_BX$ be a B-module with $\mathrm{proj.dim}({}_BX) < \infty$. Then we consider the first syzygy $\Omega_B(X)$. This is also an A-module. Let $P_A(\Omega(X))$ be an A-projective cover of $\Omega_B(X)$, and let Q be a B-projective cover of $\Omega_B(X)$. Then we have the following commutative exact diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}_B\Omega_B^2(X) & \longrightarrow & Q & \longrightarrow & \Omega_B(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & {}_B\Omega_A(\Omega_B(X)) & \longrightarrow & {}_BP_A(\Omega_B(X)) & \longrightarrow & \Omega_B(X) & \longrightarrow & 0 \end{array}$$

Since $\mathrm{top}_B(Q) \simeq \mathrm{top}_B(\Omega_B(X))$ and $\mathrm{top}_B P_A(\Omega_B(X)) \simeq {}_B\mathrm{top}_A P_A(\Omega_B(X)) \simeq {}_B\mathrm{top}_A \Omega_B(X) \simeq \mathrm{top}_B({}_B\Omega_B(X))$, we see that as B-modules, Q and ${}_BP_A(\Omega_B(X))$ have the same tops, and this implies that the map β is surjective. Moreover, it is a B-projective cover of ${}_BP_A(\Omega_B(X))$. Hence, the snake lemma yields the following exact sequence

$$0 \to \Omega_B({}_BP_A(\Omega_B(X))) \to \Omega_B^2(X) \to \Omega_A(\Omega_B(X)) \to 0.$$

By assumption, $\mathrm{add}\Omega_A(A\text{-mod})$ is finite, let $Y_1,\ldots,Y_s$ form a complete list of non-isomorphic indecomposable modules in $\mathrm{add}\Omega_A(A\text{-mod})$. Then $\Omega_A(\Omega_B(X))=\bigoplus_j Y_j^{t_j}$ for some non-negative integers t_j. Now we have the following estimation

$$\begin{aligned}
\mathrm{proj.dim}({}_BX) &\leqslant \mathrm{proj.dim}({}_B\Omega_B^2(X))+2\\
&= \Psi({}_B\Omega_B^2(X))+2\\
&\leqslant \Psi\left(\Omega_B(\Omega_B({}_BP_A(\Omega_B(X))))\oplus\Omega^2\left(\bigoplus_j Y_j^{t_j}\right)\right)+2+2\\
&= \Psi\left(\Omega_B^2({}_BP_A(\Omega_B(X)))\oplus\Omega_B^2\left(\bigoplus_j Y_j^{t_j}\right)\right)+2+2\\
&= \Psi\left(\Omega_B^2({}_BP_A(\Omega_B(X)))\oplus\bigoplus_j Y_j^{t_j}\right)+4\\
&\leqslant \Psi\left(\Omega_B^2\left({}_BA\oplus\bigoplus_{j=1}^{s} Y_j\right)\right)+4.
\end{aligned}$$

This proves what we wanted. □

5. Questions

The results in this note suggest that the following questions related to the finitistic dimension and representation dimension might be answered.

Question 1. Let C and B be two representation-finite algebras over a field. Does the trivially twisted extension of C and B at S has the representation dimension at most 3?

Question 2. Let A be an artin algebra and J an ideal in A with $J^3=0$. If A/J is representation-finite, is the finitistic dimension conjecture true for A?

Note that if A/J^2 is representation-finite then the finitistic dimension of A is finite. This follows easily from 3.2. It is also well-known that if J is the Jacobson radical of A then the finitistic dimension conjecture for A is true.

Question 3. Let A and B be two artin algebras, and let $f:B\to A$ be a surjective homomorphism of algebras such that the square of $\ker(f)$ vanishes. If the representation dimension of A is at most 3, is the finitistic dimension conjecture true for B?

Question 4. Let A be an artin algebra and I an ideal in A with $I^2=0$. If A/I is representation-finite, does the algebra A has the representation dimension at most 3?

This question has the positive answer in the case $I=\mathrm{rad}(A)$ or $I=\mathrm{rad}^n(A)$ with $n+1$ the nilpotency index of $\mathrm{rad}(A)$.

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