Journal of Algebra 321 (2009) 1632–1642

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The Isaacs–Navarro conjecture for the alternating groups

Rishi Nath

Department of Mathematics, York College-CUNY, Jamaica, NY 11418, United States

ARTICLE INFO

Article history:
Received 31 October 2007
Available online 21 January 2009
Communicated by Ronald Solomon

Keywords:
McKay conjecture
Isaacs–Navarro conjecture
Alternating groups

ABSTRACT

A recent refinement of the McKay conjecture is verified for the case of the alternating groups. The argument builds upon the verification of the conjecture for the symmetric groups [P. Fong, The Isaacs–Navarro conjecture for symmetric groups, J. Algebra 250 (1) (2003) 154–161].

1. Introduction

Let G be a finite group, p be a prime number and $G^{\vee}_{p'}$ be the set of *p'-irreducible* characters of G i.e. the complex irreducible characters whose degree is relatively prime to p. The McKay conjecture asserts that

$$|G^{\vee}_{p'}| = |N_G(P)^{\vee}_{p'}|$$

where P is a Sylow p-subgroup of G and $N_G(P)$ is the normalizer of P in G.

The McKay conjecture has been verified for many families of groups including the symmetric groups and alternating groups (see [8]). However the underlying reason for this phenomenon remains a mystery.

One approach to a further understanding of the McKay conjecture is to refine the statement of the conjecture as precisely as possible. The Alperin–McKay conjecture [1] is one such refinement. Let B be a Brauer p-block of G and D be the defect group of B. Let b be the p-block of $N_G(D)$ that is the Brauer correspondent of B. Let ν be the exponential valuation of $\mathbb{Z}$ associated with p, normalized so that $\nu(p) = 1$. The *height* $h(\chi)$ of a character χ in B is a non-negative integer such that

E-mail address: rnath@york.cuny.edu.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.11.041

$$\nu\big(\chi(1)\big) = \nu\big(|G|\big) - \nu\big(|D|\big) + h(\chi).$$

Similarly, the height of a character ϕ in b is the non-negative integer $h(\phi)$ such that $\nu(\phi(1)) = \nu(|N_G(D)|) - \nu(|D|) + h(\phi)$. Let $M(B)$ and $M(b)$ be the sets of characters in B and b of height 0. The Alperin–McKay conjecture asserts that $|M(B)| = |M(b)|$.

M. Isaacs and G. Navarro proposed a new refinement of the McKay conjecture. Their first formulation (Conjecture A, [4]) requires defining $M_k(G)$ as the set of irreducible characters of G whose degrees are congruent to $\pm k \pmod p$ where k is an integer relatively prime to p.

Conjecture 1.1 *(Isaacs–Navarro). For each integer k not divisible by p*

$$\big|M_k(G)\big| = \big|M_k\big(N_G(P)\big)\big|.$$

Their second formulation (Conjecture B, [4]) requires defining $M_k(B)$ the set of height zero characters in a p-block B for which the p'-part of the degree is congruent to $\pm k \pmod p$.

Conjecture 1.2 *(Block Isaacs–Navarro). Let B be a p-block of G and suppose that b is the Brauer correspondent of B with respect to some defect group D. Then for each integer k not divisible by p,*

$$\big|M_{ck}(B)\big| = \big|M_k(b)\big|, \quad \text{where } c = \big|G : N_G(D)\big|_{p'}.$$

Let Π be a set of size n, and $S(\Pi)$ and $A(\Pi)$ respectively be the symmetric and alternating groups on Π. The *splitting characters* $S(\Pi)^\vee_*$ are irreducible characters of $S(\Pi)$ that split into two conjugate characters when restricted to $A(\Pi)$. The *p'-splitting characters* $S(\Pi)^\vee_{p',*}$ are characters in $S(\Pi)^\vee_*$ whose degree is prime to p. Let B be a p-block of $S(\Pi)$ with defect group D. Now suppose $H = N_{S(\Pi)}(D)$ and $H^+ = N_{S(\Pi)}(D) \cap A(\Pi)$. Then $H^\vee_*$ and $H^\vee_{p',*}$ are the irreducible characters and p'-irreducible characters that split over H^+ respectively. In this paper we describe a bijection between $S(\Pi)^\vee_{p',*}$ and $N_{S(\Pi)}(D)^\vee_{p',*}$ of which the Isaacs–Navarro conjecture for $A(\Pi)$ is a consequence.

2. Combinatorial description of splitting characters of $S(n)$

Given a partition λ, we denote its *dual* (in the sense of Eq. (1.4.3), [5]) by λ^*. Then λ is *symmetric* or *self-dual* if $\lambda = \lambda^*$. By a classical result of Frobenius $S(\Pi)^\vee_*$ are labeled precisely by symmetric partitions of n. We recall every partition λ can be expressed uniquely in terms of its *p-core* λ^0 and *p-quotient* $\{\lambda_\gamma\}_{1\leqslant\gamma\leqslant p}$ (see Chapter 2 in [5] for details). There is the following relationship between the p-core and p-quotient of λ and λ^* (p. 3481, [3]).

Theorem 2.1. *Let $(\lambda^*)^0$ and $\{\lambda^*_1, \ldots, \lambda^*_p\}$ be the p-core and p-quotient of λ^* respectively. Then $(\lambda^*)^0 = (\lambda^0)^*$ and $(\lambda^*)_\gamma = (\lambda_{p+1-\gamma})^*$ for $1 \leqslant \gamma \leqslant p$. In particular, $\lambda = \lambda^*$ if and only if $\lambda^0 = (\lambda^0)^*$ and $\lambda_\gamma = (\lambda_{p+1-\gamma})^*$ for $1 \leqslant \gamma \leqslant p$.*

Let ν_p be the p-adic valuation on $\mathbb{Z}$ (so that $\nu_p(q) = \mu$ if p^μ divides q but $p^{\mu+1}$ does not). Each diagram λ_i has in turn a p-core λ_i^0 and a p-quotient $(\lambda_{i1}, \ldots, \lambda_{ip})$. Let $c_1 = \sum |\lambda_i^0|$ and $n_2 = \sum |\lambda_{ij}|$, where (λ_{ij}) is the sequence of p partitions that form the p-quotient of λ_i.

At the rth level we have p^r partitions $(\lambda_{i_1 i_2 \cdots i_r})^0$, each a p-core. In addition we inherit p^{r+1} (p for each of the p^r) partitions $\lambda_{i_1\cdots i_{r+1}}$. Then $(i_1, \ldots, i_r) \in I^r$, indexes the partitions $\lambda_{i_1\cdots i_r}$ at the rth level. Let $c_r = \sum_{(i_1,\ldots,i_r)\in I^r} |(\lambda_{i_1\cdots i_r})^0|$, $n_r = \sum_{(i_1,\ldots,i_r)\in I^r} |\lambda_{i_1 i_2\cdots i_r}|$. We define the *rth level p-core* $C_{\lambda,r}$ to be the set

$$C_{\lambda,r} := \big\{(\lambda_{i_1\cdots i_r})^0 \text{ where } (i_1, \ldots, i_r) \in I^r\big\}.$$

Then the *p-core tower* C_λ is $\bigcup_{r\geqslant 0} C_{\lambda,r}$ where $C_{\lambda,0}=\{\lambda^0\}$. The sum

$$c_r=\sum \left|(\lambda_{i_1\cdots i_r})^0\right|$$

will be called the *p-core sum at level r*. Now given $i\in I=\{1,\ldots,p\}$, we let $i^*=p+1-i$. Given $\underline{i}=(i_1,\ldots,i_r)\in I^r$ let $\underline{i}^*=(p+1-i_1,\ldots,p+1-i_r)$. The *dual* C^*_λ of a p-core tower $C_\lambda=\bigcup_{r\geqslant 0}C_{\lambda,r}$ is defined as follows:

$$C^*_\lambda=\bigcup_{r\geqslant 0} C^*_{\lambda,r}$$

where $C^*_{\lambda,r}=\{\gamma_{\underline{i}}\colon\ \underline{i}\in I^r,(\gamma_{\underline{i}})^*=(\lambda_{\underline{i}^*})^0\}$. A p-core tower C_λ is *self-dual* if $C_{\lambda,r}=C^*_{\lambda,r}$ for all r. C_λ has *height* k if k is the minimal non-negative integer k such that $(\lambda_{i_1\cdots i_r})^0=\emptyset$ for all $r>k$. [Note that if $\lambda=\lambda^0$ is a p-core, C_λ has height 0.]

Theorem 2.2. *Let γ and λ be partitions of n. Then $\gamma=\lambda^*$ if and only if $C^*_\gamma=C_\lambda$. In particular, $C_\lambda=C^*_\lambda$ if and only if $\lambda=\lambda^*$.*

Proof. By induction on the height k. Suppose C_λ has height $k=1$. Then $C_{\lambda,r}$ consists of empty sets for $r\geqslant 2$. By Theorem 2.1, $\gamma=\lambda^*$ if and only if $\gamma_i^*=\lambda_{p+1-i}$ and $(\gamma^0)^*=\lambda^0$.

Suppose that the theorem is true for height $k-1$ and consider C_λ with height k. Then we have the following equivalences.

$$C^*_\gamma=C_\lambda \iff C^*_{\gamma_i}=C_{\lambda_{p+1-i}} \quad \text{for all } i, \quad \text{and}$$
$$C^*_{\gamma,0}=C_{\lambda,0} \iff \gamma_i=\lambda^*_{p+1-i} \quad \text{for all } i, \quad \text{and} \quad \left(\gamma^0\right)^*=\lambda^0.$$

The first equivalence follows by the definition of two core towers being self-dual. The second follows by the induction hypothesis since γ_i and λ_{p+1-i} are partitions whose core towers have height at most $k-1$. □

We let $n=n_0+n_1p+n_2p^2+\cdots+n_rp^r$ be the p-adic decomposition of n so the n_i satisfy $0\leqslant n_i<p$. The following theorem appears in Section 4 of [6].

Theorem 2.3 *(MacDonald Criterion). Let $n=\sum_{r\geqslant 0}n_rp^r$ be the p-adic decomposition of n. Let C_λ be the p-core tower of λ, and c_r be the p-core sum at each level $r\geqslant 0$. Suppose $\chi_\lambda\in S(\Pi)^\vee$. Then $\nu_p(\chi_\lambda(1))=0$ if and only if $c_r=n_r$ for all r.*

Corollary 2.4. *Let p be an odd prime. Then $S(\Pi)^\vee_{p',*}$ is the set $\{\chi_\lambda\}$ such that the p-core tower C_λ is self-dual and $\sum_{\underline{i}}|\lambda^0_{i_1\cdots i_k}|=n_k$ for all k.*

Proof. This follows from Theorem 2.2 and Theorem 2.3. □

3. Block theory for the symmetric groups

Following [2], we describe a bijection between height zero irreducible characters of $S(\Pi)$ and $N_{S(\Pi)}(X)$. We partition Π as $\Pi_-\cup\Pi_+$ where

$$\Pi_-=\{x\in\Pi \mid Dx=x\} \quad \text{and} \quad \Pi_+=\Pi-\Pi_-.$$

Let $|\Pi| = n$, let $|\Pi_-| = n_-$ and $|\Pi_+| = n_+$ where $n = n_- + n_+$. Note that $n_+ \equiv 0 \pmod p$. Let B_+ be the *principal block* of $S(\Pi_+)$, i.e. the block containing the identity character. Recall the Nakayama conjecture: Two ordinary irreducible representations χ_λ and $\chi_{\lambda'}$ of $S(\Pi)$ belong to the same p-block if and only if the λ and λ' have the same p-core (see Chapter 6.1, [5]).

Thus the Nakayama conjecture implies that B and B_+ are parametrized respectively by a p-core partition $\kappa \vdash n_-$ and the empty partition. So

$$B = \{\chi_\lambda \in S(\Pi) \mid \lambda^0 = \kappa\},$$
$$B_+ = \{\chi_{\lambda_+} \in S(\Pi_+) \mid (\lambda_+)^0 = \emptyset\}.$$

D is then a Sylow p-subgroup of $S(\Pi_+)$ and a defect group of B_+. Given a partition $\lambda \vdash n$ such that $\lambda^0 = \kappa$ there exists a partition $\lambda_+ \vdash n_+$ with empty p-core and the same p-quotient as λ. Conversely, given a partition λ_+ of n_+ with empty p-core, we let λ be the partition of n with p-core κ and p-quotient the same as λ_+. The correspondences $\mu \to \lambda_+$ and $\lambda_+ \to \mu$ are inverses to each other and induce a bijection $\beta : B \mapsto B_+$ such that

$$\beta(\chi_\lambda) = \chi_{\lambda_+}.$$

The following is Lemma 1.3 in [2].

Lemma 3.1. *The bijection $\beta : B \to B_+$ where $\beta(\chi_\lambda) = \chi_{\lambda_+}$ is height-preserving. In particular, $\chi_\lambda \in M(B)$ if and only if $\sum_{k\geqslant 1} c_k(\lambda)p^k$ is the p-adic expansion of n_+.*

We now consider $n_+ = n_1 p + n_2 p^2 + \cdots$. Let Δ_k be a set of size n_k for $n_k \geqslant 1$. Let $I = \{1, \ldots, p\}$, $\Pi_k = (I^k)^{\Delta_k}$ and $\Pi_+ = \bigsqcup_{k\geqslant 1} \Pi_k$. Notice $S(I^k)^{\Delta_k}$ and $\prod_{k\geqslant 1} S(I^k)^{\Delta_k}$ act componentwise on Π_k and Π_+ respectively. Given $X_1 \in \mathrm{Syl}_p(S(I))$ that $X_k = X_1 \wr \cdots \wr X_1$ (the k-fold wreath product) is a Sylow p-subgroup of $S(I^k)$. Hence $X_k^{\Delta_k} \in \mathrm{Syl}_p(\Pi_k)$ and $X = \prod_{k\geqslant 1} {X_k}^{\Delta_k} \in \mathrm{Syl}_p(S(\Pi_+))$. Since D is a Sylow p-subgroup of $S(\Pi_+)$, we may assume $D = X$. Now consider $Y_k = N_{S(I^k)}(X_k)$, and let $Y = N_{S(\Pi)}(X)$. By Proposition 1.5 in [8] we have the isomorphism α_k where

$$\alpha_k : Y_k/[X_k, X_k] \simeq Y_1^k.$$

Hence $Y/[X, X] = \prod_{k\geqslant 1} Y_k/[X_k, X_k] \simeq \prod_{k\geqslant 1} Y_1^k$. Since

$$N_{S(\Pi)}(X) = S(\Pi_-) \times \prod_{k\geqslant 1} Y_k \wr S(\Delta_k)$$

then

$$N_{S(\Pi)}(X)/X' \simeq S(\Pi_-) \times \prod_{k\geqslant 1} Y_1^k \wr S(\Delta_k).$$

Let b be the Brauer correspondent of B. Then b consists of characters $\omega_\kappa \times \psi$ where ω_κ is the character of $S(\Pi_-)$ corresponding to the p-core partition κ labeling B and $\psi = \prod \psi_k$ where $\psi_k \in [Y_k \wr S(\Delta_k)]^\vee$. Suppose $(\psi(1), p) = 1$. By a result of Clifford, $\mathrm{Res}_X^{N_{S(\Pi)}(X)}(\psi) = e\sum_{i=1}^t \theta_t$, where $\{\theta_t\}$ are the conjugates of $\theta \in \mathrm{Irr}(X)$ and e is a constant. Since $|X| = p^j$ and $\theta_t(1)$ divides $|X|$, we have $\theta_t(1) = 1$, for all t. Hence

$$\{\psi \mid \psi \in N_{S(\Pi_+)}(X)_{p'}^\vee\} = (N_{S(\Pi_+)}(X)/X')^\vee = \prod_{k\geqslant 1} (Y_1^k \wr S(\Delta_k))^\vee.$$

Since Y_1 is a Frobenius group, Y_1 has p characters, $p-1$ of which $\{\xi_i\}_{1\leqslant i\leqslant p-1}$ have degree 1, and one of which ξ_p has degree $p-1$. Hence we can label the elements of $(Y_1^k)^\vee$ as k-tuples $(\xi_{i_1},\ldots,\xi_{i_k})$ or $\xi_{\underline{i}}$, where $\underline{i}=(i_1,\ldots,i_k)\in I^k$.

Let Λ_k be a partition-valued function on $(Y_1^k)^\vee$ whose values $\Lambda_k(\xi_{\underline{i}})$ satisfy

$$\sum_{\underline{i}\in I^k}\bigl|\Lambda_k(\xi_{\underline{i}})\bigr|=n_k.$$

In particular, each $\Lambda_k(\xi_{\underline{i}})$ is a p-core since $n_k<p$. We partition Δ_k into disjoint subsets $\Delta_{k,\underline{i}}$ of size $|\Lambda_k(\xi_{\underline{i}})|$ for $\underline{i}\in I^k$. Let ξ_{Λ_k} be the character of the base group $Y_k^{\Delta_k}$ with component $\xi_{\underline{i}}$ in positions indexed by elements of $\Delta_{k,\underline{i}}$. We note $\xi_k(1)\equiv\pm1 \pmod p$ since $\xi_{\underline{i}}(1)\equiv\pm1 \pmod p$ for all $\underline{i}$. Then the stabilizer of ξ_{Λ_k} in $S(\Delta_k)$ is:

$$S(\Delta_k)_{\xi_{\Lambda_k}}=\prod_{\underline{i}\in I^k}S(\Delta_{k,\underline{i}})$$

and ξ_{Λ_k} extends to a character $E(\xi_{\Lambda_k})$ of $Y_k^{\Delta_k}\cdot S(\Delta_k)_{\xi_{\Lambda_k}}$. Let ω_{Λ_k} be the character of $S(\Delta_k)_{\xi_{\Lambda_k}}$ with component $\omega_{\Lambda_k(\xi_{\underline{i}})}$ on $S(\Delta_{k,\underline{i}})$. We describe the p'-irreducible characters of $Y_k\wr S(\Delta_k)$ by an application of Clifford's theory and is Theorem 4.3.34 in [5].

Theorem 3.2.

$$\psi_k=Ind_{Y_k^{\Delta_k}S(\Delta_k)_{\xi_{\Lambda_k}}}^{Y_k^{\Delta_k}S(\Delta_k)}\bigl(E(\xi_{\Lambda_k})\omega_{\Lambda_k}\bigr)$$

is a p'-irreducible character of $(Y_k\wr S(\Delta_k))^\vee$. Moreover, every p'-irreducible character of $Y_k\wr S(\Delta_k)$ is of this form.

The following is Eq. 2.7 in [2].

Theorem 3.3. *$M(B)$ and $M(b)$ are in bijection via where*

$$\omega_\lambda\mapsto\omega_\kappa\times\psi_{\lambda_+}.$$

4. Equivalence of sign characters

We seek to describe the p'-splitting characters of $Y_k\wr S(\Delta_k)$ combinatorially. We must first describe the relevant sign character of Y_k/X_k' (by the discussion following Lemma 3.1 and the isomorphism α_k: $Y_k/X_k'\simeq Y_1^k$). Let sgn_k be the sign function of $S(I^k)$ with respect to the alternating group $A(I^k)$. In particular, since $X_k'\subseteq A(I^k)$, sgn_k is constant on cosets of X_k'.

Here $\mathrm{sgn}_{Y_1^k}$ is the sign function of Y_1^k that is, $\mathrm{sgn}_{Y_1^k}(y_1\times\cdots\times y_k)=\prod_{i=1}^{i=k}\mathrm{sgn}_{Y_1}(y_i)$. We normalize $(Y_1^k)^\vee=\{\xi_{\underline{i}}\mid \underline{i}\in I^k\}$ so that $\xi_{(1,\ldots,1)}=\mathrm{sgn}_{Y_1^k}$ and $\xi_{(i_1,\ldots,i_k)}\cdot\xi_{(1,\ldots,1)}=\xi_{(i_1^*,\ldots,i_k^*)}$ where $i^*=p+1-i$.

Lemma 4.1. *Consider $gX_k'\in Y_k/X_k'$, let α_k be defined as above, and suppose $\alpha_k(gX_k')=(y_1,\ldots,y_k)$ where $y_i\in Y_1^k$. We claim that*

$$\mathrm{sgn}_k(g)=\mathrm{sgn}_{Y_1{}^k}(y_1,\ldots,y_k).$$

Proof. Following the argument in Proposition (1.5) of [8] we write elements of $S(I^{k-1}) \wr S(I) = S(I^{k-1})^I \rtimes S(I)$ as $(g_1, g_2, \dots, g_p; y)$ where $g_i \in S(I^{k-1})$ and $y \in S(I)$. Then

$$X'_k = \{(g_1, g_2, \dots, g_p; 1)\colon\ g_i \in X_{k-1} \text{ and } g_1 g_2 \cdots g_p \in X'_{k-1}\},$$
$$Y_k = \{(g_1, g_2, \dots, g_p; y)\colon\ g_i \in Y_{k-1}, g_i \equiv g_j \pmod{X_{k-1}}\ i, j \text{ and } y \in Y_1\}.$$

In particular, Y_k contains subgroups M and F where

$$M = \{(g_1, g_2, \dots, g_p; 1)\colon\ g_i \in Y_{k-1}, g_i \equiv g_j \pmod{X_{k-1}} \text{ for all } i \text{ and } j\},$$
$$F = \{(1, 1, \dots, 1 : y)\colon\ y \in Y_1\},$$

such that $M \lhd Y_k$, $Y_k = MF$, and $M \cap F = 1$. Now $[M, F] \leqslant X'_k \leqslant M$, so that

$$\phi : Y_k / X'_k \simeq M / X'_k \times F,$$

where $\phi((g_1, g_2, \dots, g_p; y)X'_k) = ((g_1, g_2, \dots, g_p; 1)X'_k, y)$. We claim

$$M/X'_k \simeq Y_{k-1}/X'_{k-1}.$$

By the Schur–Zassenhaus theorem, $Y_{k-1} = X_{k-1}T$ for some subgroup T of Y_{k-1} such that $T \cap X_{k-1} = 1$. Let $xt \in Y_{k-1}$, where $x \in X_{k-1}$ and $t \in T$. Let

$$\Phi : Y_{k-1} \to M/X'_k, \quad xt \to (xt, t, \dots, t; 1)X'_k.$$

Note $(xt, t, \dots, t; 1) \in M$ by the above description of M. We have $\Phi(xt) \in X'_k$ if and only if $xt^p \in X'_{k-1}$, that is, if and only if $x \in X'_{k-1}$ and $t = 1$. Moreover, Φ is surjective. For an element of M has the form $(x_1 t, x_2 t, \dots, x_p t; 1)$, where $x_i \in X_{k-1}$ and $t \in T$. Now

$$\begin{aligned}(x_1 t, x_2 t, \dots, x_p t; 1) &= (x_1, x_2, \dots, x_p; 1)(t, t, \dots, t; 1)\\ &\equiv (x, 1, 1, \dots, 1; 1)(t, t, \dots, t; 1) \pmod{X'_k}\\ &\equiv (xt, t, t, \dots, t; 1) \pmod{X'_k}\end{aligned}$$

where $x = x_1 x_2 \cdots x_p$. Then $\Phi(xt) = (x_1 t, x_2 t, \dots, x_p t; 1)X'_k$ and Φ induces an isomorphism of $Y_{k-1}/X'_{k-1} \simeq M/X'_k$. Hence α_k: $Y_k/X'_k \simeq Y_1^k$ since α_{k-1}: $Y_{k-1}/X'_{k-1} \simeq Y_1^{k-1}$ (by induction) and $F \simeq Y_1$. In summary, α_k is the composite of the isomorphisms

$$Y_k/X'_k \simeq M/X'_k \times F \to Y_{k-1}/X'_{k-1} \times Y_1 \to Y_1^k.$$

If $\alpha_k(gX'_k) = (y_1, y_2, \dots, y_k)$ and $g = (g_1, g_2, \dots, g_p; 1)(1, 1, \dots, 1; y)$, where the factors are respectively in M and F, then $y_k = y$. Moreover, if

$$(g_1, g_2, \dots, g_p; 1)X'_k = \Phi(xt)$$

for $xt \in Y_{k-1}$, then $\alpha_{k-1}(xt) \in X'_{k-1} = (y_1, y_2, \dots, y_{k-1})$.

We now prove the relation $\mathrm{sgn}_k(g) = \mathrm{sgn}_{Y_1^k}(y_1, \dots, y_k)$ by induction on k (following an argument of P. Fong). Suppose g in Y_k and

$$g = (g_1, g_2, \dots, g_p; 1)(1, 1, \dots, 1; y)$$

where $(g_1, g_2, \dots, g_p; 1) \in M$ and $(1, 1, \dots, 1; y) \in F$. Then

$$\begin{aligned} \operatorname{sgn}_k(g) &= \operatorname{sgn}_k(g_1, g_2, \dots, g_p; 1) \operatorname{sgn}_k(1, 1, \dots, 1; y) \\ &= (\operatorname{sgn}_{Y_1} y) \prod_{i=1}^{p} \operatorname{sgn}_{k-1}(g_i). \end{aligned}$$

For p odd, we have $\operatorname{sgn}_k(1, 1, \dots, 1; y) = \operatorname{sgn}_Y(y)$. This follows by viewing y as a permutation matrix of degree p and $(1, 1, \dots, 1; y)$ as the permutation matrix of degree p^k obtained from y by replacing 0 and 1 respectively by the zero matrix and the identity of degree p^{k-1}. Taking determinants then gives $\operatorname{sgn}_k(1, 1, \dots, 1; y) = \operatorname{sgn}_{Y_1}(y)$. On the other hand, if we view $(g_1, g_2, \dots, g_p; 1)$ as a block diagonal matrix with permutation matrices $g_1, g_2, \dots, g_p$ of degree p^{k-1} along the diagonal, we see that

$$\operatorname{sgn}_k(g_1, g_2, \dots, g_p; 1) = \prod_{i=1}^{p} \operatorname{sgn}_{k-1}(g_i).$$

Now sgn_k is constant on cosets of X_k' and $(g_1, \dots, g_p; 1)X_k' = (xt, t, t, \dots, t; 1)X_k'$ where $x \in X_{k-1}$, $t \in T$, and $\Phi(xt) = (g_1, g_2, \dots, g_p; 1)X_k'$. Thus

$$\prod_{i=1}^{p} \operatorname{sgn}_{k-1}(g_i) = \operatorname{sgn}_{k-1}(xt)\big(\operatorname{sgn}_{k-1}(t)\big)^{p-1} = \operatorname{sgn}_{k-1}(xt).$$

Since $\operatorname{sgn}_{Y_1}(y) = \operatorname{sgn}_{Y_1}(y_k)$ and $\operatorname{sgn}_{k-1}(xt) = \prod_{i=1}^{k-1} \operatorname{sgn}_{Y_1}(y_i)$ we have, by induction, $\operatorname{sgn}_k(g) = \operatorname{sgn}_{Y_1^k}(g_1, \dots, g_k)$.

Suppose $p = 2$. The result follows since $Y_{k-1} = X_{k-1}$. □

5. A criterion for splitting characters of the normalizer

Let $|\Pi| = n_1 p$, $I = \{1, \dots, p\}$ and Δ be a set of size $n_1 < p$ so $\Pi = (I)^{\Delta}$. Let $X \in \operatorname{Syl}_p(S(\Pi))$. In this case, P. Fong and M. Harris (see Proposition (4D), [3]) obtained a criterion for a character of $H = N_{S(\Pi)}(X)$ to split when restricted to the subgroup $H^+ = H \cap A(\Pi)$: For $\psi_\Lambda \in H^\vee$ $\operatorname{Res}^H_{H^+} \psi_\Lambda = \psi_\Lambda^+ + \psi_\Lambda^-$ if and only if $\Lambda^*(\xi_i) = \Lambda(\xi_{p+1-i})$ for all i.

We extend this to the case where $n = \sum_{i \geqslant 1} n_i p^i$. First consider $n = n_k p^k$, where $n_k < p$ for $k > 1$ so $|\Pi| = n_k p^k$. From Section 3, elements of $N_G(X_k)^\vee_{p'} = [Y_k \wr S(\Delta_k)]^\vee_{p'}$ can be labeled by maps $\Lambda_k : I^k \to \{p\text{-core partitions}\}$. Consider

$$(f, \sigma) \in \big(Y_1^k\big)^{\Delta_k} \cdot S(\Delta_k)_{\xi_{\Lambda_k}}$$

where $f \in (Y_1^k)^{\Delta_k}$ and $\sigma \in S(\Delta_k)_{\xi_{\Lambda_k}}$. We calculate $E(\xi_{\Lambda_k})(f, \sigma)$. First decompose $\sigma = \sigma_1 \cdots \sigma_d$ into a product of its disjoint cycles in $S(\Delta_k)_{\xi_{\Lambda_k}}$. Let $\Delta^\sigma_{k,\delta}$ be the support of σ_δ, for $1 \leqslant \delta \leqslant d$. Let $n_\delta = |\Delta^\sigma_{k,\delta}|$, and $h_\delta \in \Delta^\sigma_{k,\delta}$. Let $\rho_{h_\delta}(f, \sigma) = f(h_\delta) \cdot f(\sigma^{-1}(h_\delta)) \cdots f(\sigma^{-(n_\delta - 1)}(h_\delta))$. Let $\Gamma_{\underline{i}}$ be the index sets such that $\Delta_{k,\underline{i}} = \bigsqcup_{\delta \in \Gamma_{\underline{i}}} \Delta^\sigma_{k,\delta}$ (since σ stabilizes each $\Delta_{k,\underline{i}}$). Then by an extension of Lemma 4.3.9 in [5]

$$E(\xi_{\Lambda_k})(f, \sigma) = \prod_{\underline{i} \in I^k} \prod_{\delta \in \Gamma_{\underline{i}}} \xi_{\underline{i}}\big(\rho_{h_\delta}(f, \sigma)\big).$$

We want to find the partition-valued function labeling the sign function $\mathrm{sgn}_{(k)}$ of $Y_1^k \wr S(\Delta_k)$ with respect to $(Y_1^k \wr S(\Delta_k))^+$.

Let $\Lambda_{(k)}$ be the map that sends $(\xi_1, \ldots, \xi_1)$ to $\{1^{n_k}\}$ and all other p^k-tuples to $\emptyset$. Since the stabilizer of $\xi_{\Lambda_{(k)}}$ is $(Y_1^k)^{\Delta_k} \cdot S(\Delta_k) = Y_1^k \wr S(\Delta_k)$,

$$\begin{aligned} E(\xi_{\Lambda_{(k)}})(f,\sigma) &= \prod_{\delta \in \Gamma_{(1,1,\ldots,1)}} \xi_{(1,1,\ldots,1)}\big(\rho_{h_\delta}(f,\sigma)\big) \\ &= \prod_{\delta \in \Gamma_{(1,1,\ldots,1)}} \prod_{j \in \Delta^{\sigma}_{k,\delta}} \mathrm{sgn}_{Y_1^k}\big(f(j)\big) \\ &= \prod_{j \in \Delta_k} \mathrm{sgn}_{Y_1^k}\big(f(j)\big). \end{aligned}$$

Since $\omega_{\Lambda_{(k)}} = \mathrm{sgn}_{S(\Delta_k)}$, we have $\psi_{\Lambda_{(k)}}(f,\sigma) = \mathrm{sgn}_{S(\Delta_k)}(\sigma) \cdot \prod_{j \in \Delta_k} \mathrm{sgn}_{Y_1^k}(f(j))$. It remains to show that this is the restriction to $Y_1^k \wr S(\Delta_k)$ of the usual sign function on $S(\Pi)$. View $f \in (Y_1^k)^{\Delta_k}$ as a mapping from Δ_k to Y_1^k and $\sigma \in S(\Delta_k)$. Now, since $(f,\sigma) \in (Y_1^k)^{\Delta_k} \cdot S(\Delta_k)$ we have $\mathrm{sgn}_{S(\Pi)}(f,\sigma) = \mathrm{sgn}_{(Y_1^k)^{\Delta_k}}(f) \cdot \mathrm{sgn}_{S(\Delta_k)}(\sigma) = \prod_{i \in \Delta_k} \mathrm{sgn}_{Y_1^k}(f(i)) \cdot \mathrm{sgn}_{S(\Delta_k)}(\sigma)$.

Given a partition-valued function Λ_k, consider the mapping $*: \Lambda_k \to \Lambda_k^*$ such that

$$\Lambda_k^* : \xi_{\underline{i}} \to \Lambda_k(\xi_{\underline{i}^*})^*.$$

Then Λ_k^* is the *dual* of Λ_k. If $\Lambda_k = \Lambda_k^*$, we say Λ_k is *self-dual* or *symmetric*. If $\Lambda = \bigsqcup_{k \geqslant 1} \Lambda_k$, we say that Λ is *self-dual* or *symmetric* if $\Lambda_k = \Lambda_k^*$ for all k. The following generalizes Proposition (4D) in [3].

Proposition 5.1. *Let $\psi_{\Lambda_k} \in (Y_1^k \wr S(\Delta_k))^\vee$. Then $\mathrm{sgn}_{(k)} \cdot \psi_{\Lambda_k} = \psi_{\Lambda_k^*}$. In particular, ψ_{Λ_k} is a splitting character if and only if Λ_k is self-dual.*

Proof.

$$\begin{aligned} \psi_{\Lambda_k} \cdot \mathrm{sgn}_{(k)} &= \Big[\mathrm{Ind}_{Y_k^{\Delta_k} S(\Delta_k)_{\xi_{\Lambda_k}}}^{Y_k^{\Delta_k} \cdot S(\Delta_k)} \cdot E(\xi_{\Lambda_k})\omega_{\Lambda_k}\Big] \cdot \mathrm{sgn}_{(k)} \\ &= \mathrm{Ind}_{Y_k^{\Delta_k} \cdot \prod_{\underline{i} \in I^k} S(\Delta_{k,\underline{i}})}^{Y_k^{\Delta_k} \cdot S(\Delta_k)} \Big(E(\xi_{\Lambda_k})\omega_{\Lambda_k} \cdot \mathrm{Res}_{(Y_1^k)^{\Delta_k} \cdot S(\Delta_k)_{\xi_{\Lambda_k}}}^{Y_1^k \wr S(\Delta_k)} (\mathrm{sgn}_{(k)})\Big). \end{aligned}$$

Decompose $(f,\sigma) \in (Y_1^k)^{\Delta_k} \cdot S(\Delta_k)_{\xi_{\Lambda_k}}$ into $(f,\sigma) = \prod_{\underline{i} \in I^k} (f_{\underline{i}}, \sigma_{\underline{i}})$, where $(f_{\underline{i}}, \sigma_{\underline{i}}) \in Y_1^k \wr S(\Delta_{k,\underline{i}})$. Let $\Delta_{k,\underline{i}} = \bigsqcup_{\delta \in \Gamma_{\underline{i}}} \Delta_{k,\delta}^{\sigma_{\underline{i}}}$ be the orbit decomposition of $\sigma_{\underline{i}}$ on $\Delta_{k,\underline{i}}$. Since

$$\mathrm{Res}_{(Y_1^k)^{\Delta_k} \cdot S(\Delta_k)_{\xi_{\Lambda_k}}}^{Y_1^k \wr S(\Delta_k)} (\mathrm{sgn}_{(k)})(f,\sigma) = \prod_{\underline{i} \in I^k} \Big[\prod_{\delta \in \Gamma_{\underline{i}}} \mathrm{sgn}_{Y_1^k}\big(\rho_{h_\delta}(f_{\underline{i}}, \sigma_{\underline{i}})\big)\Big] \cdot \mathrm{sgn}_{S(\Delta_{k,\underline{i}})}(\sigma_{\underline{i}})$$

we have

$$\begin{aligned} \big(E(\xi_{\Lambda_k})\omega_{\Lambda_k} \cdot \mathrm{sgn}_{(k)}\big)(f,\sigma) &= \prod_{\underline{i} \in I^k} \Big[\prod_{\delta \in \Gamma_{\underline{i}}} \xi_{\underline{i}}\big(\rho_{h_\delta}(f_{\underline{i}}, \sigma_{\underline{i}})\big) \mathrm{sgn}_{Y_1^k}\big(\rho_{h_\delta}(f_{\underline{i}}, \sigma_{\underline{i}})\big)\Big] \big(\omega_{\Lambda_k(\xi_{\underline{i}})} \cdot \mathrm{sgn}_{S(\Delta_{k,\underline{i}})}\big)(\sigma_{\underline{i}}) \\ &= \prod_{\underline{i} \in I^k} \Big[\prod_{\delta \in \Gamma_{\underline{i}}} (\xi_{\underline{i}^*})\big(\rho_{h_\delta}(f_{\underline{i}}, \sigma_{\underline{i}})\big)\Big] \cdot \omega_{\Lambda_k^*(\xi_{\underline{i}})}(\sigma_{\underline{i}}) = E(\xi_{\Lambda_k^*}) \cdot \omega_{\Lambda_k^*}(f,\sigma). \end{aligned}$$

Hence, $\psi_\Lambda \cdot \mathrm{sgn}_{(k)} = \psi_{\Lambda^*}$. In particular, if $\psi_\Lambda = \psi_{\Lambda^*}$ then ψ_Λ splits when restricted to $A(\Pi)$. If $\psi_\Lambda \neq \psi_{\Lambda^*}$ then ψ_Λ does not split. □

Theorem 5.2. *Every $\chi \in H^\vee_{p',*}$ can be written as $\omega_\kappa \times \psi_\Lambda$ such that κ is a symmetric p-core partition, $\omega_\kappa \in S(\Pi_-)^\vee$ and Λ is a self-dual partition-valued function. That is, $\Lambda = \bigsqcup_{k\geqslant 1} \Lambda_k$ where $\Lambda_k : Y_1^k \longrightarrow \{p\text{-core partitions}\}$ and $\Lambda_k = \Lambda_k^*$ for all k.*

Proof. $H = S(\Pi_-) \times \prod Y_1^k \wr S(\Delta_k)$ and let $\psi_\Lambda \in N_{S(\Pi_+)}(X)^\vee_{p',*}$ be a splitting character with respect to H^+, $\Lambda = \bigsqcup \Lambda_k$, and $\mathrm{Res}^H_{Y_1^k \wr S(\Delta_k)} \mathrm{sgn}_H = \mathrm{sgn}_{Y_1 \wr S(\Delta_k)}$. Then $\chi \in H_{p',*}$ implies $\chi = \omega_\kappa \times \psi_\lambda$ where

$$\begin{aligned}\omega_\kappa \times \psi_\Lambda &= [\omega_\kappa \times \psi_\Lambda] \cdot \mathrm{sgn}_H\\ &= \omega_{\kappa^*} \times \bigsqcup_{k\geqslant 1} (\psi_{\Lambda_k} \cdot \mathrm{sgn}_{(k)})\\ &= \omega_{\kappa^*} \times \bigsqcup_{k\geqslant 1} \psi_{\Lambda_k^*} = \omega_{\kappa^*} \times \psi_{\Lambda^*}.\end{aligned}$$

Hence $\psi_{\Lambda_k} = \psi_{\Lambda_k^*}$ and $\Lambda_k = \Lambda_k^*$ for all $k \geqslant 1$ and $\kappa = \kappa^*$. □

6. A bijection between splitting characters

Let $M_*(B)$ and $M_*(b)$ be the splitting characters of $M(B)$ and $M(b)$ respectively. We restrict the bijection $f_B : M(B) \to M(b)$ (see Theorem 3.3) to $f_{B,*}$ which acts only on the domain $M_*(B)$.

Theorem 6.1. *$f_{B,*}$ is a bijection between $M_*(B)$ and $M_*(b)$.*

Proof. Let $\chi_\lambda \in M_*(B)$ and C_λ be the associated p-core tower. Then $C_\lambda = C^*_\lambda$ by Theorem 2.4. Now let C_{λ_+} be the p-core tower of λ with $\lambda^0 = \emptyset$. The set of $\chi_\lambda \in M_*(B)$, where $\lambda = \lambda^*$, is in bijection via f with the set of $\omega_\kappa \times \psi_{\lambda_+} \in M(b)$ where $\kappa = \kappa^*$ and $\lambda_+ = \lambda_+^*$. But by Theorem 5.2 the latter are exactly the constituents of $M_*(b)$. Hence $M_*(B)$ and $M_*(b)$ are in bijection via $f_{B,*}$. □

Theorem 6.2. *If $\lambda = \lambda^0$, then every irreducible constituent of $\mathrm{Res}^{S(\Pi)}_{A(\Pi)} \chi_\lambda$ forms its own p-block. Let $\{\pi_i \vdash n \mid \pi_i \neq \lambda\}$ be the set of partitions of n distinct from λ. If $\lambda \neq \lambda^0$, then to the p-block of an irreducible constituent of $\mathrm{Res}^{S(\Pi)}_{A(\Pi)} \chi_\lambda$ there belong just the constituents of such restrictions χ_{π_i} where $\pi_i^0 = \lambda^0$ or $\pi_i^0 = (\lambda^0)^*$.*

A block B of $S(\Pi)$ *splits* over $A(\Pi)$ if each character $\chi \in B$ splits into two characters $\chi^\pm$ when restricted to $A(\Pi)$. Consider B_κ the block of $S(\Pi)$ indexed by a p-core κ. The following is Theorem 6.1.46 in [5].

Lemma 6.3. *The block B_κ of $S(\Pi)$ splits over $A(\Pi)$ if and only if $\kappa = \kappa^*$ and $\kappa \vdash n$.*

Proof. A block $B = B_\kappa$ splits over $A(\Pi)$ if and only if every character χ_λ in B splits upon restriction to $A(\Pi)$. By Theorem 6.2, this occurs if for each λ where $\chi_\lambda \in B$ and $\lambda = \lambda^*$. By Theorem 2.1, this implies $\kappa = \kappa^*$. However if $|\kappa| < n$, there will exist a $\chi_\lambda \in B$ such that $\lambda \neq \lambda^*$. □

Theorem 6.4. *The alternating groups $A(\Pi)$ satisfy the block Isaacs–Navarro conjecture.*

Proof. Theorem 3 in [2] verifies the block Isaacs–Navarro conjecture for $S(\Pi)$. Hence, for a p-block B of $S(\Pi)$ and its Brauer correspondent, a p-block b of $N_{S(\Pi)}(X)$, $|M_{ck}(B)| = |M_k(b)|$.

If B is a splitting block, then B and b are of defect 0, in which case the result follows trivially. Now suppose p is odd and B does not split. Then either

1. $\kappa = \kappa^*$ and $|\kappa| < n$ or
2. $\kappa \neq \kappa^*$.

Consider the case where $\kappa = \kappa^*$ and $|\kappa| < n$. Although the block B does not split when restricted to $A(\Pi)$, individual characters χ_λ of B may split upon restriction. By Theorem 6.2, both constituents will be in the same block, since the set of constituents of the restrictions of characters of a block of $S(\Pi)$ forms a block of $A(\Pi)$. Let B' be the block of $A(\Pi)$ formed by the constituents of $\mathrm{Res}^{S(\Pi)}_{A(\Pi)}(\chi)$ of $\chi \in B$. Hence if $\mathrm{Res}^{S(\Pi)}_{A(\Pi)}(\chi_\lambda) = \chi_\lambda^+ + \chi_\lambda^-$, $\{\chi_\lambda^+, \chi_\lambda^-\} \subseteq B'$. Then b' is defined from b in an analogous way. Let

$$s_k = \left|\left\{\chi_\lambda \in M_*(B)\colon\ \chi_\lambda(1) \equiv \pm 2ck \pmod p\right\}\right|,$$
$$2t_k = \left|\left\{\psi_\lambda \in M(B) - M_*(B)\colon\ \psi_\lambda(1) \equiv \pm ck \pmod p\right\}\right|.$$

Then $|M_{ck}(B')| = 2s_k + t_k$. Similarly, let

$$s_k^b = \left|\left\{f(\chi_\lambda) \in M_*(b)\colon\ f(\chi_\lambda)(1) \equiv \pm 2k \pmod p\right\}\right|,$$
$$2t_k^b = \left|\left\{f(\chi_\lambda) \in M(b) - M_*(b)\colon\ f(\chi_\lambda)(1) \equiv \pm k \pmod p\right\}\right|.$$

Then $|M_k(b')| = 2s_k^b + t_k^b$. Then $t_k = t_k^b$ by Theorem 3.3 and $s_k = s_k^b$ by Theorem 6.1, so $|M_{ck}(B')| = |M_k(b')|$.

Suppose $\kappa \neq \kappa^*$. In this case, no χ_λ splits when restricted to $A(\Pi)$ by Theorem 2.1. Hence no $f(\chi_\lambda)$ splits and $s = s^b = 0$. Let

$$2t_k = \left|\left\{\psi_\lambda \in M(B)\colon\ \chi_\lambda(1) \equiv \pm ck \pmod p\right\}\right|.$$

Hence $|M_{ck}(B')| = t_k$. Similarly, let

$$2t_k^b = \left|\left\{\psi_\lambda \in M(B)\colon\ \chi_\lambda(1) \equiv \pm ck \pmod p\right\}\right|.$$

Then $|M_k(b')| = t_k^b$. Since $t_k = t_k^b$ by Theorem 3.3, $|M_{ck}(B')| = |M_k(b')|$.

Now suppose $p = 2$ and $\kappa < n$. Then it is known that $\kappa = \kappa^*$ (see p. 24 in [9]). Then

$$s_0 = \left|\left\{\chi_\lambda \in M_*(B)\colon\ \chi_\lambda(1) \equiv 0 \pmod 2\right\}\right|,$$
$$2t_0 = \left|\left\{\chi_\lambda \in M(B) - M_*(B)\colon\ \chi_\lambda(1) \equiv 0 \pmod 2\right\}\right|,$$
$$2t_1 = \left|\left\{\chi_\lambda \in M_B - M_*(B)\colon\ \chi_\lambda(1) \equiv 1 \pmod 2\right\}\right|.$$

Then $|M_0(B')| = t_0$ and $|M_1(B')| = 2s_0 + t_1$. Similarly,

$$\left(s_0^b\right) = \left|\left\{\chi_\lambda \in M_*(b)\colon\ \chi_\lambda(1) \equiv 0 \pmod 2\right\}\right|,$$
$$2t_0^b = \left|\left\{\chi_\lambda \in M(b) - M_*(b)\colon\ \chi_\lambda(1) \equiv 0 \pmod 2\right\}\right|,$$
$$2t_1^b = \left|\left\{\chi_\lambda \in M(b) - M_*(b)\colon\ \chi_\lambda(1) \equiv 1 \pmod 2\right\}\right|.$$

Then $|M_0(b')| = t_0^b$ and $|M_1(B)| = 2(s_0^b) + t_1^b$. The result follows using Theorem 3.3 and Theorem 6.1. □

Acknowledgments

The author thanks his PhD advisor Paul Fong for suggesting this problem as part of his dissertation [7]. The author also thanks the anonymous referee for several valuable suggestions and corrections.

References

[1] J. Alperin, The main problem of block theory, in: Proc. of the Conference of Finite Groups, University of Utah, Park City, Utah, 1976, pp. 341–356.
[2] P. Fong, The Isaacs–Navarro conjecture for symmetric groups, J. Algebra 250 (1) (2003) 154–161.
[3] P. Fong, M. Harris, On perfect isometries and isotypies in alternating groups, Trans. Amer. Math. Soc. 349 (9) (1997) 3469–3516.
[4] I.M. Isaacs, G. Navarro, New refinements of the McKay conjecture for arbitrary finite groups, Ann. of Math. 156 (2002) 333–344.
[5] G. James, A. Kerber, Encyclopedia of Mathematics: The Representation Theory of the Symmetric Groups, Addison and Wesley, London, 1981.
[6] I. MacDonald, On the degrees of the irreducible representations of symmetric groups, Bull. Lond. Math. Soc. 3 (1971) 189–192.
[7] R. Nath, Partial results on Navarro's conjecture and the Isaacs–Navarro conjecture for the alternating groups, PhD thesis, University of Illinois at Chicago, 2006.
[8] J. Olsson, McKay numbers and heights of characters, Math. Scand. 38 (1976) 25–42.
[9] J. Olsson, Combinatorics and representations of finite groups, Vorlesungen aus dem FB Mathematik der Univ. Essen, Heft 20, Essen, 1993.