Orbital Schemes of $B_3(q)$ Acting on 2-Dimensional Totally Isotropic Subspaces

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In this paper the authors determine the intersection numbers of the orbital association schemes $B_{3,2}(q)$ by characterizing the orbitals in terms of incidence chains of specified type. It is subsequently shown that the schemes $B_{3,2}(q)$ have no nontrivial fusion schemes.

1. INTRODUCTION

Consider the transitive action of finite Chevalley group $G = X_l(q)$ on the set $\gamma^l_i = (G; P_i)$ of cosets of $P_i$ in $G$, where $l$ is the rank of $G$ (i.e., the number of nodes in the Dynkin diagram $X_l$ of $G$) and $P_i$ is the maximal parabolic subgroup corresponding to the $i$th node of the diagram. The orbitals of the permutation group $(G, \gamma^l_i)$ are the relations of an association scheme, usually denoted $X_l, i(q)$, which is an important object in algebraic combinatorics (see [1] and [4]).

In the case of such schemes which are $P$-polynomial, formulas for the intersection numbers and eigenvalues are known (see [16] and [11], respectively), and a description of all fusion schemes can be found in [9, 10].

In this paper we determine the intersection numbers, eigenvalues, and fusion schemes in the simplest case of an orbital association scheme $X_l, i(q)$ which is not $P$-polynomial. Namely, we do this for the scheme $B_{3,2}(q)$ which arises from the action of the group $B_3(q)$ (equivalently, $P\Omega_7(q)$) on the set of 2-dimensional totally isotropic subspaces of a 7-dimensional vector space over $GF(q)$ equipped with a quadratic form.

Before we begin, let us make some remarks about general methods for computing the intersection numbers of $B_{3,2}(q)$. Following the work of Tits [12], for example, it is at least theoretically possible to compute them in the following manner. Let $(W, S) = (W(B_3), S = \{s_1, s_2, s_3\})$ be a Coxeter system of Weyl group $W = W(B_3)$, and let $l(w)$ be the minimal length of any expression for $w \in W$ as a product of elements from $S$ (see [5]). Consider now the associative algebra $A_W$ with formal generators $a_w, w \in W$, and relations

$$a_{s_i} a_w = \begin{cases} a_{s_i w}, & \text{if } l(s_i w) < l(w) \\ q a_{s_i w} + (q - 1) a_w, & \text{otherwise.} \end{cases}$$

Then the intersection numbers of $B_{3,2}(q)$ are precisely the structure constants of the subalgebra of $A_W$ generated by the elements $\sum_{x \in W_2 W_2} a_x, w \in W$.

A second (and simpler) method is to compute the intersection numbers from special parameters related to the group geometry for $B_3(q)$. (See [3] for a complete discussion on how to compute these parameters in the general case.)

In the present paper we devise an independent method of computation by which we obtain direct formulas for the intersection numbers in terms of $q$ (see Section 6). All computations are carried out by hand.

The paper is organized as follows. In Section 2 all relevant terminology dealing with association schemes (e.g., orbitals, intersection numbers, eigenvalues, fusion schemes, etc.)

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is presented, and that dealing with Coxeter and Lie geometries is presented in Section 3. In Section 4 we describe a procedure of central importance to our work, that of embedding the aforementioned geometries into a corresponding Lie algebra. Section 5 is a composite of earlier sections, specialized to the case $B_{3,2}(q)$. Concrete characterizations for the relations of orbital schemes $B_{3,2}$ and $B_{3,2}(q)$ are given in this section. In Section 6 we introduce a criterion for membership of an ordered pair of type-2 objects from the $B_{3,2}(q)$ geometry to a relation of $B_{3,2}(q)$. This enables us to compute all intersection numbers $p_{ij}^k$ of $B_{3,2}(q)$; a sample computation ($p_{2j}^3, 0 \leq j \leq 4$) is provided. Finally in Section 7, we give the first eigenmatrix of the scheme $B_{3,2}(q)$, along with a proof that nontrivial fusion schemes of $B_{3,2}(q)$ do not exist.

2. Association Schemes

An association scheme $(X, \{R_i\}_{0\leq i \leq d})$ is a set $X$ together with a family of binary relations $R_0, R_1, \ldots, R_d$ such that:

(i) the relations form a partition of $X \times X$, i.e., $X \times X = \cup_{0\leq i \leq d} R_i$ and $R_i \cap R_j = \emptyset$ for $i \neq j$;

(ii) $R_0$ is the diagonal relation on $X$, i.e., $R_0 = \{(x, x) \mid x \in X\}$;

(iii) for any relation $R_i$, its transpose relation $R_i^t = \{(y, x) \mid (x, y) \in R_i\}$ is again a relation of the scheme, i.e., $R_i^t = R_i$, for some $i' \in \{0, 1, \ldots, d\}$;

(iv) for any $(x, y) \in R_k$, the number $p_{ij}^k$ of elements $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ depends only on $i, j, k$, i.e., $p_{ij}^k$ is independent of the representative $(x, y)$ from $R_k$.

The numbers $p_{ij}^k$ are called the intersection numbers of the association scheme $(X, \{R_i\}_{0\leq i \leq d})$ (see [1] or [4]).

Let $(G, X)$ be a transitive permutation group (so $G$ acts faithfully and transitively on the set $X$). An orbital of $(G, X)$ is, by definition, an orbit of $(G, X \times X)$, where the action of $G$ on $X \times X$ is the natural induced action $(x, y)^g = (x^g, y^g)$. It is convenient to consider orbitals as binary relations (or directed graphs) on $X$. When this is done, it is easy to see that set $X$ together with the family of orbitals of $(G, X)$ is an association scheme, called an orbital scheme.

Let $(X, \{R_i\}_{0\leq i \leq d})$ be an association scheme. The adjacency matrices $A_0 = I$, $A_1, \ldots, A_d$, which correspond, respectively, to the relations $R_0, R_1, \ldots, R_d$, generate a vector space $A$ over the complex numbers, which is closed under matrix multiplication (and so is also an algebra!). $A$ is called the Bose–Mesner algebra of the scheme $(X, \{R_i\}_{0\leq i \leq d})$. Moreover, the structure constants of $A$ are precisely the intersection numbers of the scheme, i.e.,

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k.$$ 

In the case of an orbital scheme, the Bose–Mesner algebra is just the Hecke algebra (see [2]) or, equivalently, the Schur $V$-ring (see [7]).

As an abstract algebra, $A$ is always semisimple. Thus, there exists a basis (unique up to order) consisting of the primitive orthogonal idempotents $E_0 = d^{-1} J$, $E_1, \ldots, E_d$ of $A$. (Here $J$ denotes the all-ones matrix.) The eigenvalues $p_i(j)$ are now defined to be the complex numbers determined by

$$A_i = \sum_{j=0}^{d} p_i(j) E_j.$$
We end the section with the definition of a fusion scheme. Let \( \mathcal{X} = (X, \{ R_i \}_{0 \leq i \leq d}) \) and \( \mathcal{Y} = (X, \{ S_j \}_{0 \leq j \leq e}) \) be association schemes on the common set \( X \). We say \( \mathcal{Y} \) is a fusion scheme of \( \mathcal{X} \) if for every \( i \), \( 0 \leq i \leq d \), relation \( R_i \) of \( \mathcal{X} \) is a subset of relation \( S_j \) of \( \mathcal{Y} \) for some \( j \), \( 0 \leq j \leq e \). That is, fusion scheme \( \mathcal{Y} \) is obtained by fusing together relations from scheme \( \mathcal{X} \) in a very restricted way (so that \( \mathcal{Y} \) is itself an association scheme).

3. COXETER AND LIE GEOMETRIES

An incidence system over type set \( \Delta \) is a triple \( (\Gamma, I, t) \), where \( \Gamma \) is a set (whose elements are called objects), \( I \) is a symmetric and reflexive binary relation on \( \Gamma \) (called the incidence relation) and \( t \) is a map from \( \Gamma \) into \( \Delta \) (called the type function). The rank of the incidence system is defined to be \( |\Delta| \). It is convenient to write \( \Gamma \) in place of \( (\Gamma, I, t) \) when doing so will not lead to confusion. Let \( \Gamma \) and \( \Gamma' \) be incidence systems defined over the same type set \( \Delta \). A morphism of \( \Gamma \) into \( \Gamma' \) is a map \( \phi : \Gamma \rightarrow \Gamma' \) which preserves incidence. We say \( \phi \) is type-preserving if, in addition, \( t(A) = t(\phi(A)) \) for all \( A \in \Gamma \).

An important example of the above is the so-called group incidence system \( \Gamma(G, G)_{s} \in S \). Here \( G \) is an abstract group and \( \{ G_{s} \}_{s} \in S \) is a family of distinct subgroups of \( G \). The objects of \( \Gamma(G, G)_{s} \in S \) are the cosets of \( G_{s} \) in \( G \) for all possible \( s \in S \). Cosets \( \alpha \) and \( \beta \) are incident precisely when \( \alpha \cap \beta \neq \emptyset \). The type function is defined by \( t(\alpha) = s \) where \( \alpha = xG_{s} \) for some \( x \in G \).

Let \( (W, S) \) be a Coxeter system, i.e., \( W \) is a group with set of distinguished generators given by \( S = \{ s_{1}, s_{2}, \ldots, s_{l} \} \) and generic relations \( (s_{i}s_{j})^{m_{ij}} = 1 \). Here \( M = (m_{ij}) \) is a symmetric \( l \times l \) matrix with \( m_{ii} = 1 \) and off-diagonal entries satisfying \( m_{ij} \geq 2 \) (allowing \( m_{ij} = \infty \) as a possibility, in which case the relation \( (s_{i}s_{j})^{m_{ij}} = 1 \) is omitted). Letting \( W_{i} = \langle S \setminus \{ s_{i} \} \rangle \), \( 1 \leq i \leq l \), we obtain a group incidence system \( \Gamma_{W} = \Gamma(W, W_{i})_{1 \leq i \leq l} \) called the Coxeter geometry of \( W \). The \( W_{i} \) are referred to as the maximal standard parabolic subgroups of \( W \) (see [2]).

Let \( G \) be a group, \( B \) and \( N \) subgroups of \( G \), and \( S \) a collection of cosets of \( B \cap N \) in \( N \). We call \( (G, B, N, S) \) a Tits system (or we say that \( G \) has a BN-pair) if

(i) \( G = \langle B, N \rangle \) and \( B \cap N \) is normal in \( N \),
(ii) \( S \) is a set of involutions which generate \( W = N / B \cap N \),
(iii) \( sBwB \subset BwB \cup BsB \) for any \( s \in S \) and \( w \in W \),
(iv) \( sBs \neq B \) for all \( s \in S \).

Properties (i)–(iv) imply that \( (W, S) \) is a Coxeter system (see [2]). Whenever \( (G, B, N, S) \) is a Tits system, we call group \( W \) the Weyl group of the system, or, more usually, the Weyl group of \( G \). The subgroups \( P_{i} \) of \( G \) defined by \( P_{i} = BW_{i}B \) are called the standard maximal parabolic subgroups of \( G \). The group incidence system \( \Gamma_{G} = \Gamma(G, P_{i})_{1 \leq i \leq l} \) is commonly referred to as the Lie geometry of \( G \) (see [12]). Note that the Lie geometry of \( G \) and the Coxeter geometry of the corresponding Weyl group \( W \) have the same rank. In fact there is a type-preserving morphism from \( \Gamma_{G} \) onto \( \Gamma_{W} \) given by \( gP_{i} \mapsto wW_{i} \), where \( w \) is determined from the equality \(BgP_{i} = BwP_{i} \) of double cosets. This morphism is called retraction (see [12]).

For \( (G, \Delta) \) a general permutation group with orbits \( \Delta_{1}, \ldots, \Delta_{r} \) and corresponding one-point stabilizers \( G_{1}, \ldots, G_{r} \), the orbitals of \( (G, \Delta) \) are in one-to-one correspondence with the double cosets \( G_{i}gG_{j}, 1 \leq i, j \leq r, g \in G \). In the case where \( G \) is a Chevalley group and \( G_{i} = P_{i} \) are the maximal parabolic subgroups of \( G \) (so that \( \Delta \) coincides with the set of objects of \( \Gamma_{G} \)), properties (i)–(iv) of a Tits system give a natural one-to-one correspondence between double cosets \( P_{g}P_{i} \) of \( G \) and double cosets \( W_{i}w/W_{j} \) of corresponding Weyl group \( W \). As a consequence we have a natural one-to-one correspondence between the respective orbitals of
(G, Γ_G) and (W, Γ_W). Finally, this gives a bijection between the respective relations of the orbital schemes X_{i,j}(q) and X_{i,j}, the latter coming from the action of W = W(X_i) acting on the cosets of W_i in W.

More explicitly, we can identify Γ_W as a subgeometry of Γ_G in the following manner. For a fixed Borel subgroup B of G, define T = \bigcap_{w \in W} B^w. (We also have T = B \cap N in the language of Tits systems. T is called a maximal torus.) Consider now the subset Ω of Γ_G consisting of all cosets stabilized by T, i.e., \text{tg} F_t = gP_i for all t \in T. Then the incidence system (Ω, I_2, I_3) —where I_2 and I_3 denote the respective restrictions of incidence and type in Γ_G to Ω—is isomorphic to Γ_W. In fact, restriction to Ω of the retraction morphism defined above yields an isomorphism. (See [12] for a full discussion on this.) Thus, not only are the orbitals of (G, Γ_G) and (W, Γ_W) in one-to-one correspondence as mentioned above, but, more strongly, we can represent each orbital of (G, Γ_G) by an ordered pair of objects from Γ_W.

4. Embeddings in the Lie Algebra

Throughout this section we assume (G, B, N, S) is a Tits system which arises in connection with Chevalley group G, although we point out that the results of this section remain valid in a far more general setting (see [13–15]). We write G = X_1(K) to signify that G is a Chevalley group over the field K, with associated Dynkin diagram X_1. We are most interested in the case when K is finite, and we shall write X_i(q) instead of X_i(GF(q)) in that case.

So, fix Chevalley group G = X_1(K) with corresponding Weyl group W. As in the previous section, denote by Γ_W and Γ_G their associated Coxeter and Lie geometries. Let L = H \oplus L^+ \oplus L^- be the Lie algebra corresponding to G. Following convention, we refer to H, L^+, L^- and H \oplus L^+ as, respectively, the Cartan subalgebra, positive root space, negative root space and (positive) Borel subalgebra with respect to the given decomposition of L. We also use the familiar bracket notation [ , ] to indicate the Lie product.

Below, we turn our attention to a method for embedding Γ_W and Γ_G in L. As the reader shall see, this method actually embeds Γ_W in the Cartan subalgebra H and Γ_G in the Borel subalgebra U = H \oplus L^+.

It is well known (see [8], for example) that the Coxeter geometry Γ_W of W can be embedded in l-dimensional Euclidean space, which, in the case when K is the real number field, can be identified with the Cartan subalgebra H of L. Let us consider this embedding more precisely.

Let A = (a_{ij}) be the Cartan matrix corresponding to root system Φ of W. We consider the lattice R which is generated by the simple roots α_1, ..., α_l, and the reflections r_1, ..., r_l of R defined by the equality

(a_i)^{r_j} = a_i - a_{ij} a_j.

The set S = \{r_1, ..., r_l\} is a set of Coxeter generators of Weyl group W. Let \{α_1^*, ..., α_l^*\} be a dual basis of \{α_1, ..., α_l\}, i.e., α_i^* is the linear functional on R which satisfies α_i^*(α_j) = δ_{ij}. We define the action of W on the dual lattice R^* by \text{t}^*(v) = t(v^*) for all t \in R^*, v \in V, and s \in S. Consider the orbit H_i = \{(α_i^*)^w | w \in W\} of permutation group \text{}(W, R^*) which contains α_1^*. We give the set H = \bigcup_i H_i the structure of an incidence system as follows. Linear functionals l_1 and l_2 are incident if and only if \text{l}_1(α)l_2(α) ≥ 0 for all α ∈ Φ. The type function is defined by t(l) = i where l ∈ H_i. It can be shown that (H, I, t) is isomorphic to the Coxeter geometry Γ_W. (In fact, there is a unique isomorphism of Γ_W with \text{(H, I, t)} which sends W_i to α_i^*. 1 ≤ i ≤ l.) This gives the desired embedding since H ⊂ R^* ⊂ H. Moreover, this embedding obtains for K a field of sufficiently large characteristic since, in that case, H ⊂ R^* ∩ K = H. This latter fact is crucial to what follows.

We now consider an analogous embedding of the Lie geometry Γ_G of G into the Borel
subalgebra \( \mathcal{U} = \mathcal{H} \oplus \mathcal{L}^+ \) of \( \mathcal{L} \). Let \( d = \sum_{i=1}^l \alpha_i^* \). Then we can take

\[
\Phi^+ = \{ \alpha \in \Phi \mid d(\alpha) \geq 0 \}
\]
to be our set of positive roots in \( \Phi \). For any \( l(x) \in \mathcal{R}^* \), define

\[
\eta^-(l) = \{ \alpha \in \Phi^+ \mid l(\alpha) < 0 \}.
\]

Let \( \mathcal{L}_\alpha \) be the root space corresponding to the positive root \( \alpha \), so that \( \mathcal{L}^+ = \sum_{\alpha \in \Phi^+} \mathcal{L}_\alpha \). For each \( h \in H \) we define the subalgebra \( \mathcal{L}_h = \sum_{\alpha \in \eta^-(h)} \mathcal{L}_\alpha \). Let \( U_i = (h + v \mid h \in H_i, v \in \mathcal{L}_h) \) and \( U = \bigcup_i U_i \). We give \( U \) the structure of an incidence system as follows. Elements \( h_1 + v_1 \) and \( h_2 + v_2 \) are incident if and only if each of the following hold:

(i) \( h_1(\alpha)h_2(\alpha) \geq 0 \) for all \( \alpha \in \Phi \), i.e., \( h_1 \) and \( h_2 \) are incident in \( (H, I, t) \),

(ii) \( |h_1 + v_1, h_2 + v_2| = 0 \).

Element \( h + v \) has type \( i \) if \( h + v \in U_i \). In [14] it is shown that this newly defined incidence system \( (U, I, t) \) is isomorphic to the Lie geometry \( \Gamma_G \), provided the characteristic of \( K \) is sufficiently large to ensure isomorphism at the level of the subgeometries \( (H, I, t) \) and \( \Gamma_W \).

So let us make the assumption that the characteristic is sufficiently large. Then, analogous to the Weyl case, there exists a unique isomorphism of \( \Gamma_G \) onto \( (U, I, t) \) which sends \( P_i \) to \( \alpha_i^* \), \( 1 \leq i \leq l \). This gives an obvious embedding of \( \Gamma_G \) in \( \mathcal{U} \), in which the image of the subset \( \Omega \) is \( H \). From our discussion in Section 3, it is clear that each orbital of \( (G, \Gamma_G) \) can be represented by an ordered pair \( (h, h') \) of objects from \( H \). Moreover, by transitivity on objects of fixed type, we can further choose \( h = \alpha_i^* \) where \( t(h) = i \).

Finally, observe that the map \( h + v \mapsto h \) (that is, the canonical projection of \( \mathcal{U} \) onto \( \mathcal{H} \)) is a type-preserving morphism of incidence systems from \( (U, I, t) \) onto \( (H, I, t) \); in fact, it is essentially the retraction of \( \Gamma_G \) onto \( \Gamma_W \) introduced in Section 3.

5. Characterizing Relations in \( B_{3,2}(q) \)

In this section we restrict our attention to the association scheme \( B_{3,2}(q) \), \( q \) a prime power. That is, we consider the orbital scheme arising from the action of Chevalley group \( B_3(q) \) on cosets of the maximal parabolic subgroup which corresponds to the middle node of the diagram \( B_3 \). In classical terms this action can be described as follows.

Let \( O_7(q) \) be the 7-dimensional orthogonal group over \( GF(q) \), that is, the group of all 7 \times 7 invertible matrices which preserve a quadratic form \( f \) on the 7-dimensional space \( \mathcal{V} \) over \( GF(q) \). (When \( q \) is odd, \( f \) can be chosen to be \( x_0^2 + x_1x_2 - x_3 \).) When \( q \) is even a description of \( f \) is more complicated; however in this case one has the isomorphism \( O_7(q) \cong Sp_6(q) \), where \( Sp_6(q) \) is the 6-dimensional symplectic group.) Let \( \Omega_7(q) \) be the commutator subgroup of \( O_7(q) \). One can now describe \( B_3(q) \) as the factor group \( P\Omega_7(q) = \Omega_7(q)/Z(\Omega_7(q)) \) where \( Z(\Omega_7(q)) \) is the center of \( \Omega_7(q) \). The aforementioned action of \( B_3(q) \) can now be realized as the action of \( P\Omega_7(q) \) on the 2-dimensional totally isotropic subspaces of \( \mathcal{V} \).

It is convenient to use the following model of the Coxeter geometry \( \Gamma_W \), \( W = W(B_3) \), to directly compute the intersection numbers of the scheme \( B_{3,2} \). (Note that these numbers can be recovered by setting \( q = 1 \) in Tables 6–10.) Let \( Y \) be a fixed set of three elements. The objects of \( \Gamma_W \) are pairs \( (A, f) \), where \( A \) is a nonempty subset of \( Y \) and \( f \) is a function from \( A \) into \( [0, 1] \). Objects \( (A, f) \) and \( (B, g) \) are incident if and only if each of the following hold:

(i) \( A \subseteq B \) or \( B \subseteq A \),

(ii) \( f(x) = g(x) \) for all \( x \in A \cap B \).
The set of positive roots, \( \alpha \) is bilinear, skew-symmetric, i.e., \([ \alpha, \beta ] = -[\beta, \alpha] \) for all \( \alpha, \beta \in \mathbb{R} \).

Let \( \Phi = \{ \alpha_1, \alpha_2, \alpha_3 \} \) be a fundamental basis of root system \( \Phi \) of \( W \), so that we obtain, as the set of positive roots,

\[ \Phi^+ = \{ \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3 \}. \]

For each \((h, h') \in H^2\), define

\[ \rho(h, h') = |\{ \alpha \in \Phi^+ : h(\alpha)h'(\alpha) < 0 \}|. \]

Then \( \rho(h, h') \) uniquely determines the relation of \( B_{3,2} \) to which \((h, h')\) belongs:

From this characterization one can readily determine all \( h' \in H_2 \) for which \((\alpha^+_2, h')\) belongs to relation \( R_0 \) of \( B_{3,2}(q) \) (see Table 1). This suffices to characterize the relations of \( B_{3,2}(q) \) since, for any \( v \in L_{h'}, (\alpha^+_2, h' + v) \) and \((\alpha^+_2, h')\) belong to the same relation.

In what follows, it will be convenient to sometimes represent the root \( a\alpha_1 + b\alpha_2 + c\alpha_3 \) (resp., functional \( da_1^+ + ea_2^+ + f\alpha_3^+ \)) by its coordinate vector \((a, b, c)\) (resp., \([d, e, f]\)) with respect to the basis \( \Pi \) (resp., dual basis of \( \Pi \)).

Recall that incidence in \((U, I, i)\) (and so in \( \Gamma_G \)) is defined in terms of Lie product. For completeness, we list below those properties of Lie product which suffice in determining incidence.

(a) \([, ]\) is bilinear,

(b) \([, ]\) is skew-symmetric, i.e., \([x, y] = -[y, x]\) for all \(x, y \in U\),

(c) \([h, h'] = 0\) for all \(h, h' \in H\),

(d) \([h, e_\alpha] = h(\alpha)e_\alpha\) for all \(h \in H, \alpha \in \Phi^+\).
Orbital schemes of $B_3(q)$

**Table 2.**

Chevalley structure constants $k_{αβ}$ ($α, β \in \Phi^+$) for the Lie algebra $L$ of the group $B_3(q)$.

<table>
<thead>
<tr>
<th>Relation</th>
<th>Number of Chains</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_0$</td>
<td>$(q + 1)^4$</td>
</tr>
<tr>
<td>$R_1$</td>
<td>$2q + 1$</td>
</tr>
<tr>
<td>$R_2$</td>
<td>$q + 1$</td>
</tr>
<tr>
<td>$R_3$</td>
<td>1</td>
</tr>
<tr>
<td>$R_4$</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 3.**

The number of chains for each relation.

<table>
<thead>
<tr>
<th>Relation of $(z, y)$</th>
<th>Number of Chains</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_0$</td>
<td>$(q + 1)^4$</td>
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</tr>
<tr>
<td>$R_3$</td>
<td>1</td>
</tr>
<tr>
<td>$R_4$</td>
<td>0</td>
</tr>
</tbody>
</table>

(e) For $α, β \in \Phi^+$,

$$[e_α, e_β] = \begin{cases} k_{αβ}e_{α+β}, & \text{if } α + β \in \Phi^+; \\ 0, & \text{otherwise} \end{cases}$$

where $k_{αβ}$ is the $(α, β)$-entry in the array of Table 2.

In the above, $e_α, α \in \Phi^+$, are appropriately chosen root vectors (see discussion on Chevalley basis in [5]).

6. Computing Intersection Numbers

Let us say that incidence chain $x_1 x_2 \ldots x_m$ has type $t_1 \to t_2 \to \ldots \to t_m$ if $t_i = t(x_i)$ is the type of object $x_i$, $1 \leq i \leq m$. We begin with a proposition which provides a criterion for membership in relation $R_j$ of $B_3(q)$.

**Proposition 1.** Let $y, z \in U_2$, with $(z, y) \in R_j$. Then $j$ is uniquely determined by the number of chains from $z$ to $y$ of type $2–1–3–2$. The correspondence is given in Table 3.

**Proof.** It is clear that the number of such chains is an invariant of the relation $R_j$. To complete the proof, one merely needs to compute the number of chains for a representative pair from each relation. For example, one can do this for the pairs $(α^*_s, h')$ which appear in Table 1. (Of course for $R_1$, $R_2$ and $R_3$, one has a choice of $h'$.) Since a sample computation is performed below, we omit the details.

It is possible to count the number of chains of type $2–1–3–2$ in $(H, I, t)$ as well (the Coxeter case) and set up a table similar to Table 3. (As a check, set $q = 1$ in Table 3.) This can also be done for arbitrary rank, i.e., for the geometry of $B_l$. Since the numbers will again be distinct
(as in the case \( l = 3 \)) we conclude that the proposition remains valid for all \( l \). This allows the possibility of developing a similar criterion to that given by Table 3 for the relations of \( B_{1,2}(q) \), and so to compute the intersection numbers of these schemes.

Recall that in \((U, I, \tau)\), incidence depends, in part, on the vanishing of the Lie product in Borel subalgebra \( \mathcal{U} \). Thus, if \( zI_s Ir I_y \) is an incidence chain of type \( 2–1–3–2 \), we must have \([z, s] = [s, r] = [r, y] = 0\) where \( y, z \in U_{2, s} \in U_1, r \in U_3 \). Since \([U, U] \subset \mathcal{L}^+\), these conditions on the Lie product give rise to a system of equations in the vector space \( \mathcal{L}^+ \). In fact, the number of solutions to this system equals the number of chains from \( z \) to \( y \) of prescribed type. In this manner our criterion for membership to a relation is transformed to a problem of counting solutions to a well-defined system of equations.

Finally, we remark that it is advantageous for us to partition the set \( \{zI_s Ir I_y\} \) of type \( 2–1–3–2 \) chains into ‘classes’ before counting them (see below). Indeed, this enables a much clearer description of those subchains \( sIr I_y \) for which \( z \) is incident to \( s \).

**Example (Calculating \( p_{2j}^4 \)).** Recall that the intersection number
\[
p_{2j}^4 = |\{z : (x, z) \in R_2, (z, y) \in R_j\}|
\]
is independent of the representative \((x, y)\) chosen from \( R_4 \). Thus we can choose \( x = [0, 1, 0] \) and \( y = [0, -1, 0] \) (see Table 1). Now \( z \) must have the form \( z = h + v \), where \( h \in H_2 \) and \( v \in \mathcal{L}_h \). Since \((x, h) \in R_2\) (indeed, \((x, h)\) belongs to the same relation as \((x, z)\)), we have either \( h = [2, -1, 0] \) or \([-2, 1, 0]\) (Table 1 again). Computing \( \eta^-(h) \) in each case, we obtain two possible forms for \( z \):
\[
\begin{align*}
z &= [2, -1, 0] + a_1 e_{(0,1,0)} + a_2 e_{(0,2,1)} + a_3 e_{(0,1,1)} \\
z &= [-2, 1, 0] + b_1 e_{(1,0,0)} + b_2 e_{(1,1,0)} + b_3 e_{(1,1,1)} + b_4 e_{(2,2,1)}
\end{align*}
\]
where \( a_1, a_2, a_3, b_1, b_2, b_3, b_4 \in GF(q) \).

We next determine, for each \( z \), the relation to which \((z, y)\) belongs, and for this we use the criterion of the proposition. We first calculate the possible choices for \( r \in U_3 \) so that \( r \) is incident to \( y \). There are two choices:
\[
r = [0, -1, 1], \quad r = [0, 0, -1] + ce_{(0,0,1)},
\]
where \( c \in F \). We next determine all objects \( s \in U_1 \) which are incident to each such \( r \). For notational convenience, we list these incidences \( sIr \) as ordered pairs \((s, r)\). They fall into six classes:
\[
\begin{align*}(1) \ &([0, -1, 2], [0, -1, 1]) \\
(2) \ &([1, -1, 0] + de_{(0,1,1)}, [0, -1, 1]) \\
(3) \ &([0, 1, -2] + 2fe_{(0,0,1)}, [0, 0, -1] + ce_{(0,0,1)}) \\
(4) \ &([0, -1, 0] + de_{(1,0,0)} + fe_{(1,1,1)}, [0, -1, 1]) \\
(5) \ &([0, 1, 0] + de_{(1,1,0)} + de_{(1,1,0)} - cde_{(0,1,1)}, [0, 0, -1] + cd_{(0,0,1)}) \\
(6) \ &([0, 1, 0] + de_{(1,0,0)} + fe_{(1,1,0)} + ce_{(1,1,1)}, [0, 0, -1] + ce_{(0,0,1)})
\end{align*}
\]
here \( c, d, f \in GF(q) \).

We are now ready to count chains, and we proceed as follows. For each choice of \( z \) above, the relation containing \((z, y)\) can be determined by counting the number of incidences \( sIr \) in the above list in which \( s \) is incident to \( z \). Tables 4 and 5 give the number of incidences from each
Orbital schemes of $B_3(q)$

The number of incidences $sIr$, by classes, with $s$ incident to $z = [2, -1, 0] + a_1 e_{(0,1,0)} + a_2 e_{(0,2,1)} + a_3 e_{(0,1,1)}$.

<table>
<thead>
<tr>
<th>$a_1 = 0?$</th>
<th>$a_2 = 0?$</th>
<th>$a_3 = 0?$</th>
<th># Class (2)</th>
<th># Class (5)</th>
<th>Total #</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>1</td>
<td>$q$</td>
<td>$q+1$</td>
</tr>
<tr>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>—</td>
<td>No</td>
<td>—</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>No</td>
<td>Yes</td>
<td>—</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The number of incidences $sIr$, by classes, with $s$ incident to $z = [-2, 1, 0] + b_1 e_{(1,0,0)} + b_2 e_{(1,1,0)} + b_3 e_{(1,1,1)} + b_4 e_{(2,2,1)}$.

<table>
<thead>
<tr>
<th>$b_1 = 0?$</th>
<th>$b_2 = 0?$</th>
<th>$b_3 = 0?$</th>
<th>$b_4 = 0?$</th>
<th># Class (3)</th>
<th># Class (6)</th>
<th>Total #</th>
</tr>
</thead>
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<td>Yes</td>
<td>Yes</td>
<td>1</td>
<td>$q$</td>
<td>$q+1$</td>
</tr>
<tr>
<td>—</td>
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<td>No</td>
<td>Yes</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>—</td>
<td>No</td>
<td>—</td>
<td>Yes</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>—</td>
<td>—</td>
<td>—</td>
<td>No</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Intersection numbers $p_{ij}^0$ of $B_{3,2}(q)$ ($\epsilon = q + 1$).

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$q\epsilon^2$</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$q^3\epsilon$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$q^4\epsilon^2$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$q^7$</td>
</tr>
</tbody>
</table>

Intersection numbers $p_{ij}^1$ of $B_{3,2}(q)$ ($\delta = q - 1$).

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>$q^2 + q - 1$</td>
<td>$q^2$</td>
<td>$q^3$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$q^2$</td>
<td>$q^2\delta$</td>
<td>$q^4$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$q^3$</td>
<td>$q^4$</td>
<td>$q^3(2q^2 - 1)$</td>
<td>$q^6$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$q^6$</td>
<td>$q^6\delta$</td>
</tr>
</tbody>
</table>

Intersection numbers $p_{ij}^2$ of $B_{3,2}(q)$ ($\delta = q - 1, \epsilon = q + 1$).

<table>
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<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$\epsilon$</td>
<td>$\delta\epsilon$</td>
<td>$q^2\epsilon$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\delta\epsilon$</td>
<td>$q^2\delta$</td>
<td>0</td>
<td>$q^4$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$q^2\epsilon$</td>
<td>0</td>
<td>$q^2(2q^2 - 1)\epsilon$</td>
<td>$q^4\delta\epsilon$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>$q^4$</td>
<td>$q^4\delta\epsilon$</td>
<td>$q^6\delta$</td>
</tr>
</tbody>
</table>
class for different choices of $z$. Note that in Table 4 we do not have any incidences $s \lambda r$ from classes (1), (3), (4) and (6). The reason for none from class (3), for example, is that $[-1, 0, 0]$ and $[2, -1, 0]$ are not incident, so that we can never have $[-1, 0, 0] + d e_{(1, 0, 0)} + f e_{(1, 1, 1)}$ incident to $[2, -1, 0] + a_1 e_{(0, 1, 0)} + a_2 e_{(0, 2, 1)} + a_3 e_{(0, 1, 1)}$. In a similar way, we see that Table 5 need not include columns corresponding to classes (1), (2), (4) and (5).

It is now an easy matter to use Tables 4 and 5 to count the number of elements $z \in U_2$ such that $(z, y)$ belongs to $R_j$, thereby determining $p_{2j}^4$ for each $j$. For example, the number of choices of $z$ such that there is a unique subchain $s \lambda r I y$ with $s$ incident to $z$ is

$$(q - 1) + q(q - 1) + q(q - 1) + q^2(q - 1) = (q - 1)(q + 1)^2$$

so, using our criterion, we have $p_{23}^4 = (q - 1)(q + 1)^2$. We thus obtain $p_{2j}^4$, $0 \leq j \leq 4$, as follows:

\[
p_{2j}^4: \begin{array}{cccc}
0 & 1 & 2 & 3 \\
0 & 0 & q + 1 & (q - 1)(q + 1)^2 & q^2(q - 1)(q + 1)
\end{array}
\]

A complete listing of intersection numbers is given in Tables 6–10. Observe that although our technique of embedding in Lie algebras is valid only for $GF(q)$ of sufficiently large characteristic, the formulas so obtained are correct for all $q$ because intersection numbers $p_{ij}^k$ arise as specializations of the structure constants $p_{ij}^k(x)$ of the Tits generic algebra (see, for example, [6]).

7. Eigenvalues and Fusion Schemes

The eigenvalues of $B_{3, 2}(q)$ are presented in Table 11 as entries of the first eigenmatrix $P = (p_{ij}(q))$. They were obtained by standard methods: each row of $P$ corresponds to a common left eigenvector of the intersection matrices $B_i = (p_{ij}^k)$. The corresponding multiplicities are given by $m_0 = 1, m_1 = q(q + 1)(q^2 + 1)/2, m_2 = q^2(q^4 + q^2 + 1), m_3 = q^2(q^4 + q^2 + 1)$ and $m_4 = q(q^2 + 1)(q^2 + q + 1)$, as can be computed directly from the table (see [1]).

We now investigate the possible existence of fusion schemes for $B_{3, 2}(q)$. In [17] all nontrivial fusion schemes of $B_{1, 2}$ are classified. We see from that article that there are three such fusion schemes for $l = 3$: the first results from fusing relations 1 and 3; the second from simultaneously fusing relations 1 and 3 and relations 2 and 4; the third from fusing relations 1, 2 and 3. Each of these can easily be checked in the Lie case, and none works. For example, if the first fusion pattern were to work for general $q$, we would need

$$p_{12}^1 + p_{32}^1 = p_{12}^3 + p_{32}^3,$$

which yields (see Tables 7 and 9)

$$q^2 + q^4 = 2q^3.$$
Moreover, the intersection numbers of $B_3$ and $C_3$ are identical. By taking advantage of such ‘duality’ one can easily obtain a characterization of the orbitals of $C_3(q)$ analogous to the one obtained for $B_3(q)$ in this paper.

**References**


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