Europ. J. Combinatorics (1993) **14**, 17–22

Orbits of Antichains in Ranked Posets

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We consider the permutation f of antichains of a ranked poset P, moving the set of lower units of any monotone boolean function on P to the set of its upper zeros. A duality relation on orbits of this permutation is found, which is used for proving a conjecture by M. Deza and K. Fukuda. For P a direct product of two chains, possible lengths of orbits are completely determined.

Let $(P, <)$ be a finite poset, and let A be an antichain in P (i.e. a set of mutually incomparable elements). We define the following sets:

$$A^+ = \{x: x \geqslant a \text{ for some } a \in A\};$$

$$A^- = \{x: x \leqslant a \text{ for some } a \in A\};$$

$$f(A) = \{x: x \notin A^+ \text{ and } y \in A^+ \text{ for any } y > x\}.$$

It is clear that the set $f(A)$ is also an antichain, and that f is a permutation of the set of all antichains in P. We can equivalently define $f(A)$ as the only antichain B for which the equalities $A^+ \cup B^- = P$, $A^+ \cap B^- = \varnothing$ hold. The operation f can also be defined in terms of monotone boolean functions on P; namely, $f(A)$ is the set of upper zeros of the monotone boolean function having A as the set of lower units.

The operation f has attracted much attention in the case when $P = P(\Omega)$ is the lattice of subsets of a finite set Ω. In [1] the following conjectures are formulated:

(a) If $A \subseteq P_k(\Omega)$ (i.e. A is a set of k-element subsets of Ω), and $B = f^n(A) \subseteq P_l(\Omega)$, then

$$P_k(\Omega)\backslash A = f^{n+l-k}(P_l(\Omega)\backslash B).$$

(b) If $A \subseteq P_k(\Omega)$, $B = P_k(\Omega)\backslash A$ and $(A, A_1, \ldots, A_{n-1})$, $(B, B_1, \ldots, B_{m-1})$ are cycles of the permutation f, then $n = m$ and

$$|A| + |A_1| + \cdots + |A_{n-1}| = |B| + |B_1| + \cdots + |B_{m-1}|.$$

When $|\Omega| \leqslant 6$, these conjectures were checked with the help of a computer [2]. In this paper we shall prove them in a more general form (Theorem 1).

In Theorem 2 we consider orbits of antichains in a direct product of two chains. It turns out that in this case all possible orbit lengths can be defined.

For any poset P we define the following operations (here $x \in P$, $X \subseteq P$):

$$x\uparrow = \{y \in P; y > x \text{ and there is no } z \text{ such that } y > z > x\};$$

$$x\downarrow = \{y \in P: y < x \text{ and there is no } z \text{ such that } y < z < x\};$$

$$X\uparrow = \bigcup_{x \in X} x\uparrow;$$

$$X\downarrow = \bigcup_{x \in X} x_\downarrow.$$

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We will call P a ranked poset of height r if there exists a function $rk: P \to \{0, 1, \ldots, r\}$ such that
(a) $rk(x) = 0$ iff $x\downarrow = \varnothing$;
(b) $rk(x) = r$ iff $x\uparrow = \varnothing$;
(c) if $y \in x\uparrow$ then $rk(y) = rk(x) + 1$.

For $i \in \{0, \ldots, r\}$ we define the set $P_i = \{x \in P: rk(x) = i\}$—the ith layer of P. For $i < 0$ and $i > r$ we let $P_i = \varnothing$ by definition.

The lattice $P(\Omega)$ is an example of a ranked poset with $r = |\Omega|$ and $rk(X) = |X|$ for $X \subseteq \Omega$. Its ith layer is the set $P_i(\Omega)$.

LEMMA 1. (a) *For any* $X \subseteq P$ *we have* $X^+ \cap P_{k+1} = (X^+ \cap P_k)\uparrow \cup (X \cap P_{k+1})$.
(b) X *is an antichain iff* $(X^+ \cap P_k)\uparrow \cap (X \cap P_{k+1}) = \varnothing$ *holds for all* k.

PROOF. Immediate from definitions. □

THEOREM 1. *Let* P *be a ranked poset and* $A \subseteq P_a$. (a) *If* $B = f^k(A) \subseteq P_b$, *then* $P_a \backslash A = f^{k+b-a}(P_b \backslash B)$.
(b) *If* $(A, A_1, \ldots, A_{n-1})$ *and* $(P_a \backslash A, X_1, \ldots, X_{m-1})$ *are cycles of the permutation* f, *then* $n = m$ *and any element of* P *is contained in the same number of antichains from both cycles. In particular,* $|A| + |A_1| + \cdots + |A_{n-1}| = |P_a \backslash A| + |X_1| + \cdots + |X_{m-1}|$.

Before proving this theorem, we shall consider an arbitrary cycle of the permutation f. From now on, we fix a ranked poset P of height r.

Let $(A_0, A_1, \ldots, A_{n-1})$ be a cycle of f. For convenience, we continue the numeration in both directions: $A_n = A_0$, $A_{n+1} = A_1$, $A_{-1} = A_{n-1}, \ldots$. By definition, for any i the sets A^+_{i-1} and A^-_i form a partition of P. So we can define the ith co-ordinate $e_i(x)$ of any element $x \in P$:

$$e_i(x) = \begin{cases} 0 & \text{if } x \in A_i^-, \\ 1 & \text{if } x \in A^+_{i-1}. \end{cases}$$

We denote by $P_l(i_1, \varepsilon_1; \ldots; i_k, \varepsilon_k)$ the set

$$\{x \in P_l: e_{i_1}(x) = \varepsilon_1, \ldots, e_{i_k}(x) = \varepsilon_k\}.$$

If $l < 0$ or $l \geqslant r$, then let it be empty by definition.

Using this notation, we can express the sets A_i as follows:

$$A_i = \bigcup_{0 \leqslant l \leqslant r} P_l(i, 0; i+1, 1),$$

because $A_i = A_i^- \cap A_i^+$.

LEMMA 2. *The following equalities hold for any* i, k:
(a) $P_k(i, 1)\uparrow = P_{k+1}(i-1, 1; i, 1)$;
(b) $P_k(i, 0)\downarrow = P_{k-1}(i, 0; i+1, 0)$.
In particular, $P_0(i-1, 1; i, 1) = P_r(i, 0; i+1, 0) = \varnothing$.

PROOF. We prove the first equality; the second one is proved dually. By Lemma 1, we have $(A^+ \cap P_k)\uparrow = (A^+ \cap P_{k+1}) \backslash (A \cap P_{k+1})$. Applying this to the antichain A_{i-1}, we obtain

$$P_k(i, 1)\uparrow = (A^+_{i-1} \cap P_k)\uparrow = (A^+_{i-1} \cap P_{k+1}) \backslash (A_{i-1} \cap P_{k+1})$$
$$= P_{k+1}(i, 1) \backslash P_{k+1}(i-1, 0; i, 1) = P_{k+1}(i-1, 1; i, 1). \quad \square$$

Now we define the sets X_s, playing the key role in our proof of the theorem:

$$X_s = \bigcup_{0 \leq i \leq r} P_{r-i}(i+s, 1; i+s+1, 0), \qquad s \in \mathbb{Z}.$$

LEMMA 3. *Any set X_s is an antichain, and $(X_{n-1}, X_{n-2}, \ldots, X_0)$ is a cycle of the permutation f.*

PROOF. We shall show that $X_s^+ = \bigcup_{0 \leq i \leq r} P_{r-i}(i+s, 1)$. At the same time it will be shown that X_s is an antichain. To do this, we shall find the sets $X_s^+ \cap P_i$ consecutively for $i = 0, \ldots, r$.

For $i = 0$ we have

$$\begin{aligned} X_s^+ \cap P_0 = X_s \cap P_0 &= P_0(r+s, 1; r+s+1, 0) \\ &= P_0(r+s, 1) \backslash P_0(r+s, 1; r+s+1, 1) = P_0(r+s, 1) \end{aligned}$$

by Lemma 2. Now, by Lemma 2(a), we have $(X_s^+ \cap P_0)\!\uparrow\, = P_1(r+s-1, 1; r+s, 1)$. Hence $(X_s^+ \cap P_0)\!\uparrow\, \cap (X_s \cap P_1) = \emptyset$ and $X_s^+ \cap P_1 = P_1(r+s-1, 1; r+s, 1) \cup P_1(r+s-1, 1; r+s, 0) = P_1(r+s-1, 1)$.

Continuing in the same manner, we obtain, for any i:

$$(X_s^+ \cap P_i)\!\uparrow\, \cap (X_s \cap P_{i+1}) = \emptyset \quad \text{and} \quad X_s^+ \cap P_i = P_i(r+s-i, 1).$$

So by Lemma 1(b), X_s is an antichain, and the formula for X_s^+ is proved. The dual argument gives $X_s^- = \bigcup_{0 \leq i \leq r} P_{r-i}(i+s+1, 0)$.

Now we see that $X_s^+ \cup X_{s-1}^- = P$ and $X_s^+ \cap X_s^- = \emptyset$ for any s. It means that $X_{s-1} = f(X_s)$. So, all the sets X_s comprise a cycle of the permutation f.

To finish the proof, we only need to find the length of the cycle, i.e. the least positive number m such that $X_{s+m} = X_s$ for any s. Since $X_{s+n} = X_s$ for any s, we have $m \leq n$. On the other hand, we have $P_i(s, \varepsilon) = P_{i+m}(s, \varepsilon)$ for any choice of i, s and ε. Indeed,

$$P_i(s, 0) = P_i \cap X_{s-(r-i+1)}^- = P_i \cap X_{m+s-(r-i+1)}^- = P_i(s+m, 0),$$

and also

$$P_i(s, 1) = P_i \backslash P_i(s, 0) = P_i \backslash P_i(s+m, 0) = P_i(s+m, 1).$$

But then $A_{s+m} = A_s$ for any s, because of the expression of the set A_s in terms of sets $P_i(k, \varepsilon)$. So, $m = n$ and the lemma is proved. □

The correspondence between the cycles $(A_0, A_1, \ldots, A_{n-1})$ and $(X_{n-1}, X_{n-2}, \ldots, X_0)$ is a symmetric one. It can be shown easily with the help of formulas obtained in the proof of Lemma 3, but we will not use this fact here.

LEMMA 4. *Any element of P is contained in the same number of antichains from the cycles $(A_0, A_1, \ldots, A_{n-1})$ and $(X_{n-1}, X_{n-2}, \ldots, X_0)$.*

PROOF. Let us construct for an element $x \in P$ the cyclical word of its co-ordinates $w = (e_0(x), \ldots, e_{n-1}(x))$. Then the number of antichains $A_0, A_1, \ldots, A_{n-1}$ containing x equals the number of subwords 01 appearing in the word w. On the other hand, the number of antichains $X_{n-1}, X_{n-2}, \ldots, X_0$ containing x equals the number of subwords 10 in w. It is clear that these numbers coincide. □

LEMMA 5. *If $A_i \subseteq P_a$ then $P_a \backslash A_i = X_{i+a-r}$.*

Proof. We shall prove the following three claims:
(1) $P_{a+k}(i-k+1, 0)=\varnothing$ for $1 \leqslant k \leqslant r-a$;
(2) $P_{a-k}(i-k, 1)=\varnothing$ for $1 \leqslant k \leqslant a$;
(3) $P_a(i, 0; i+1, 0)=P_a(i, 1; i+1, 1)=\varnothing$.
The lemma follows from them immediately. Indeed,

$$X_{i+a-r}=P_a(i, 1; i+1, 0) \cup \bigcup_{1 \leqslant k \leqslant r-a} P_{a+k}(i-k, 1; i-k+1, 0)$$

$$\cup \bigcup_{1 \leqslant k \leqslant a} P_{a-k}(i+k, 1; i+k+1, 0).$$

Claim 3 implies that $P_a \backslash P_a(i, 0; i+1, 1)=P_a(i, 1; i+1, 0)$.
Claim 1 implies that

$$P_{a+k}(i-k, 1; i-k+1, 0) \subseteq P_{a+k}(i-k+1, 0)=\varnothing.$$

Claim 2 implies that

$$P_{a-k}(i+k, 1; i+k+1, 0) \subseteq P_{a-k}(i+k, 1)=\varnothing.$$

To prove the claims we note first that $A_i^+ \cap P_{a-1}=A_i^- \cap P_{a+1}=\varnothing$. So, $P_{a-1}(i+1, 1)=\varnothing$, and $P_{a-1}=P_{a-1}(i+1, 0)$. Then, by Lemma 2(b), $P_{a-2}=(P_{a-1})\downarrow=P_{a-2}(i+1, 0; i+2, 0)$; hence $P_{a-2}(i+2, 0)=P_{a-2}$ and $P_{a-2}(i+2, 1)=\varnothing$. Continuing in the same manner, we obtain Claim 2. The dual argument, starting from the equality $P_{a+1}(i, 0)=\varnothing$, gives us claim 1. Applying Lemma 2(a) to $P_{a-1}(i+1, 1)=\varnothing$, and Lemma 2(b) to $P_{a+1}(i, 0)=\varnothing$, we prove claim 3. □

The proof of the theorem is now immediate. Let $A=A_0 \subseteq P_a$ generate the orbit $(A_0, A_1, \ldots, A_{n-1})$, and $B=f^k(A)=A_k \subseteq P_b$. Then, by Lemma 5, $P_a \backslash A=X_{a-r}$ and $P_b \backslash B=X_{k+b-r}$ and, by Lemma 3, $P_a \backslash A=f^{(k+b-r)-(a-r)}(P_b \backslash B)$: part (a) is proved. Part (b) follows directly from Lemma 3 and 4. □

Theorem 2. *If P is a direct product of two chains of orders n and m, then the length of any orbit of the permutation f is $(n+m)/d$ for some d dividing both n and m. Any number of this form is the length of some orbit.*

Proof. Let $P=\{(i, j): 1 \leqslant i \leqslant n; 1 \leqslant j \leqslant m\}$; $(i, j) \leqslant (k, l)$ iff $i \leqslant j$ and $k \leqslant l$.

Let us represent the set P as the set of cells of a rectangular chessboard $m \times n$ (with n columns and m rows), the left lower cell corresponding to the minimal element of P. Then any antichain A is uniquely determined by a broken line on the board which separates the sets A^+ and $P \backslash A^+=f(A)^-$. This line consists of n horizontal and m vertical pieces and joins the left upper corner of the board with the right lower one (Figure 1).

For our purposes, it is convenient to encode such lines in the following manner.

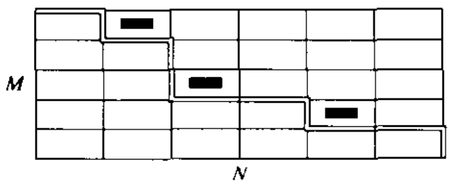

Figure 1.

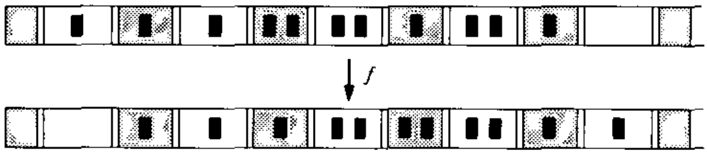

FIGURE 2.

Take an infinite strip of squares coloured alternately in black and white; going along the line from the left upper corner to the right lower one, put pebbles on the strip (from left to right): for a horizontal segment of length x, put x pebbles on a white square, for a vertical segment of length y, put y pebbles on the next black square and so on. The colour of the first square is defined by the direction of the first segment; the exact position of this square is irrelevant. An example is shown in Figure 2.

The transformation from A to $f(A)$ can be described in these terms as follows: at first, all pebbles on black squares are moved two squares to the right; then one pebble from the leftmost non-empty square (which is necessarily white) is moved one square to the right, and one pebble from the rightmost non-empty square (a black one) is moved one square to the left. We leave to the reader to check that this is indeed so.

First we shall show that after $m+n$ such operations the distribution of pebbles will be the initial one shifted $2m$ squares to the right.

To do this, let us suppose that all pebbles are different and that in each cell they are placed in a row. Furthermore, we specify that at the first stage of the transformation each such row is moved as a whole, without any permutations inside it, and at the second stage the leftmost pebble of the leftmost square is moved to the leftmost position in the neighbouring square (the rightmost pebble is moved similarly). Now we are able to trace the movements of each pebble individually, and it is easily seen that during $n+m$ moves it is moved m times by two cells to the right, once by one cell to the right and once by one cell to the left. So, the total shift of any pebble equals $2m$.

We have proved that the length of any orbit divides $n+m$. Let an orbit have length $k=(n+m)/d$. The total shift of all pebbles after one operation is equal to $2m$, so it is equal to $2m(m+n)/d$ after k operations. However, the distribution of pebbles after k operations becomes the initial one shifted by some even number of cells to the right. So, $2m(m+n)/d = 2l(m+n)$, and $m = ld$. We obtain that d divides m. Reversing the roles of m and n, we obtain that d divides n.

X_1			X_2			X_3		
0	4	3	2	1	0	4	3	2
1	0	4	3	2	1	0	4	3

(a)

0	4	3						
1	0	4						
			2	1	0			
			3	2	1			
						4	3	2
						0	4	3

(b)

FIGURE 3. $m=6$, $n=9$, $d=3$.

To finish the proof of the theorem, we need to show that for any d dividing (m, n) there is an orbit of length $(m+n)/d$.

Let $m = dm_1$ and $n = dn_1$. Let $X = (x_{ij})$ be an $m_1 \times n$ matrix with elements $x_{ij} = (i-j)(\operatorname{mod} m_1 + n_1)$ (Figure 3(a)). Partition it into d submatrices $X_1, \ldots, X_d$ of order $m_1 \times n_1$. Now partition our $m \times n$ chessboard into d^2 blocks of size $m_1 \times n_1$; and place matrices $X_1, \ldots, X_d$ consecutively in the diagonal blocks, starting from the left upper corner to the right lower one (Figure 3(b)). For each $i = 0, 1, \ldots, m_1 + n_1 - 1$, let A_i be the set of cells marked by the number i. It is easy to check that each set A_i is an antichain; and that $(A_0, A_1, \ldots, A_{m_1+n_1-1})$ is an orbit of the permutation f of length $m_1 + n_1 = (m+n)/d$. So the theorem is proved. □

References

1. M. Deza and K. Fukuda, Loops of clutters, in: *Coding Theory and Design Theory, part* 1, *IMA Vol. Math. Appl.*, 20, Springer-Verlag, 1989, pp. 72–92.
2. M. Deza, I. Faradjev and K. Fukuda, On symmetry of clutters, to appear.

Received 16 March 1992 and accepted 28 June 1992

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