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METHODS AND TRADITIONS OF BABYLONIAN MATHEMATICS

Plimpton 322, Pythagorean Triples, and the Babylonian Triangle Parameter Equations

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SUMMARIES

The remarkable Old Babylonian clay tablet which is commonly called Plimpton 322 *was published originally by Neugebauer and Sachs in their now-classical* Mathematical Cuneiform Texts *of 1945. It contains, in a table with three preserved columns, a list of values of three quantities, which in the present paper are referred to as* $\bar{c}^2$, b, *and* c. *It is easy to verify that the listed values (expressed in the usual Babylonian sexagesimal notation) are precisely the ones that can be obtained by use of the* triangle parameter equations

$$b = a\bar{b}, \quad c = a\bar{c}; \quad \bar{b} = \tfrac{1}{2}(t' - t), \quad \bar{c} = \tfrac{1}{2}(t' + t),$$

if one allows the parameter t *(with the reciprocal number* $t' = 1/t$*) to vary over a conveniently chosen set of 15 rational numbers* $t = s/r$, *and if the multiplier* a *is chosen in such a way that* b *and* c *become integers with no common prime factors. Hence, for every pair* (b, c) *appearing in the second and third columns of Plimpton 322, the corresponding triple* (a, b, c) *is a* positive primitive Pythagorean triple, *i.e., the coprime integers* a, b, *and* c *are the sides of a right triangle and therefore a solution of the indeterminate equation* $a^2 + b^2 = c^2$, *the so-called Pythagorean equation). After its publication by Neugebauer and Sachs the Plimpton tablet was further discussed and interpreted by a number of other authors (Bruins, Price, et al.) from several different points of view. It is the purpose of the present paper to try to extract and extend the best ideas from these various discussions and interpretations in order to achieve a unified and comprehensive analysis of the construction and meaning of this unique and important Babylonian mathematical text. In the paper a few comparisons with related texts are also made, for the purpose of showing that the table on Plimpton 322 is intimately*

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associated with several other interesting aspects of Babylonian mathematics.

Den märkliga lertavlan Plimpton 322 *från den s k gammal-babyloniska perioden publicerades för första gången av Neugebauer och Sachs i deras nu klassiska bok med matematiska kilskriftstexter (MCT) som utkom 1945. Lertavlan innehåller en tabell med tre bevarade kolumner där en serie av värden är angivna för tre storheter som i detta arbete kommer att kallas för* $\bar{c}^2$, *b och c. Det är en lätt uppgift att verifiera att de angivna värdena (som är uttryckta i det sedvanliga babyloniska sexagesimal-systemet) är precis de värden som kan erhållas med hjälp av* triangelparameterekvationerna

$$b = a\bar{b}, \quad c = a\bar{c}; \quad \bar{b} = \tfrac{1}{2}(t' - t), \quad c = \tfrac{1}{2}(t' + t),$$

om man tillåter parametern *t (med det inversa talet* $t' = 1/t$) *att variera över en lämpligt vald mängd omfattande 15 rationella tal* $t = s/r$, *och om faktorn a väljs så att b och c blir heltal utan någon gemensam primfaktor. För varje talpar (b, c) i andra och tredje kolumnerna på Plimpton 322 blir då den motsvarande trippeln (a, b, c) en* positiv primitiv pytagoreisk trippel, *dvs. de relativt prima heltalen a, b, c blir då sidor i en rätvinklig triangel och utgör alltså en lösning till den obestämda ekvationen* $a^2 + b^2 = c^2$ *(Pythagoras' ekvation). Efter det att Plimton-lertavlan hade publicerats av Neugebauer och Sachs blev den föremål för ytterligare studier och tolkningsförsök i en rad arbeten av andra författare (Bruins, Price, m fl.) från ett antal olika utgångspunkter. Avsikten med det föreliggande arbetet är att försöka samla och vidareutveckla de bästa uppslagen från alla dessa studier och tolkningsförslag för att kunna komma fram till en enhetlig och ingående analys av konstruktion och syfte hos denna unika och betydelsefulla babyloniska matematiska text. Dessutom innehåller arbetet ett antal jämförelser med andra kilskriftstexter, i avsikt att visa att tabellen på Plimpton 322 har ett intimt samband med flera andra fenomen inom den babyloniska matematiken.*

Замечательная древне-вавилонская глиняная табличка, обычно называемая "Плимптон 322", была первоначально опубликована Нейгебауэром и Заксом в их ставших теперь классическими Математических Клинописных Текстах в 1945. Она содержит в таблице с тремя сохранившимися столбцами список

значений трех величин, которые в данной статье обозначены соответственно $\bar{c}^2$, b и c . Легко проверить, что указанные значения (выраженные в обычной вавилонской 60-ричной системе) точно те же, которые можно получить, используя следующие уравнения параметра треугольников:

$$b = a\bar{b}, \quad c = a\bar{c}; \quad \bar{b} = \tfrac{1}{2}(t' - t), \quad \bar{c} = \tfrac{1}{2}(t' + t),$$

если допустить, что параметр t (с обратной величиной $t' = 1/t$) может меняться на подходяще выбранном множестве 15-ти рациональных чисел $t = s/r$ и если множитель а выбран так, что b и c будут целыми числами без общих простых множителей. Следовательно, для каждой пары (b, c) встречающейся во втором и третьем столбцах Плимптона 322, соответствующая тройка (a, b, c) есть положительно простая пифагорова тройка, т. е. взаимно-простые целые a, b, c будут сторонами прямоугольного треугольника и, следовательно, решением неопределенного уравнения $a^2 + b^2 = c^2$ (так называемого уравнения Пифагора).

После опубликования Нейгебауэром и Заксом плимптонская табличка в дальнейшем обсуждалась и интерпретировалась другими авторами (Бруинс, Прайс и т. д.) с различных точек зрения. Цель настоящей статьи-попытаться извлечь и обобщить наиболее плодотворные идеи этих дискуссий и интерпретаций для того, чтобы получить единый и исчерпывающий анализ этого уникального и важнейшего вавилонского математического текста. В статье сделано также несколько сравнений указанных текстов с целью показать, что таблица Плимптон 322 тесно связана с некоторыми другими интересными аспектами вавилонской математики.

1. PLIMPTON 322; A GENERAL PRESENTATION OF THE CLAY TABLET WITH ITS TEXT

The clay tablet "Plimpton 322" acquired its name from its registration number in the George A. Plimpton Collection of the Rare Book and Manuscript Library, Columbia University, New York. The tablet was originally bought by George A. Plimpton ca. 1923 from Edgar James Banks of Eustis, Florida, and had allegedly been found at Senkereh. At the time of acquisition it was dated at 2250 B.C. (I owe these data to Kenneth A. Lohf, Librarian of the Rare Book and Manuscript Library of Columbia University.) It was first recognized as an extremely important document from the early history of mathematics by Neugebauer and Sachs [1945]. The tablet had been provisionally classified as a "commerical account," but Neugebauer and Sachs established its identity as a mathematical table text from the Old Babylonian period (1900-1600 B.C.). Since they were able to show that the table on the tablet could only have been constructed by the systematic use of a *generating formula* for (integral or rational) Pythagorean triples, Neugebauer and Sachs enthusiastically pronounced Plimpton 322 "the oldest preserved document in ancient number theory" [Neugebauer and Sachs 1945, 37]. The text on the obverse of Plimpton 322 is reproduced in Figure 1.1, in a hand copy made after photographic originals kindly put at the author's disposal by the Department of Rare Books and Manuscripts of the Butler Library of Columbia University. The reverse is uninscribed. The format of the tablet is about 13 × 9 × 3 cm. A sign-by-sign transliteration of the cuneiform text is provided in Figure 1.2. This transliteration is a *conformal* transliteration, in the sense that the numerical or phonetic values of the signs on the original have been reproduced in their respective positions within the outline of the tablet. The cuneiform numbers, by the way, are easy to read even without any previous acquaint-

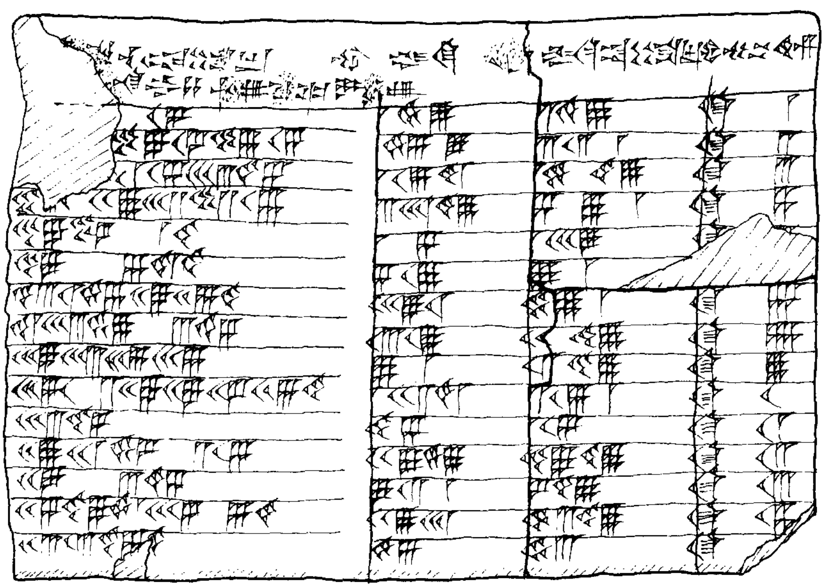


Figure 1.1. The cuneiform text of Plimpton 322.

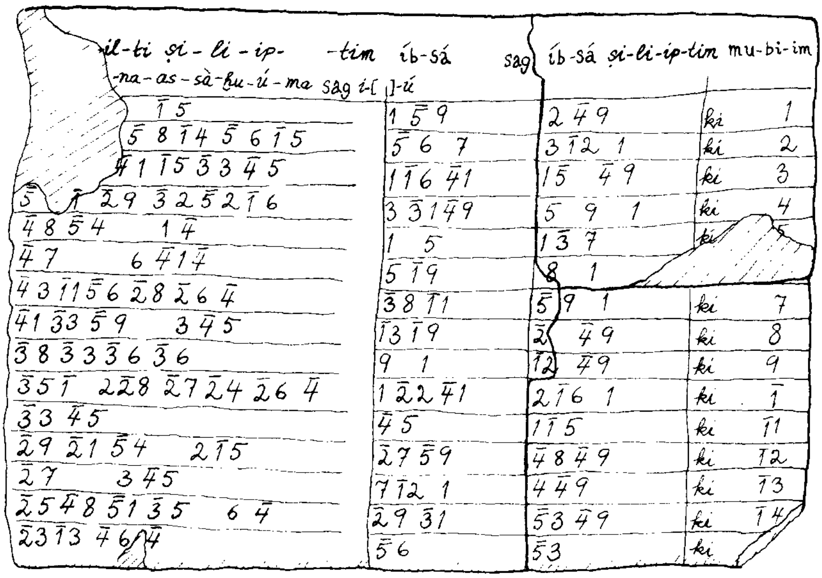
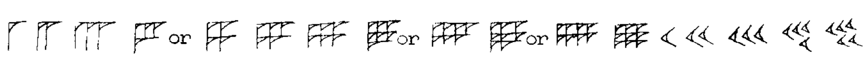


Figure 1.2. A *conformal* transliteration of Plimpton 322.

tance with cuneiform writing, if only one knows that the signs for the units, 1, ..., 9, and the "tens," 10, ..., 50, in this text are the following:



It has to be remarked here, however, that such a conformal transliteration is possible only if the use of zeros (which do not occur in the cuneiform original) is avoided in the transliteration. For this reason, the cuneiform signs for the tens have been transliterated in Figure 1.2 by the use of the ad hoc notations $\bar{1}$, ..., $\bar{5}$. Thus, for instance, in the conformal transliteration the entries in row 5 of the first three columns are the sexagesimally written numbers $\bar{4}$ 8 $\bar{5}$ 4 1 $\bar{4}$, 1 5, 1 $\bar{3}$ 7, which in a nonconformal transliteration with zeros would look like this: 48 54 01 40, 1 05, 1 37. In this connection it is important to remember that in the Babylonian sexagesimal notation *missing tens and units* are sometimes indicated by an empty space. It is convenient to indicate such missing tens and units in a nonconformal transliteration by use of zeros, as in the examples 01, 40, and 05 above. As a matter of fact, this slight *departure from Neugebauer's standard sexagesimal notation* (in which the number $\bar{4}$ 8 $\bar{5}$ 4 1 $\bar{4}$ would be transliterated as 48, 54, 1, 40) becomes a necessity when one wants to make a program for sexagesimal multiplication on a pocket calculator, as in the Appendix of the present paper.

Consider again the three entries in row 5 of the first three columns of the tablet. It is clear that if we set the second of them equal to b = 1 05 (=65), and the third one to c = 1 37 (= 97), then $c^2 - b^2 = a^2$, where a = 1 12 (= 72). In addition, if we set $\bar{c} = c/a$, then it can be shown that $\bar{c}$ = 1.20 50, which means that $\bar{c}$ is a *rational* number with a *terminating sexagesimal expansion.* In the Babylonian sexagesimal notation, however, no "fractional point" was used, with the result that such terminating rational numbers cannot be distinguished from integers in this notation. Thus, if we make use of the Babylonian notation, we can write $\bar{c}$ simply as $\bar{c}$ = 1 20 50, and its square as $\bar{c}^2$ = 1 48 54 01 40. Similarly, if we write $\bar{b} = b/a$ in the Babylonian way as 54 10, then we can show that its square is $\bar{b}^2 = \bar{c}^2 - 1$ = 48 54 01 40. Consequently, the entry in row 5 of the first column can be either $\bar{c}^2$ or $\bar{b}^2$, depending on whether we do or do not believe that we can see the traces of a vertical cuneiform wedge, standing for an initial digit (1), preceding the number 48 54 01 40 precisely where a break has occurred along the left edge of the (preserved part of the) tablet. Repeating the same arguments for each of the 15 lines on the tablet, we can conclude that if the entries in the second and third columns are called b and c, respectively, then in each of the 15 cases it is true that $c^2 - b^2 = a^2$, where a is a sexagesimal "integer," and that the number in the first column is equal either to $\bar{c}^2$ or $\bar{b}^2$, with $\bar{c} = c/a$ and $\bar{b} = b/a$. Presumably the heading over the first column, if intact, would have informed us about the precise nature of the numbers listed in that column. However, the heading in question is damaged to such an extent that it is not possible to read and translate it with any appreciable degree of certainty. This fact is regrettable since such headings over the columns of Babylonian mathematical table texts are otherwise virtually unknown. Fortunately, however, it is really not very crucial for our interpretation of the text, or for our reconstruction of the algorithm used in the preparation of it, to know whether the first column listed the values of $\bar{c}^2$ or of $\bar{b}^2$.

Nevertheless, the choice between the two possibilities mentioned is not an easy one. In fact, Neugebauer and Sachs, in their original transliteration of the text on Plimpton 322, assumed that every number in the first column was preceded by an initial (1), of which only a trace of the impression is left on the preserved part of the tablet. On the other hand, it has been maintained by Bruins [1955] with considerable emphasis that the initial digits (1) should *not* be there. Bruins differs with Neugebauer and Sachs also in that he does not find it likely that the intact tablet contained any columns in addition to those on the preserved part of the tablet. It seems to me, however, that much speaks in favor of the view held by Neugebauer and Sachs. In fact, the schematic drawing of the outlines of the tablet presented in Figure 1.3, which is made after the photographic originals at my disposal, shows quite clearly that the tablet in its present condition is thicker at its left edge than at the opposite edge. Presumably an intact tablet would have had a more symmetric appearance.

The nature of the break at the left edge is also such that it would be difficult to understand how it could have come about if only a small margin to the left of the first column is missing. On the other hand, the break would be easy to explain if we are allowed to assume that as much as perhaps a third of the whole tablet is missing, and that all the numbers in the first preserved column began with a vertical wedge for the digit (1). In fact, a look at the front and rear views of the preserved part of the tablet in Figure 1.3 and a comparison with the more detailed copy of the obverse in Figure 1.1 show quite clearly that the tablet has a throughgoing crack following the straight line which is indicated by a series of initial vertical wedges (the first of the wedges in the sexagesimal notations for 2, 3, 1, 5 ...) in rows

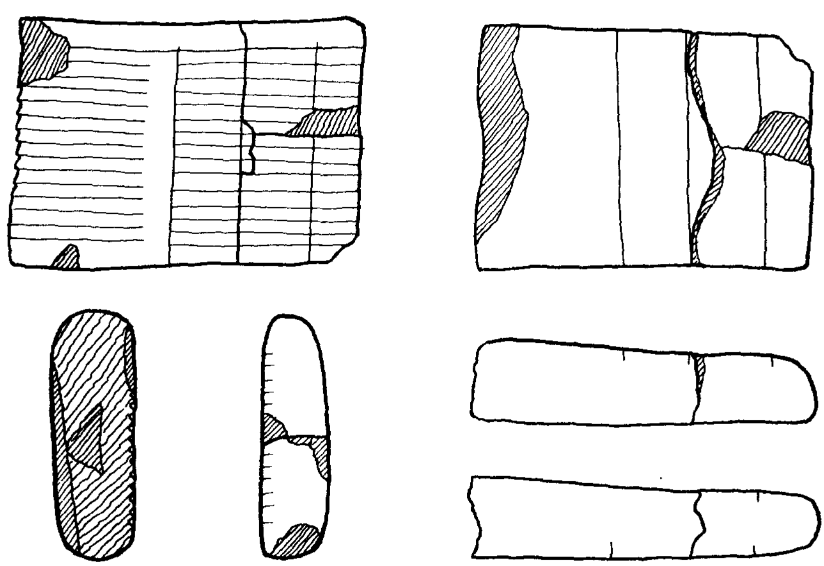


Figure 1.3. Front, rear, and side views of Plimpton 322.

1-6, 10-11, and 13 of the "c column," but deviating from this straight line in rows 7-9 where there is no initial vertical wedge. Hence the obvious implication is that there must have been a line of initial digits (1) in the "$\bar{c}^2$ column," producing a weakness in the surface of the tablet which ultimately caused the now missing portion to be broken off. All that is left of this presumed line of initial wedges is the series of inclining depressions which can still be seen along the left edge of the tablet.

Our tentative conclusion must therefore be that a substantial part of the original tablet is missing. Judging from the curvature of the tablet, it seems reasonable to conjecture that the first column of Plimpton 322 must have been situated roughly in the middle of the intact tablet. This would give room for an additional two or, possibly, three short columns on the missing part. In this connection it is interesting to recall that according to Neugebauer and Sachs "the presence of modern glue, until the recent baking of the tablet, on the left (broken) edge shows that the missing part must have been lost after the excavation of the tablet" [1945, 39]. Therefore the possibility cannot be excluded that the missing complement to Plimpton 322 still exists in some private collection or, perhaps, uncatalogued in the collections of some public institution.

2. *THE RESTRICTIONS ON THE PARAMETERS*

As observed above, if we choose the notations $\bar{c}^2$, b, and c for the three numbers in any of the 15 rows on Plimpton 322, then it can be shown (for instance using the multiplication algorithms offered in the Appendix) that $c^2 - b^2 = a^2$, where a is a sexagesimal "integer" such that $\bar{c} = c/a$. Therefore, if we set, in addition, $\bar{b} = b/a$, then it follows that $\bar{c}^2 - \bar{b}^2 = (c^2 - b^2)/a^2 = 1$, a relation which can be written also in the factorized form $(\bar{c} + \bar{b})(\bar{c} - \bar{b}) = 1$. This is an indeterminate equation for the pair $(\bar{c}\ ,\ \bar{b})$. Looking for positive *rational* solutions to this indeterminate equation, we can let it be replaced by the equivalent system of equations

$$\bar{c} + \bar{b} = t',\ \bar{c} - \bar{b} = t, \quad t \text{ rational},\ tt' = 1,\ t' > t > 0. \quad (2.1)$$

Hence, the general positive rational solution of the indeterminate equations $\bar{c}^2 - \bar{b}^2 = 1$ is given by the following generating formula (well known in classical number theory):

$$\bar{b} = \tfrac{1}{2}(t' - t),\ \bar{c} = \tfrac{1}{2}(t' + t), \quad t \text{ rational},\ tt' = 1,\ t' > t > 0. \quad (2.$$

Let us now see what results we can obtain by making the assumption that the equations in (2.2), which I call in the following the *Babylonian triangle parameter equations*, were used in the original compilation of the table on Plimpton 322. Then our first problem is to try to determine how the 15 values of the parameter t were chosen in order to yield those values of $\bar{c}^2 = (\frac{1}{2}(t' + t))^2$ which are listed in the first column of the tablet. As observed already by Neugebauer and Sachs (see also [Bruins 1949, 1955]), every admissible parameter t must be such that not only t itself but also its reciprocal number $t' = 1/t$ can be written in the Babylonian sexagesimal notation as a number with finitely many sexagesimal places. But this is possible only if t is what Neugebauer calls a *regular sexagesimal number,* i.e., only if t can be factorized into powers of 2, 3, and 5, the three prime factors in the base 60 of the sexagesimal number system. In other words, t must be of the form

$$t = 2^{\alpha}3^{\beta}5^{\gamma}, \quad \alpha,\beta,\gamma \text{ integers (not necessarily positive).}$$

Let us call the triple (α,β,γ) the *index* of any regular number of this type. Then it is clear that a regular number which is also an integer must have a nonnegative index, and that any given regular number t can be written in a unique way as a product $sr'(= s/r)$, where s and r are regular *integers* without common prime factors. In fact, the index of s is then equal to the positive part of the index of t, and the index of r' is equal to the negative part of the index of t. More generally, a given positive rational number can be written as a terminating sexagesimal number in the Babylonian sexagesimal notation if and only if it is of the form p/q, where p is an arbitrary integer, while q is a regular sexagesimal integer. In the following, I call such numbers *semiregular* sexagesimal numbers.

Now it is clear that one possible way of beginning a *systematic* enumeration of all positive semiregular solutions of the indeterminate equation $\bar{c}^2 - \bar{b}^2 = 1$ would be to let the parameter t in the generating formula (2.2) vary in a systematic way over all regular sexagesimal numbers of the form $t = s/r$, where s and r are "sufficiently small" regular integers without common prime factors (see [Price 1964]). Let us consider, for instance, all such "parameter pairs" with $0 < s < r \leq 2\ 05$ (= 125), where the condition $0 < s < r$ is included in order to ensure that $0 < t < t'$ (see (2.1)). If we indicate the respective positions of all these parameter pairs in an (r,s) plane with logarithmic scales on both coordinate axes, we obtain a diagram like the one in Figure 2.1. The use of logarithmic scales, which is necessary in order to obtain an easily readable diagram, has the additional advantage that all pairs (r,s) with $s/r = t$ lie on a straight line with slope 1. Thus, noting where the straight line with slope 1 through a given point (r,s) intersects the vertical line $r = 1\ 00$, one can read off the corresponding value of $t \times 1\ 00$ as the s

coordinate of the point of intersection. The three lines corresponding to the t values $t = 1$, $t = 5/9 = .33\ 20$, and $t = 2^{\frac{1}{2}} - 1 = .24\ 51\ 10 \ldots$ are explicitly indicated in the diagram. The line $t = 2^{\frac{1}{2}} - 1$ has the property that it separates the set of parameter pairs for Pythagorean triples $(1, \bar{b}, \bar{c})$ with $\bar{b} < 1$ from the set of parameter pairs for triples with $\bar{b} > 1$. In fact, $\bar{b} < 1$ if and only if $\frac{1}{2}(t' - t) < 1$, i.e., if and only if $t^2 + 2t - 1 > 0$, so that $2^{\frac{1}{2}} - 1 < t$ (< 1). The line $t = 5/9$, on the other hand, passes through the point $(9, 5)$, and it can be shown that $t = 5/9$ is the t value which generates the Pythag-

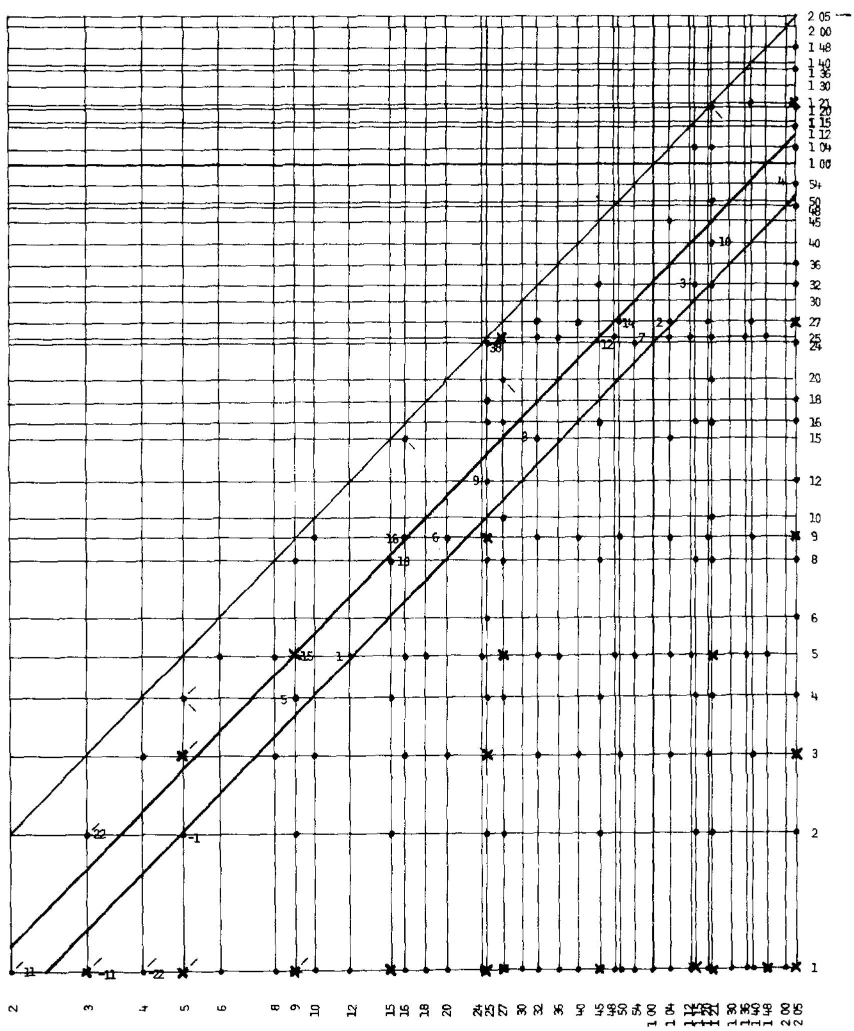


Figure 2.1. A Diagram, with logarithmic scales on both coordinate axes, showing the distribution of all pairs (r, s) of coprime regular integers with $0 < s/r < 1$ and $r < 2\ 05$. The 15 pairs associated with Plimpton 322 are separately indicated.

orean triple associated with the smallest of the $\bar{c}^2$ values listed on Plimpton 322. As a matter of fact, it can be shown that all the $\bar{c}^2$ values in the first column on Plimpton 322 are associated with Pythagorean triples corresponding to t values in the interval $2^{\frac{1}{2}} - 1 < t < 5/9$. This observation, originally due to Neugebauer and Sachs, was made more precise by Price [1964], who realized that the t values involved in the construction of the tables on Plimpton 322 can be characterized as *all* values of $t = s/r$ which correspond to parameter pairs (r, s) with $2^{\frac{1}{2}} - 1 < s/r \leqq 5/9$, $1 \leqq s < 1\ 00$ $(= 60)$. The corresponding points in Figure 2.1 are indicated by means of the numbers 1 through 15.

Moreover, it can be seen from the diagram that there are precisely 23 points in the complementary set that is characterized by the restriction to parameter pairs (r, s) with $5/9 < s/r < 1$, $1 \leqq s < 1\ 00$. In this connection Price made the interesting and very plausible conjecture that the original intention of the Babylonian mathematician, who was the author of

r	s	t = s/r	t' = r/s	n
2	1	30	2	11 ←
3	1	20	3	-11
	2	40	1 30	22
4	1	15	4	-22
	3	45	1 20	29
5	2	24	2 30	-1
	3	36	1 40	18
	4	48	1 15	31
6	5	50	1 12	32
8	3	22 30	2 40	-5
	5	37 30	1 36	20
9	4	26 40	2 15	5 ←
	5	33 20	1 48	15 ←
	8	53 20	1 07 30	34
10	3	18	3 20	-15
	9	54	1 06 40	35
12	5	25	2 24	1 ←
15	4	16	3 45	-20
	8	32	1 52 30	13 ←
16	5	18 45	3 12	-13
	9	33 45	1 46 40	16
	15	56 15	1 04	37
18	5	16 40	3 36	-19
20	9	27	2 13 20	6 ←
25	8	19 12	3 07 30	-12
	9	21 36	2 46 40	-7
	12	28 48	2 05	9 ←
	16	38 24	1 33 45	21
	18	43 12	1 23 20	27
	24	57 36	1 02 30	38

r	s	t = s/r	t' = r/s	n
27	8	17 46 40	3 22 30	-16
	10	22 13 20	2 42	-6
	16	35 33 20	1 41 15	17
	20	44 26 40	1 21	28
	25	55 33 20	1 04 48	36
32	9	16 52 30	3 33 20	-18
	15	28 07 30	2 08	8 ←
	25	46 52 30	1 16 48	30
	27	50 37 30	1 11 06 40	33
36	25	41 40	1 26 24	24
40	27	40 30	1 28 53 20	23
45	16	21 20	2 48 45	-8
	32	42 40	1 24 22 30	26
48	25	31 15	1 55 12	12 ←
50	27	32 24	1 51 06 40	14 ←
54	25	27 46 40	2 09 36	7 ←
1 04	25	23 26 15	2 33 36	-3
	27	25 18 45	2 22 13 20	2 ←
	45	42 11 15	1 25 20	25
1 12	25	20 50	2 52 48	-9
1 15	32	25 36	2 20 37 30	3 ←
1 20	27	20 15	2 57 46 40	-10
1 21	25	18 31 06 40	3 14 24	-14
	32	23 42 13 20	2 31 52 30	-2
	40	29 37 46 40	2 01 30	10 ←
	50	37 02 13 20	1 37 12	19
2 05	32	15 21 36	3 54 22 30	-21
	36	17 16 48	3 28 20	-17
	48	23 02 24	2 36 15	-4
	54	25 55 12	2 18 53 20	4 ←

Figure 2.2. A list of all values of the parameter $t = s/r$ and its reciprocal $t' = r/s$ obtained by letting (r, s) vary over all admissible parameter pairs satisfying the additional condition that $s < 1\ 00$, $r \leq 2\ 05$, and for instance, $0.15 \leq s/r < 1$.

Plimpton 322, was to include in his table *all* those values of $\bar{c}^2$, b, c (etc.) with $1 < \bar{c}^2 < 2$ (i.e., with $0 < \bar{b}^2 < 1$ and consequently with $0 < \bar{b} < 1$), which can be obtained from the Babylonian triangle parameter equations (2.2) when $t = s/r$, where s is a *single-place* regular sexagesimal integer. Price based this conjecture on the fact (see Figure 1.3) that the guide-lines separating the columns on the inscribed obverse of Plimpton 322 are continued onto the uninscribed reverse in a way suggesting that the writer of the tablet had hoped to be able to add perhaps as many as 23 more rows on the edges and the reverse to the 15 rows on the obverse.

By writing the parameter t and its reciprocal t' as $t = s/r$, and $t' = r/s$, and by letting the pair (r, s) vary over all admissible parameter pairs (coprime pairs of regular sexagesimal integers) within a bounded "strip" in the (r, s) plane, one can generate an arbitrarily large set of parameter values in a systematic and straightforward way. This type of procedure has been followed in the construction of the table in Figure 2.2. In order to obtain a sufficiently rich set of parameter values, I have chosen to include in the table all pairs (r, s) with $0 < s < 1\ 00$, $r \leqq 2\ 05$, and $0.15 \leqq s/r < 1$. An index n has been assigned to each pair (r, s) in the table in such a way that $t = s/r$ becomes an increasing function of n. In this way, the 38 pairs considered by Price (including the 15 pairs associated with Plimpton 322) have indices $n = 1, \ldots, 38$, while for 22 additional pairs with $0.15 \leqq t < 0.24$, the indices are $n = -22, \ldots, -1$.

$\bar{b} = \frac{1}{2}(t'-t)$	$\bar{c} = \frac{1}{2}(t'+t)$	$\bar{c}^2 (= 1+\bar{b}^2)$	b	c	n
1 52 30	2 07 30	4 30 56 15	15	17	-22
---	---	---	---	---	---
1 03	1 27	2 06 09	21	29	-1
59 30	1 24 30	1 59 00 15	1 59	2 49	1
58 27 17 30	1 23 46 02 30	1 56 56 58 14 50 06 15	56 07	1 20 25	2
57 30 45	1 23 06 45	1 55 07 41 15 33 45	1 16 41	1 50 49	3
56 29 04	1 22 24 16	1 53 10 29 32 52 16	3 31 49	5 09 01	4
54 10	1 20 50	1 48 54 01 40	1 05	1 37	5
53 10	1 20 10	1 47 06 41 40	5 19	8 01	6
50 54 40	1 18 41 20	1 43 11 56 28 26 40	38 11	59 01	7
49 56 15	1 18 03 45	1 41 33 45 14 03 45	13 19	20 49	8
48 06	1 16 54	1 38 33 36 36	8 01	12 49	9
45 56 06 40	1 15 33 53 20	1 35 10 02 28 27 24 26 40	1 22 41	2 16 01	10
45	1 15	1 33 45	3	5	11
41 58 30	1 13 13 30	1 29 21 54 02 15	27 59	48 49	12
40 15	1 12 15	1 27 00 03 45	2 41	4 49	13
39 21 30	1 11 45 20	1 25 48 51 35 06 40	29 31	53 49	14
37 20	1 10 40	1 23 13 46 40	28	53	15
36 27 30	1 10 12 30	1 22 09 12 36 15	2 55	5 37	16
---	---	---	---	---	---
2 27	1 00 03	1 00 06 00 09	49	20 01	38

Figure 2.3. A corrected and extended version of the table on the preserved part of Plimpton 322, using the t and t' values listed in the table of Figure 2.2.

Using the values of t and t' listed in the table of Figure 2.2, it is easy to compute the corresponding values of $\bar{b} = \frac{1}{2}(t' - t)$ and $\bar{c} = \frac{1}{2}(t' + t)$ according to the triangle parameter equations (2.2). These values are listed in the first two columns of the table in Figure 2.3. Once the values of $\bar{b}$ and $\bar{c}$ are known, it is also quite easy to compute the corresponding values of $\bar{c}^2$, as well as those of b and c. These values are listed in the third, fourth, and fifth columns of the table in Figure 2.3, ordered by increasing values of the index n, which itself is given in the last column. In this way we obtain a comprehensive table containing the table on the preserved part of the Plimpton tablet (after corrections) as a subset. This subset is identified in Figure 2.3 by means of a frame around it.

3. THE NUMERICAL ALGORITHM USED IN THE CONSTRUCTION OF THE TABLE

I will show below that the values of $\bar{c}^2$, b, and c can be obtained fairly easily from the values of $\bar{b}$ and $\bar{c}$, using only methods which would have been available also to a mathematician of the Old Babylonian period. Granted that this can be done, it is then possible to *conjecture* that the columns supposedly "broken off" from Plimpton 322 were columns for $\bar{b}$ and $\bar{c}$, in other words that the intact tablet contained columns for the variables

$$\bar{b} \quad \bar{c} \quad \bar{c}^2 \ (= 1 + \bar{b}^2) \quad b \quad c \quad n, \qquad (3.1)$$

just as in the extended table in Figure 2.3. (Note that there is no need to postulate two extra columns on the tablet for t and t', because these values would be written on a second tablet corresponding to the table in Figure 2.2. Also, the values of t and t' can be computed quite easily, if so desired, by use of the simple relations $t = \bar{c} - \bar{b}$, $t' = \bar{c} + \bar{b}$, as soon as the values of $\bar{b}$ and $\bar{c}$ are known.)

Our first task is to show how the values of $\bar{c}^2$ may have been computed. A possible clue to the answer to this question is the fact that, remarkably enough, the $\bar{c}^2$ column contains only two errors, which, as we shall see, have quite simple explanations. One would, a priori, have imagined it to be very hard to compute, mentally or manually, the squares of so many two- to five-place (i.e., 4- to 10-digit) (non-regular) sexagesimal numbers without making several mistakes. The columns for b and c on the Plimpton tablet, on the other hand, contain as many as four errors of various types. (This fact shows also, incidentally, that it is hardly likely that the values in the $\bar{c}^2$ column were obtained from the values in the c column and in an unrecorded a column by setting $\bar{c}^2 = (c/a)^2$ $(= (r^2 + s^2)/2rs)$, as claimed in [Price 1964].)

The explanation I propose for the virtual absence of errors in the $\bar{c}^2$ column is that the values written in this column had been checked beforehand by being computed in two different ways: in view of the fact that $\bar{c}^2 = 1 + \bar{b}^2$, the checking of the values of $\bar{c}^2$ may have consisted in computing the values of $\bar{b}^2$ as well, to see whether they would be the same sexagesimal numbers except for the initial "1". Note, in this connection, that my interpretation of the first column on Plimpton 322 as representing at the same time both $\bar{c}^2$ and $\bar{b}^2 = \bar{c}^2 - 1$ is in fairly good agreement with the tentative translation in [Neugebauer and Sachs 1945, 40] of the heading of this column: "the [...] of the diagonal which has been subtracted such that the width [...]." (A similar translation can be found in [Bruins 1949]. See also [Price 1964, 8].)

It might be objected that it seems unlikely that anybody would really have gone to the trouble of computing the squares of both $\bar{c}$ and $\bar{b}$ in all 15 cases, in particular since $\bar{c}$ and $\bar{b}$ are represented by several-place *nonregular* sexagesimal numbers in all cases but one. (The Babylonian square tables for several-place numbers which are known to have existed listed, for several reasons, only squares of *regular* sexagesimal numbers.) This objection, however, loses much of its strength if we assume that a certain Babylonian *factorization method*, well known in other connections, was involved in the computation of the squares in question in a way which is described below. In fact, the relation between a *normalized* Pythagorean triple $(1, \bar{b}, \bar{c})$ and the corresponding *primitive* Pythagorean triple (a, b, c) is simply that (a, b, c) is the uniquely determined multiple of $(1, \bar{b}, \bar{c})$ which can be interpreted as a triple of coprime integers. Since, according to (2.2),

$$(\bar{b}, \bar{c}) = (\tfrac{1}{2}(t' - t), \tfrac{1}{2}(t' + t))$$
$$= ((r^2 - s^2)/2rs, (r^2 + s^2)/2rs) \quad \text{if } t = s/r, \qquad (3.2)$$

it follows, therefore, that

$$(a, b, c) = \lambda(2rs, r^2 - s^2, r^2 + s^2),$$
$$\text{where } \lambda = 1 \text{ if } rs \text{ is even and } \lambda = \tfrac{1}{2} \text{ if } rs \text{ is odd} \qquad (3.3)$$

(see [Friberg 1980a, 13]). In particular, we see that $a = 2\lambda rs$ for every given regular value of the parameter $t = s/r$, and consequently a must be a *regular sexagesimal integer*. Hence, we have that (see (3.2))

$$(\bar{b}, \bar{c}) = (ba', ca'), \qquad aa' = 1 \text{ and } a \text{ regular.} \qquad (3.2)'$$

Therefore we obtain the pair (b, c) from the pair $(\bar{b}, \bar{c})$ simply by *removing the common regular factor a' from the numbers* $\bar{b}$ *and* $\bar{c}$. This can be done in a number of simple steps, irrespective of whether the value of a (or a') is known beforehand or not.

The method is most easily explained by means of an example. Consider, for instance, row 10 of the extended Plimpton table in Figure 2.3, with

$$(\bar{b}, \bar{c}) = (45\ 56\ 06\ 40\ ,\ 1\ 15\ 33\ 53\ 20), \qquad (3.4)$$

where the numbers are expressed in the Babylonian sexagesimal notation. What we want to do now is to remove successively, in several small steps, all the common regular factors in the two numbers. A first such factor is 20, for the obvious reason that

$$\bar{b} = 45\ 56\ 06 \times 60 + 40 = (45\ 56\ 06 \times 3 + 2) \times 20,$$

and similarly for $\bar{c}$. As is well known, however, in Old Babylonian mathematics division by a regular number was habitually replaced by the equivalent operation of multiplication by the reciprocal number. For this reason, it is preferable to say, not that $\bar{b}$ and $\bar{c}$ in our example have the common regular factor 20, but rather that they have the common *reciprocal factor* 3. (3 is the reciprocal of 20 in the Old Babylonian sexagesimal notation, because $3 \times 20 = 1$ (= 1 00).) This fact can be expressed by introducing the new "reduced pair"

$$(b_1, c_1) = 3 \times (\bar{b}, \bar{c}) = (3 \times 45\ 56\ 06 + 02\ ,\ 3 \times 1\ 15\ 33\ 53 + 01)$$

$$= (2\ 17\ 48\ 20\ ,\ 3\ 46\ 41\ 40). \qquad (3.4)_1$$

The reader who wants to check the multiplications can easily do so by use of the algorithms in the Appendix. In particular, the calculator program offered there, given for an HP-25, is particularly well suited for (repeated) multiplication of a given sexagesimal number of four or fewer places by (a sequence of) single-place reciprocal factors. Thus, the program gives directly the product $3 \times 45\ 56\ 06\ 40 = 2\ 17\ 48\ 20$, but somewhat less readily the product

$$3 \times 1\ 15\ 33\ 53\ 20 = 3 \times 1\ 15\ 33\ 53 + 01$$

$$= 3\ 46\ 41\ 39 + 01 = 3\ 46\ 41\ 40.$$

Since b_1 and c_1 of the reduced pair (b_1, c_1) also have 3 as a common reciprocal factor, we can repeat the process, obtaining

$$(b_2, c_2) = 3 \times 3 \times (\bar{b}, \bar{c}) = (6\ 53\ 25\ ,\ 11\ 20\ 05). \qquad (3.4)_2$$

b_2 and c_2 of the new reduced pair (b_2, c_2) have 12 (the reciprocal of 5) as a common reciprocal factor. Hence,

$$(b_3, c_3) = 12 \times 3 \times 3 \times (\bar{b}, \bar{c}) = (1\ 22\ 41\ ,\ 2\ 16\ 01). \qquad (3.4)_3$$

Since 41 is a nonregular sexagesimal number, the process stops here; b_3 and c_3 of the pair (b_3, c_3) have no common reciprocal factor. Consequently,

$$a = 12 \times 3 \times 3, \qquad (b, c) = (b_3, c_3) = (1\ 22\ 41\ ,\ 2\ 16\ 01).$$

Continuing further with the same example, we can now go on to compute the squares of both $\bar{b}$ and $\bar{c}$ in (3.4) by use of the following variant of the Babylonian "factorization method": computing the squares of the "maximally reduced" pair (b, c) by any method available (for instance, by use of the calculator program), we obtain the pair of squares,

$$(b^2, c^2) = (1\ 53\ 56\ 32\ 01\ ,\ 5\ 08\ 20\ 32\ 01). \qquad (3.5)$$

But it follows from $(3.4)_3$ that $(\bar{b}, \bar{c}) = 5 \times 20 \times 20 \times (b, c)$ $= 33\ 20 \times (b, c)$, hence that

$$(\bar{b}^2, \bar{c}^2) = (33\ 20)^2 \times (b^2, c^2)$$

Consequently we can compute $(\bar{b}^2, \bar{c}^2)$ in two simple steps (using once more the program in the Appendix):

$$(33\ 20) \times (b^2, c^2)$$
$$= (1\ 03\ 18\ 04\ 27\ 13\ 20\ ,\ 2\ 51\ 18\ 04\ 27\ 13\ 20),$$

and

$$(\bar{b}^2, \bar{c}^2) = (33\ 20)^2 \times (b^2, c^2)$$
$$= (35\ 10\ 02\ 28\ 27\ 24\ 26\ 40\ ,\ 1\ 35\ 10\ 02\ 28\ 27\ 24\ 26\ 40).$$

We recognize in one or the other of these two numbers the entry in the first column of row 10 on the Plimpton tablet, and we notice in addition that the factorization method has allowed us to compute the square of the *five-place* number $\bar{c}$ in several easy steps, of which the first one was the computation of the square of the *three-place* reduced number c. The importance of such a reduction from, say, a five-place to a three-place number becomes obvious as soon as we start trying to use the calculator program, with its built-in limitation to at most four-place numbers in the register that is reserved for the current value of a. We can very well imagine that the computations of the Old Babylonian mathematician who constructed the table on Plimpton 322 were subject to similar limitations, especially if he used some kind of "abacus" which would set a restriction on the number of sexagesimal places in the numbers of the computation.

It is possible to prove that the factorization method, illustrated by the example just described, will always lead to the desired result in a finite number of steps. The proof, which is not very difficult, will be ommitted since it is of no particular importance in the present connection.

Of more specific interest, however, is the fact that there exists a well-known example of a cuneiform text in which a variant of the Babylonian factorization method is used in order to extract the square root of a several-place semiregular sexagesimal number. In fact, the foregoing discussion of the way in which the squares $\bar{c}^2$ on the Plimpton tablet may have been computed was conducted in imitation of the (reverse) procedure in this example. The text in question is the Sippar text Ist.S 428 [Neugebauer 1935, Ch. I.6.c; Huber 1957]. (A similar example is in the text IM 54472 [Bruins 1954]). The problem posed on Ist.S 428 is to compute the square root of the number

$(x^2 =)$ 2 02 02 02 05 05 04.

The solution, as given on the tablet, is the following. In a first phase of the computation obvious regular square factors in the given number are removed, one at a time, through repeated multiplication with the corresponding reciprocal factors:

$(x_1^2 = 30^2 \times x^2 = 15 \times x^2 =)$ 30 30 30 31 16 16,

$(x_2^2 = 15^2 \times x_1^2 = 3\ 45 \times x_1^2 =)$ 1 54 24 24 27 16,

$(x_3^2 = 15^2 \times x_2^2 = 3\ 45 \times x_2^2 =)$ 7 09 01 42 15,

$(x_4^2 = 2^2 \times x_3^2 = 4 \times x_1^2 =)$ 28 36 06 06 49.

Note that the multiplication here by the *two-place* number 3 45 may have been carried out in a single operation, multiplying successively each sexagesimal place of $x_1{}^2$, or of $x_2{}^2$, by 3 45, which could be done easily by use of a Babylonian *combined multiplication table* of standard type. In fact, such combined multiplication tables as a rule contained multiplication tables for, in particular, the *reciprocals* of almost all single-place regular numbers (See [Neugebauer 1935, I,9].) The next step in the computation is to find the square root x_4 of the last reduced number, by any method available. The result can be shown to be:

$$(x_4 =) \qquad 5\ 20\ 53.$$

And finally, in the last line of the text, the correct value of the desired square root x is obtained by reversing the relation $x_4 = 2 \times 15 \times 15 \times 30 \times x$:

$$(x = 30 \times 4 \times 4 \times 2 \times x_4 = 16 \times x_4 =) \quad 1\ 25\ 34\ 08.$$

(Remark: Interestingly enough, the number 1 25 34 08 is the "integral part" of the square root of the very special sexagesimal "integer" 2 02 02 02 02 02 02. This observation explains the appearance of the seemingly strange number 2 02 02 02 05 05 04--a square!--in the text we have just considered.)

Analogous applications of the factorization method seem to have been employed regularly by Babylonian mathematicians throughout the entire period of cuneiform writing. In late (Seleucid) times a particularly elaborate model of the factorization method may have been used in order to construct the well-known Babylonian tables of reciprocals and squares of "six-place" regular numbers. (See [Neugebauer 1935, 14-22; Friberg 1980b].)

4. THE NATURE OF THE ERRORS

In the preceding section of this paper I conjectured that the following algorithm was used for the construction of the columns on the Plimpton tablet, including the missing part: First a list was made, one way or another, of admissible t values $\geq 2^{\frac{1}{2}} - 1$, which were then ordered in an increasing sequence and labeled by an index n. Next, the corresponding values of $\bar{b} = \frac{1}{2}(t' - t)$ and $\bar{c} = \frac{1}{2}(t' + t)$ were computed, and it was checked, by the factorization method, that the Pythagorean equation $\bar{c}^2 = 1 + \bar{b}^2$ was satisfied. This, incidentally, gave a valuable verification of the computations so far. It was observed that, in the process of computing $\bar{b}^2$ and $\bar{c}^2$, the maximally reduced pair

(b, c) had also been computed, so that the three columns for $\bar{c}^2$, b, and c were the outcome of a single series of computations. Of the 38 rows of values $\bar{b}$, $\bar{c}$, $\bar{c}^2$, b, and c, presumably obtained in this way, the first 15 were written on the obverse of the tablet, while the remaining 23 were saved to be written later on the reverse and on the edge (although this never happened). Later, the part of the tablet containing the first columns was broken off and lost, possibly in modern times.

Let us now see how the algorithm proposed above can be reconciled with the errors appearing in the text of the Plimpton tablet. These errors are (see Figure 1.2 and the corresponding part of the corrected table in Figure 2.3):

in the $\bar{c}^2$ column:

row 2: [1 56 56]58 14 *56* 15 instead of 1 56 56 58 14 *50 06* 15

row 8: [1]41 33 *59* 03 45 instead of 1 41 33 *45 14* 03 45

in the b column:

row 9: *9* 01 instead of *8* 01

row 13: *7 12 01* instead of *2 41*

row 15: *56* instead of *28*

in the c column:

row 2: *3 12 01* instead of *1 20 25*

(Remark: Not exactly an error, but a puzzling departure from the standard practice in the first three columns, is the use in the n column of some unusual sign forms for the numbers 4, 7, and 8. A possible explanation is that the tablet is a copy of an older original, and that the person who made the copy happened to imitate the sign forms of the original in the n column where the individual number signs are relatively isolated from each other.)

It is also customary to point out as an "error" that the pair (b, c) = (45 , 1 15) in row 11 is not coprime; if it is assumed that the purpose of Plimpton 322 is the enumeration of a certain set of *primitive* Pythagorean triples (a, b, c) by listing the last two coordinates (b, c) of each such triple, then the correct entry in row 11 would have been (*3*, *5*), rather than (45 , 1 15). In my interpretation, however, the pair (45 , 1 15), while not maximally reduced as are the corresponding pairs in all the other 14 rows, is still *sufficiently* reduced, in the sense that it is easy to compute the squares of the pair (45 , 1 15) directly. Besides, it may well be true that the *standard example* of a Pythagorean triple which *we* know as the triple (3, 4, 5) (or 4, 3, 5)) was known to the Babylonian mathematicians as the triple (1, 45 , 1 15). In other words, in this particular case, the writer simply preferred to keep the original pair $(\bar{b}, \bar{c})$, because it was a pair with which he was already familiar. This means that the presence of this particular "error" is very close to a proof that no application of "number theory", in the proper sense of the word, was involved in the construction of the table on Plimpton 322. What the true purpose of the table may have been is discussed in the next section.

Of the errors listed above, two occur in the column of $\bar{c}^2$ values. Of these two errors, the one in row 2 is most easily explained. It consists in the *absence of an empty space to denote a couple of missing* tens and units and may have been caused, for example, by a misreading of the notes which must have been scribbled on a piece of clay in the course of the actual computations. The second error in the $\bar{c}^2$ column seems to be more serious. In fact, the text gives in row 8 the incorrect $\bar{c}^2$ value, 1 41 33 *59* 03 45, instead of the correct value, 1 41 33 *45 14* 03 45. Hence, at some point in the computation a mistake must have been made, probably due to the inadequacy of the Babylonian sexagesimal notation in certain situations, with the effect that the two sexagesimal places 45 14 "telescoped" into the single place 59 (see [Bruins 1955]).

In order to see how this may have happened without the error being discovered when the relation $\bar{c}^2 = \bar{b}^2 + 1$ was tested (according to the algorithm suggested in the preceding section), the computation of $\bar{c}^2$ and $\bar{b}^2$ by the Babylonian factorization method must be examined. First the common regular factors of the pair (b, c) are removed:

$(\bar{b}, \bar{c}) =$ (49 56 15 , 1 18 03 45),

$(b_1, c_1) = 4 \times (b, c) =$ (3 19 45 , 5 12 15),

$(b_2, c_2) = 4 \times (b_1, c_1) =$ (13 19 , 20 49).

Next the squares of the reduced pair (b_2, c_2) are found:

$(b_2{}^2, c_2{}^2) =$ (2 57 20 01 , 7 13 20 01)

Finally, the squares of the pair $(\bar{b}, \bar{c})$ are computed by reversing the facorization process:

$$(\bar{b}^2, \bar{c}^2) = 3\ 45 \times 3\ 45 \times (2\ 57\ 20\ 01\ ,\ 7\ 13\ 20\ 01)$$
$$= 3\ 45 \times (11\ 05\ 00\ 03\ 45\ ,\ 27\ 05\ 00\ 03\ 45)$$
$$= (41\ 33\ 45\ 14\ 03\ 45\ ,\ 1\ 41\ 33\ 45\ 14\ 03\ 45).$$

In the Babylonian sexagesimal notation there was no special symbol for missing tens or units, not even for entire missing sexagesimal places (this is true at least for most texts from the Old Babylonian period), although an empty space was used, when needed, to indicate the absence of one or more consecutive tens or units. Hence, the difference between the sexagesimal numbers 11 05 00 03 45 and 11 05 03 45, in the Babylonian notation, must have been (at most) that the empty space indicating the *three* missing tens and units in the first number was larger than the empty space in the second number, which indicated only one missing ten. Therefore, it may well be that the author of the text misread his notes and worked with the number 11 05 03 45, instead of the correct number 11 05 00 03 45. Similarly, he would have calculated with the number 27 05 03 45 instead of the correct 27 05 00 03 45. In this way the error introduced in the second step of the computation of $(\bar{b}^2, \bar{c}^2)$ (see above) would then have spread to the third step of the same computation, causing the "telescoping" of the two places 45 14 into the single place 59. In fact,

$$3\ 45 \times 11\ 05\ 03\ 45 = 3\ 45 \times (11\ 05\ 00\ 00 + 3\ 45)$$
$$= 41\ 33\ 45\ 00\ 00 + 14\ 03\ 45 = 41\ 33\ 59\ 03\ 45,$$

while

$$3\ 45\ 11\ 05\ 00\ 03\ 45 = 3\ 45 \times (11\ 05\ 00\ 00\ 00 + 3\ 45)$$
$$= 41\ 33\ 45\ 00\ 00\ 00 + 14\ 03\ 45 = 41\ 33\ 45\ 14\ 03\ 45.$$

Since the same kind of telescoping would have occurred in the two sexagesimal numbers expressing the values of $\bar{b}^2$ and of $\bar{c}^2$, respectively, it is clear that the relation $\bar{c}^2 = \bar{b}^2 + 1$ would still hold for the telescoped numbers, and the mistake could therefore not have been detected, as would other types of mistakes which would have occurred in just one of the two numbers.

Now consider the errors in the b column of Plimpton 322. Of these the error in row 9 was simply a misreading of the notes, copying a "9" as an "8," which would have happened quite easily because of the small difference between the two number signs in question. The error in row 13, which consists in writing the value 7 12 01 of b^2, instead of the value 2 41 of b, must also be a mistake made in copying from the notes, where, according to the assumptions we have made, *the successive computation of the values of b and b^2 were two of the steps in the computation of the corresponding value of $\bar{b}^2$.* The error in row 15, finally, consists in writing 56 = 2 × 28 instead of 28. A possible explanation is that the step-by-step factorization of $\bar{b}$ and $\bar{c}$ was carried out, not simultaneously, but separately, for $\bar{b}$ and for $\bar{c}$, in the following way:

(b =)	37 20	($\bar{c}$ =)	1 10 40
($b_1 = 3 \times \bar{b}$ =)	1 52	($c_1 = 3 \times c$ =)	3 32
($b_2 = 30 \times b_1$ =)	56	($c_2 = 15 \times c_1$ =)	53
...			

Since here the factorization of $\bar{c}$ was completed after two steps, the factorization of $\bar{b}$ would also have to be interrupted after two steps, in order to obtain the two components of the pair ($\bar{b}$, $\bar{c}$) reduced to the same degree. The mistake would consist in the fact that the reciprocal factor 30 had been used in the second reduction step of the first case, whereas the reciprocal factor 15 was used in the second case, perhaps due to the circumstance that it is not quite so obvious that 4 is a factor of 52 as it is that 4 is a factor of 32.

The error in row 2 of the c column is of a similar type, but with an interesting additional complication. The following explanation is essentially due to Bruins [1955]: The successive steps in the factorization of $\bar{b}$ and $\bar{c}$ would be the following:

($\bar{b}$ =)	58 27 17 30	($\bar{c}$ =)	1 23 46 02 30
...	...	...	...
($b_4 = 12 \times b_3$ =)	56 07	($c_4 = 12 \times c_3$ =)	1 20 25
		($c_5 = 12 \times c_4$ =)	16 05
		($c_6 = 12 \times c_5$ =)	3 13

Now clearly the last two steps in the factorization of $\bar{c}$ have to be canceled in order to have $\bar{b}$ and $\bar{c}$ reduced to the same degree. This would be noticed by the author of the text when he used the reduced values for the computation of the squares $\bar{b}^2$ and $\bar{c}^2$; but later, when he referred to the same notes for the reduced values b and c of $\bar{b}$ and $\bar{c}$, it is easy to see how he can have read the wrong values (56 07 , 3 13) instead of the correct ones (56 07 , 1 20 25). The additional complication is that the value 3 13 would have been obtained through multiplication of 16 05 by the reciprocal factor 12, in the following way:

$$12 \times 16\ 05 = 3\ 12 + 1 = 3\ 13.$$

Thus when the correct value 3 13 was misread as 3 12 01 (or rather as 3 $\bar{1}$ 2 1, see Figure 1.1), it was because the cuneiform sign for 13 had been obtained by carelessly adding a "1" to the sign for "12," with a little too much space between them.

In conclusion we can now say that all the errors occurring in the text of the Plimpton tablet can be explained as above with reference *either to the inherent inadequacies of the Babylonian sexagesimal notation,* with its absence of zeros and its way of writing the units by juxtaposition of "ones" rather than with individual signs as we do it, *or to the difficulties in carrying out the factorization process for computing the squares of* $\bar{b}^2$ *and* $\bar{c}^2$. In particular, the preceding analysis of the way in which the errors may have come about lends strong support to the suggestions made in the previous sections about how the computations were carried out when the table on Plimpton 322 was constructed.

5. THE PURPOSE OF THE TEXT

In the discussion of the alleged error in row 11 of the Plimpton tablet, where the unreduced pair (45 , 1 15) is listed rather than the corresponding maximally reduced pair (3, 5), I suggested that the author of the text did not, after all, have a number-theoretical application in mind. If that is so, what then may the purpose of the text have been? In trying to answer this question, I base my arguments on the assumption that the complete tablet contained columns for the values of the variables

$$\bar{b} = \tfrac{1}{2}(t' - t), \quad \bar{c} = \tfrac{1}{2}(t' + t), \quad \bar{c}^2 = 1 + \bar{b}^2, \quad b = a\bar{b}, \quad c = a\bar{c}$$

(see the "extended Plimpton table" in Figure 2.3), or, at least, that the values of $\bar{b}$ and $\bar{c}$ had been computed and were available to the author of the text.

A first observation is that there can be no doubt about the geometrical background of the text. Indeed, the headings over the three preserved columns (see Figure 1.2), not counting the index column, are

[...-ki]-il-ti *ṣi-li-ip-tim* íb-sá sag íb-sá *ṣi-li-ip-tim*.

Here "sag" and "*ṣiliptum*" can be translated as "front" and "diagonal," respectively, while the meaning of the term "íb-sá" is less clear in the present connection. According to Neugebauer and Sachs, the term cannot here have its usual meaning of "square root," ...; hence they suggest the intentionally vague translation "solving number." For simplicity, I use the shorter but equally vague translation "root." Then the headings of the three columns can be translated in the following way:

[...] of the diagonal root of the front root of the diagonal.

The meaning of these headings is more or less clear: the "front" and the "diagonal" are, according to Babylonian standard terminology, respectively *the shortest side and the hypotenuse* of a right triangle. (The remaining longer side is called "uŝ," which may be translated by the term "flank." The damaged word in the heading over the first column ought to have the meaning "square"(?), and the "root of the front" may be assumed to mean the maximally reduced value of the front, i.e., according to our previous discussion of the algorithm for the computation, the nonregular core of the sexagesimal number appearing in the "$\bar{b}$ column" where the values of the front itself are listed. The translation of the word "sag" as front = *shortest side* is important, because it explains why the values in the $\bar{c}^2$ column decrease from a maximum value very close to 2, or, equivalently, why the range of the corresponding t values is restricted by the condition that $t > 2^{\frac{1}{2}} - 1$. Namely, if $\bar{a}$, $\bar{b}$, and $\bar{c}$ are the values of the front, the flank and the diagonal, respectively, then the fact that $\bar{a}$ has the constant value 1 implies that $\bar{b} < 1$, hence that $\bar{b}^2 < 1$ and $\bar{c}^2 < 2$. Thus we can conclude that it may have been the intention of the author of the tablet to find the front and diagonal of *all* rational right triangles *with flank = 1,* under the sole condition, for practical reasons, that the parameter $t = s/r$ in the generating formula (2.2) must be a regular sexagesimal number such that, for instance, $s < 1\ 00$. (I am assuming here, with Price [1964], that if the work on the tablet had been completed, it would have contained 38, rather than 15, rows.)

If $(1, \bar{b}, \bar{c})$ is a normalized Pythagorean triple with $\bar{b} < 1$, obtained by use of the triangle parameter equations (2.2), with t regular and $t > 2^{\frac{1}{2}} - 1$, then the corresponding primitive Pythagorean triple (a, b, c) satisfies the condition that $a = 2\lambda rs$ is a regular sexagesimal number. Hence every primitive Pythagorean *triple* obtained in this way can be identified with the sides of a primitive rational Pythagorean *triangle* with the *flank* a given by a *regular* sexagesimal number. As a rule, the *front* b $(= a \times \frac{1}{2}(t' - t))$ is only *semiregular,* because the difference between two regular sexagesimal numbers is usually not regular. On the other hand, if we use the triangle parameter equations (2.2) with t regular as before, but with $t < 2^{\frac{1}{2}} - 1$, then we can obtain, in a similar way, the sides of all primitive rational Pythagorean triangles with the *front regular* but, as a rule, the *flank* semiregular. Consider, for instance, the case $n = -1$ of the table in Figure 2.3; i.e., the case when $t = 2/5 = .24 < 2^{\frac{1}{2}} - 1 = .24\ 51 \ldots.$ Then $(a, b, c) = (20, 21, 29)$, and it is clear that the flank 21 is semiregular, while the front 20 is regular. Thus even simple Pythagorean triples like this one would be missed by the algorithm used for the construction of the table on the Plimpton tablet. This is one more reason why it does not seem likely that the intention of the author of the Plimpton text was to make a list of all (reasonably small) positive primitive Pythagorean triples. The few cases when *both* the "side" and the "front" of a primitive Pythagorean triangle are regular can be identified easily. In fact, if $b/a = \frac{1}{2}(t' - t)$, then it is easy to see that $a/b = \frac{1}{2}(t_1' - t_1)$, where $t_1 = (1 - t)/(1 + t)$. Hence, a and b are *both* regular if and only if t and t_1 are both regular, i.e., if and only if both t and $1 \pm t$ are regular. This condition is satisfied only for some very small values of r and s (if t is written in the usual way as s/r) since the condition implies that not only r and s, but also $r \pm s$, must be regular, at the same time that r and s are assumed to be coprime. It is easy to see that these conditions are satisfied by only four pairs of "conjugate parameter values," t and t_1, in the range that we are considering, namely, the pairs

$$t,\ t_1 = 1/2,\ 1/3;\ 2/3,\ 1/5;\ 3/5,\ 1/4;\ 1/9,\ 4/5.$$

The corresponding points (r, s) and (r_1, s_1) are indicated by small arrows in Figure 2.1. The associated pairs of Pythagorean triples are, respectively,

$$(4, 3, 5),\ (3, 4, 5);\ (12, 5, 13),\ (5, 12, 13);$$

$$(15, 8, 17),\ (8, 15, 17);\ (40, 9, 41),\ (9, 40, 41).$$

Thus there are essentially only four such primitive positive Pythagorean triangles with both of the shorter sides regular. It is worth mentioning that the primitive Pythagorean triples (a, b, c) with *either* a *or* b regular make up only a small fraction of the set of all primitive Pythagorean triples. Thus in a list of the 447 primitive Pythagorean triples (a, b, c) with $c < 3000$ [Martin 1912], only 9 of the 15 triples associated with the Plimpton tablet are present. The simplest of all the triples in this list with both a and b nonregular is (56, 33, 65).

It still remains to explain the reason for the presence of the "$\bar{c}^2$ column" on the Plimpton tablet. In order to be able to do this we start by making the following simple but extremely important observation. *With very few exceptions all Babylonian mathematical problem texts contain problems whose solutions are rational numbers or, more precisely, semiregular numbers which can be expressed by use of the Babylonian sexagesimal notation.* It is evident that the authors of these Babylonian mathematical texts must have devoted a lot of work and ingenuity in *choosing* the right kind of *data* in their formulation of the problems, and in *devising problems* they knew would possess solutions of the indicated kind. For brevity, I call such problems *solvable*. For example, the problem of finding the third side of a right triangle when two of the sides are given becomes "solvable" only if the sides of the given triangle are multiples of the sides of a primitive Pythagorean triangle with one of the shorter sides regular.

Thus it appears that the reason for the construction of the tables on the Plimpton tablet was not an interest in number-theoretical questions, but rather the need to *find data for a "solvable" mathematical problem.* More precisely, it is my belief that the purpose of the author of Plimpton 322 was to write a *"teacher's aid" for setting up and solving problems involving right triangles*. In fact, a typical Babylonian problem text contains not only the formulation of the problem but also the details of its numerical solution for the given data. Hence the contents of the table on the (intact) Plimpton tablet would have given a teacher the opportunity to set up a large number of solved problems involving right triangles, with full numerical details, as well as to formulate a series of exercises for his students where only the necessary data were given, although the teacher *knew* that the problem was solvable, and where he could *check* the numerical details of the students' solutions by using the numbers in the table. For example, if the given problem was to find the diagonal $\bar{c}$ of a right triangle with flank 1 and front $\bar{b}$, then the steps of the computation would be: to compute the nonregular core of $\bar{b}$ (listed in the b column of the tablet), then the square of this nonregular factor and, by means of it, the squares $\bar{b}^2$ and $\bar{c}^2 = \bar{b}^2 + 1$ (listed in the $\bar{c}^2$ column), after which a root extraction, executed (for instance) by use of the factorization method, would yield the desired value of $\bar{c}$ (listed in the c column). With this explanation the role of the $\bar{c}^2$ column as well as of the b and c columns on Plimpton 322 becomes perfectly clear.

There is also the interesting possibility of a close connection between the "Babylonian triangle parameter equations" (2.2) and the treatment of some types of quadratic equations in Babylonian mathematics. In fact, suppose that either $\bar{b}$ or $\bar{c}$ is known and that we want to find the parameter t which generates this particular $\bar{b}$ or $\bar{c}$ value when inserted into Eqs. (2.2). If $\bar{c}$ is known, then we have to solve the following equation for t:

$$t' + t = 2\bar{c} \quad (t't = 1,\ 0 < t < 1). \qquad \text{B I}$$

This type of equation, which is known from several Babylonian mathematical problem texts, is a special case of the *first Babylonian type* of "quadratic equation" [Gandz 1937, 405-406]. The equation is called "quadratic" because if t is a solution of B I, then t and t' are identical with the two (*positive*) solutions of the following true quadratic equation (also known from some Babylonian texts):

$$x^2 + 1 = 2\bar{c}x. \qquad \text{B IX}$$

Similarly, if $\bar{b}$ is known, then t can be computed as the solution of the equation

$$t' - t = 2\bar{b} \quad (t't = 1,\ 0 < t < 1). \qquad \text{B II}$$

This equation is a special case of the *second Babylonian type* of "quadratic equations," and if t is a solution of B II, then t is identical with the *positive* solution of the true quadratic equation

$$x^2 + 2\bar{b}x = 1, \qquad \text{B VII}$$

while the reciprocal number t' is identical with the *positive* solution of the quadratic equation

$$x^2 - 2\bar{b}x = 1. \qquad \text{B VIII}$$

(The remaining types of "quadratic equations" known from Babylonian texts are not discussed here. See for instance [Gandz 1937], and also [Friberg 1979].)

An elegant and typically Babylonian method of solving Eq. B I is nicely exemplified by a sequence of four solved problems in the important Babylonian problem text AO 6484 (from the

Seleucid period, 321-364 B.C.; see [Neugebauer 1935, 101-102]). The method consists in computing, in a series of simple steps, the quantities

$$\bar{c},\quad \bar{c}^2,\quad \bar{c}^2 - 1 = \bar{b}^2,\quad \bar{b},\quad \bar{c} + \bar{b} = t',\quad \bar{c} - \bar{b} = t. \qquad (5.1)$$

The given values of $2\bar{c}$ and the solutions t' and t obtained in the four cases are the following:

(1)	$2\bar{c}$ = 2 00 00 33 20 (= 2::$\bar{3}$ 3 $\bar{2}$),	t' = 1 00 45 (= 1:$\bar{4}$ 5),	t = 59 15 33 2(
(2)	$2\bar{c}$ = 2 03,	t' = 1 15,	t = 48,
(3)	$2\bar{c}$ = 2 05 26 40,	t' = 1 21,	t = 44 26 40,
(4)	$2\bar{c}$ = 2 00 15,	t' = 1 04,	t = 56 15.

(A special sign was used in astronomical and mathematical cuneiform texts from the Seleucid period in order to indicate missing tens or units in sexagesimal numbers: the sign in question is represented by a "colon" in the "conformal" transliterations of $2\bar{c}$ and t', in the first of the four cases above.)

It is important to notice that the four equations of type B I appearing in the problem text AO 6484 were not solved in the way in which they would be solved today, starting with the replacement of the given equation of type B I by the corresponding quadratic equation of type B IX. It is not even necessarily true that the Babylonian mathematicians were aware of the possibility of transforming the two types of equations into each other. In any case, equations of type B I or B II were solved by use of an algorithm proceeding via the computation of a series of intermediary values as in (5.1) above; this is illustrated by the four explicitly solved examples in the text AO 6484. The connection between this algorithm and (my proposed reconstruction of) the table on Plimpton 322 is obvious.

Thus it is conceivable that the idea of considering equations of types B I and B II and solving them in a series of steps (as in (5.1)) originated in a study of problems concerned with the generation of Pythagorean triples and with methods for finding relations between the sides of Pythagorean triangles. It is also conceivable that a secondary purpose of the Plimpton table may have been to facilitate the setting up and solving of equations of types B I and B II. Indeed, if we look at the t values used in the formulation of the four problems involving B type equations on AO 6484, and if we write these t values as rational numbers of the form s/r, then we obtain, in the respective cases,

$$s/r = 120/121, \quad 4/5, \quad 20/27, \quad 15/16.$$

The corresponding points (r, s) in Figure 2.1 are indicated by four small arrows. The distribution of these points in the diagram suggests that the author of AO 6484 may have had recourse to an "extended Plimpton table" of the type proposed by Price [1964]. Of particular interest is the location in the diagram of the point (121, 120). This point is obviously closer to the line $t = 1$ than any other point in the diagram, a circumstance which is reflected in the fact that the corresponding $\bar{c}$ value, which is 1 00 00 16 40, is very close to 1. In fact, it is possible that this particular case was considered by the author of the text for didactic reasons precisely because it involved computations with sexagesimal numbers with so many missing tens and units that the computations would have been quite hard to carry out successfully without the use of the "colon" sign for such missing units.

The tablet BM 13901 of the British Museum [Neugebauer 1937, 2-15], which is one of the oldest known Babylonian mathematical texts, is an important compendium of quadratic equations of several different types. It has been discussed in great detail by Gandz [1937] in his important work on the origin and development of quadratic equations in Babylonian, Greek, and early Arabic algebra. Gandz devoted particular attention to one of the equations, No. 23, on this tablet, because its solution was computed in a somewhat unexpected way. The equation, which we would write in the form

$$x^2 + 2px = q, \quad p = 2, \quad q = 41\ 40\ (= 50^2), \tag{5.2}$$

is clearly of type B VII, in a slightly more general case than the case $q = 1$ which we have considered before. The solution of this equation, as given in the text, proceeds in a number of steps with the successive computation of the following intermediate numbers:

$$15 \times 41\ 40 = 10\ 25\ (= 25^2),$$

$$1 + 10\ 25 = 1\ 10\ 25\ (= (1\ 05)^2), \quad 1\ 05, \ 5, \ 10.$$

Thus it appears that if we set $q = m^2$, then the method of solution of the equation $x^2 + 2px = m^2$, in this example, consists in the successive computation of the numbers

$$p^{-2} \times m^2 = (m/p)^2, \quad 1 + (m/p)^2 = (n/p)^2,$$

$$n/p, \quad n/p - 1, \quad p \times (n/p - 1) = n - p.$$

In other words, the intermediate equation $(x + p)^2 = p^2 + m^2$ is solved by writing the term $p^2 + m^2$ in the factorized form $p^2 \times (1 + (m/p)^2)$, after which a simple reference to some table of the Plimpton type makes it possible to proceed with the algorithm. It is a rather remarkable fact that Gandz, to whom this analysis is due, was led by this example to conjecture [1937, 506] that *"the Babylonians may have had a table of such fractional squares which would give a new square with ±1."* In other words, Gandz conjectured the existence of a Babylonian table of the Plimpton type 8 years before the tablet Plimpton 322 was published by Neugebauer and Sachs!

6. REFLECTIONS ON THE ORIGIN OF THE "PYTHAGOREAN THEOREM"

Although Plimpton 322 is a unique text of its kind, there are several other known texts testifying that the Pythagorean theorem was well known to the mathematicians of the Old Babylonian period. A particularly clear example is given by an Old Babylonian problem text from Tell Dhiba'i [Baqir 1962]. In this text, which is accompanied by a simple geometric drawing, the problem is to compute the sides of a rectangle, given the length of the diagonal and the area: $c = 1\ 15$, $A = 45$. The algorithm for the solution and its subsequent verification consists in the successive computation of the following quantities:

$$c^2 - 2A = (a - b)^2, \quad \tfrac{1}{2}(a - b), \quad ab + (\tfrac{1}{2}(a - b))^2 = (\tfrac{1}{2}(a + b))^2,$$

$$\tfrac{1}{2}(a + b), \quad \tfrac{1}{2}(a + b) \pm \tfrac{1}{2}(a - b) = a,\ b, \quad a^2 + b^2 = c^2, \quad c, \quad ab = A.$$

As it turns out, the given rectangle is composed of two congruent triangles with the sides (1, 45 , 1 15).

A couple of other examples (YBC 7289 [Neugebauer and Sachs 1945], and VAT 6598 [Neugebauer 1935, I, 282]) show that the Old Babylonian mathematicians were also able to handle applications of the Pythagorean theorem in the case of nonrational triangles, i.e., when the solutions of the problems involved approximate computations of square roots. As a rule, however, the Babylonians preferred to deal only with rational right triangles, in accordance with the general convention that only problems leading to rational (semiregular) solutions should be considered. Therefore, it is not surprising to find that the

Babylonian mathematicians were familiar with a certain type of cleverly devised problems (concerned with relations between the sides of a right triangle) leading to *linear* rather than quadratic equations, and, therefore, always solvable in terms of rational numbers for *arbitrarily* given rational data. One such example is problem No. 12 in the combined problem text BM 34568 from the Seleucid period [Neugebauer 1937, 14-22].

> *A cane is leaning against a wall. 3 cubits it has come down, 9 cubits it has gone out. How much is the cane, how much the wall? I do not know their numbers.*

The solution is given in the following series of steps (see Figure 6.1a)

(1) $u = 3, \quad a = 9{:}\ \tfrac{1}{2}(a^2 + u^2) = 45, \quad u' = 3' = 20,$

$u' \times \tfrac{1}{2}(a^2 + u^2) = 15 = c;$

(2) $c = 15, \quad a = 9{:}\ c^2 - a^2 = 2\ 24 = b^2, \quad b = 12.$

Thus we see that in this problem the length of the diagonal c is computed by means of the formula $c = \tfrac{1}{2}(a^2 + u^2)/u$, while the side b is found by using the Pythagorean theorem. An equivalent way of writing this expression for c is

$$c = \tfrac{1}{2}(a^2 + u^2)/u = a\bar{c}, \quad \bar{c} = \tfrac{1}{2}(t' + t), \quad t = u/a. \qquad (6.1)$$

This is precisely the formula for c which is obtained from the second of the two triangle parameter equations (2.2), if both sides of the equation are multiplied by the same factor a. In

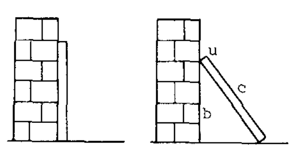


Figure 6.1a. The Babylonian "cane-against-a-wall problem", a triangle parameter problem with the parameters a and $u = c - b$.

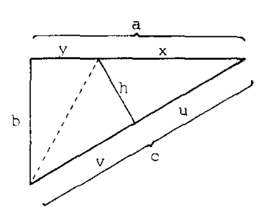


Figure 6.1b. A geometric problem which can be solved by use of the area formula and the "Babylonian similarity theorem.'

particular, (6.1) implies that $b = a\bar{b} = \frac{1}{2}(a^2 - u^2)/a$, and that the solution (b, c) of the problem will be a pair of rational (semiregular) numbers for any choice of rational *parameters* a and u (with $t = u/a$ regular). Hence the *cane-against-a-wall problem* of BM 34568:12 can be used to generate Pythagorean triples in precisely the same way as the triangle parameter equations (2.2). This problem is, therefore, a first example of what I shall refer to as various kinds of *triangle parameter problems*.

In addition to the cane-against-a-wall problem, with the parameters a and $u = c - b$, the above-mentioned Seleucid combined problem text BM 34568 contains also three other types of triangle parameter problems, with the parameter pairs $(b, c + a)$, $(a, c + b)$, and $(ab, c + a + b)$, respectively. (See the systematic description in [Neugebauer 1937, 20].) However, the cane-against-a-wall problem is the only one of the four triangle parameter problems in BM 34568 which is given a "dressed up," easily visualized form; the remaining three problems are formulated as abstract algebraic problems. Consequently, the cane-against-a-wall problem of BM 34568:12 could well be the *prototype* for the other types of triangle parameter problems appearing in BM 34568. This suspicion is confirmed by the fact that the cane-against-a-wall problem appears in a late Old Babylonian (or Kassite) text. This text is the combined problem text BM 85196 [Neugebauer 1935, II, 43-59], which out of 18 assorted problems contains only 1 (No. 9) having anything to do with right triangles. A free translation of the text of this particular problem reads as follows:

> *A wooden beam is leaning against a wall. The length of the beam is 30 length-units. If the top of the beam slides down 6 units, how much then does the lower end slide out along the ground? Conversely, if it slides out 18 units along the ground, how much does it slide down above?*

The answers to the two closely related questions in this problem are obtained by applying the Pythagorean theorem:

(1) $c = 30,\ \ u = 6$: $c^2 = 15,\ \ (c - u)^2 = b^2 = 9\ 36,$

$c^2 - b^2 = 5\ 24 = a^2,\ \ a = 18.$

(2) $c = 30,\ \ a = 18$: $a^2 = 5\ 24,\ \ c^2 - a^2 = 9\ 36 = b^2,$

$b = 24,\ \ c - b = 6 = u.$

Due to their solvability in terms of rational numbers, triangle parameter problems of a number of types seem to have enjoyed a great popularity all through the history of ancient mathematics. The only exception is their absence from Greek mathematics, although at least the generating formula (2.2) was known to Euclid and possibly also to Pythagoras. It is likely that the reason such problems were not considered by Greek mathematicians is that they, through their theory of proportions and in other ways, had developed techniques which were independent of the old ways of formulating rationally solvable mathematical problems. In ancient Hindu mathematics, on the other hand, several ingeniously dressed-up variants of the basic types of triangle parameter problems appear in, for instance, Brahmagupta's *Brāhma-sphuṭa-siddhānta* (628) and Bhāskara's *Līlāvatī* (1150) (see [Datta and Singh 1962]), and there are good reasons to believe that a method allowing the systematic generation of Pythagorean triangles was already known to the authors of the Śulba-Sūtras, several centuries B.C. Similarly, the oldest known Chinese book of mathematical problems, the *Chiu Chang Suan Shu* from the Han dynasty (202-9 B.C.), contains in its ninth and last chapter many triangle parameter problems of various types, all as ingeniously dressed-up as their counterparts in the *Līlāvatī* [Vogel 1968]. The "original" cane-against-a-wall problem also appears in the demotic Cairo Papyrus from the third century B.C. [Parker 1972, 34-40], and, for example, in the writings of Leonardo of Pisa (1170-1240 A.D.). Further references can be found in [Tropfke 1980, 616-625; Pottage 1973].

As a result of the above discussion of triangle parameter problems in general and of the Babylonian cane-against-a-wall problem in particular, it is possible to assert that Plimpton 322, with its systematic list of sides of Pythagorean triangles, is, after all, *not an outstanding and isolated achievement* of Old Babylonian mathematics, but rather *a natural complement to what was probably a well-developed geometric-algebraic theory*. In this connection it is worth mentioning that the Old Babylonian mathematicians were also familiar with a certain *trapezoid partition problem,* rationally solvable like the various types of triangle parameter problems. (See [Bruins and Rutten 1961, 114-117; Vaiman 1955, 72; Friberg 1980a, 42].) In fact, the consideration of the trapezoid partition problem leads in a straightforward way to the setting up of a generating formula similar to (2.2) for the systematic construction of rational solutions of the indeterminate equation $m^2 + n^2 = 2q^2$. (As is well known this equation is closely related to the Pythagorean equation, into which it can be transformed by means of the substitution $(a, b, c) = (\frac{1}{2}(n - m), \frac{1}{2}(n + m), q)$.) Thus the relatively well-documented appearance of the trapezoid partition problem in texts from the Old Babylonian period (the problem is not known from any later texts!) lends strong support to the conjecture that the Old Babylonian mathematicians were in possession of a well-developed theory for analyzing certain types of rationally solvable problems involving indeterminate quadratic equations and for obtaining their solution formulas of type (2.2). It must be stressed, however, that there are reasons to believe that such a theory, if it ever existed, would tend to lose its attraction and begin to degenerate after it had been used to build up a sufficiently large stock of data (the table on Plimpton 322, for instance). A typical example of this is provided by the cane-against-a-wall problem of Figure 6.1a. If a and u are chosen as the parameters in this problem, then it becomes a *true* triangle parameter problem, in the sense that it is rationally solvable for *all* choices of rational values of the parameters. If, on the other hand, c and $u = c - b$, or b and $u = c - b$ are chosen as the parameters, then the problem becomes rationally solvable only for *certain* rational values of the parameters; it is therefore an *inverted* triangle parameter problem. This distinction between true and inverted triangle parameter problems, however, is not noticeable to a person who just *copies*, without trying to change the given data, a few examples from an old text. This may be why the Babylonian text BM 85196:9 (quoted above), as well as the Cairo Papyrus, contains only variants of the cane-against-a-wall problem which can be classified as inverted triangle parameter problems. In both cases it may safely be assumed that the problems were randomly copied from more extensive and systematic older texts concerned exclusively with triangle parameter problems and related types of problems (such as the Seleucid text BM 34568).

I have given an intentionally brief survey of what is known about the early history of the Pythagorean theorem and the related theory of triangle parameter problems and generating formulas for Pythagorean triangles. One question still remains: how did the Old Babylonian mathematicians originally *discover* the Pythagorean theorem and their methods of constructing Pythagorean triangles, etc.? For obvious reasons, this is a question to which no definitive answer will ever be found. Nevertheless, it may be worth the effort to try to identify at least some way in which the Pythagorean theorem can have been discovered.

It is necessary to start by making a preliminary screening in order to eliminate the most unlikely potential answers. Indeed, there is no shortage of candidates; in [Loomis 1968] are listed 109 "algebraic" proofs, 255 "geometric" proofs, etc., of the theorem. Most of these proofs, however, fail to satisfy a first basic requirement, precisely because of their nature as constructed proofs of a known or conjectured theorem. Indeed, the Pythagorean theorem is not the kind of theorem which, while obviously true, has a proof that can be difficult to find. Also, the notion of a formal proof seems to have been unknown to the Babylonians. Thus, it appears that only one alternative remains: the Pythagorean theorem must have been *found by accident* in the course of some independent geometric investigation. Hence, we must not seek a proof of the Pythagorean theorem, but a geometric problem whose solution has the theorem as a *corollary*. Furthermore, there is the condition that any proposed original derivation of the theorem can be accepted only if it *makes use exclusively of methods and concepts that we know were available to mathematicians of the Old Babylonian period.*

Consider for instance the possibly oldest known proof of the Pythagorean theorem, found in Euclid's *Elements* I:47 [Heath 1956, I, 349]. Clearly this proof does not meet any of the requirements given above, because (1) it is without doubt a proof constructed for the purpose, and (2) it makes use of several concepts foreign to Babylonian mathematics, such as angles and parallel lines. On the other hand, it is possible to look at Euclid's proof as a clever elaboration of a certain more elementary proof, which from Euclid's point of view had the disadvantage that it makes use of the concept of similar triangles, so that it would not fit into the general plan for the first book of the *Elements*, where the theory of proportions has not yet been developed. (See Thomas [1951, 181].) The proof in question (see the proof of the generalized Pythagorean theorem in the Elements, VI:31) is based on the observation that the altitude drawn against the hypotenuse divides a given right triangle into two subtriangles similar to the given triangle. Hence, if we use the notations of Figure 6.1b in the case $y = 0$ and $x = a$, we find that

$$u:a = a:c, \quad v:b = b:c, \quad u + v = c \quad \text{implies}$$

$$u = a^2/c, \quad v = b^2/c, \quad c = (a^2 + b^2)/c.$$

This simple proof looks so much like a prototype for Euclid's proof that it has sometimes been assumed that it is essentially identical with the lost original proof usually attributed to Pythagoras (except that Pythagoras would not have used our modern algebraic notations). Suitably modified, the proof can even be made to satisfy the requirements which must be satisfied by a candidate for the original Babylonian proof or derivation of the Pythagorean theorem. In fact, such a modified proof would have had to be based on the following (reconstructed) *Babylonian similarity theorem: The inclination* (šà-gal), i.e., *the ratio between the front and the flank, is the same for a given right triangle and for every subtriangle cut out of the given triangle by a perpendicular to either the front, the flank, or the diagonal of the given triangle.* In the notations of

Figure 6.1b (once more with $y = 0$ and $x = a$) the inclination of the given triangle would be equal to $\bar{b} = b/a$, and it would follow from the similarity theorem that

$$h = \bar{b}u, \quad v = \bar{b}h = \bar{b}^2 u, \quad c = u + v \quad \text{implies}$$

$$c = (1 + \bar{b}^2)u, \quad 2A = hc = \bar{b}(1 + \bar{b}^2)u^2,$$

where A is the area of the triangle. Since $2A$ is also equal to $ba = \bar{b}a^2$, it would further follow that $a^2 = (1 + \bar{b}^2)u^2$, which is the Pythagorean equation for the subtriangle with the sides, u, $h = \bar{b}u$, and a. Hence we see that the Pythagorean theorem could, indeed, have been discovered by accident, as a corollary to the solution of the problem of finding the length of the projection u of the flank on the diagonal, when the flank a and the front b of a right triangle are given.

The documentation necessary to prove that the Old Babylonian mathematicians were familiar with a similarity theorem of the type indicated above is provided by the important text IM 55357 from Tell Harmal [Baqir 1950]. In this text (see Figure 6.2) the standard (1 , 45 , 1 15) triangle is divided into a series of subtriangles by means of lines alternately perpendicular to the diagonal and the flank of the given triangle. The areas of the subtriangles are given, and the problem consists in computing the sides of the subtriangles. This is clearly done by use of some similarity theorem. The computation proceeds in a series of steps as follows:

$$a'b \times 2B_1 = 12\ 09 = {v_1}^2, \quad v_1 = 27,$$

$$(\tfrac{1}{2}v_1)' \times B_1 = 36 = h_1, \quad c - v_1 = 48 = u_1;$$

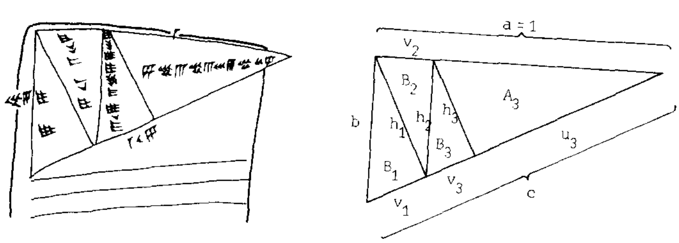


Figure 6.2. The problem of the text IM 55357: To find the values of v_1, v_2, v_3, etc., given the values (a, b, c) = (1 , 45 , 1 15) and B_1 = 8 06, B_2 = 5 11 02 24, B_3 = 3 19 03 56 09 36, A_3 = 5 53 53 39 50 24.

$$u_1'h_1 \times 2B_2 = 7\ 46\ 33\ 36 = v_2^2, \quad v_2 = 21\ 36, \quad (\tfrac{1}{2}v_2)' \times \cdots .$$

In other words, the algorithm used here is based on the two relations

$$v_i^2 = (h_{i-1}/u_{i-1}) \times 2B_i, \quad h_i = B_i/\tfrac{1}{2}v_i,$$

$$i = 1, 2, \ldots \ (u_0 = a,\ h_0 = b).$$

Since $2B_i = h_i v_i$, we can remove the common factor v_i from both sides of these two equations, obtaining an equivalent set of recursion formulas:

$$v_i = (h_{i-1}/u_{i-1}) \times h_i, \quad h_i = B_i/\tfrac{1}{2}v_i, \quad i = 1, 2, \ldots .$$

Hence we can explain how the author of IM 55357 derived his algorithm if we assume that he knew the area formula for right triangles *and* the similarity theorem, described above, for right triangles. (The numerical values of the subareas which were given in the formulation of the problem could have been computed by use of the same similarity theorem in the following way: knowing that $c = 1\ 15$, it follows that $h_1 = 2A/c = b/c$, hence that $v_1 = bh_1 = b^2/c$ (!), and $u_1 = c - v_1$. Repeating the same process one can compute h_2, v_2, u_2, etc., and, finally, the areas B_1, B_2, B_3, A_3.)

As has been observed by Bruins [1962, 312], the fact that the well-known (1 , 45 , 1 15) triangle figures in the formulation of the problem on IM 55357 is not so coincidental as it may appear. In fact, relatively few Pythagorean triangles share with the (1 , 45 , 1 15) triangle the property of having a diagonal which can be represented by a *regular* sexagesimal number; and if the diagonal c had not been regular, then the derived numbers $h_1 = b/c$, $v_1 = b^2/c$, $u_1 = c - v_1$, etc., would not have been semiregular. For precisely this reason it can be assumed that the (1 , 45 , 1 15) triangle and its multiples--the (3, 4, 5) triangle, for example--were *favorite objects for geometric experimentation* in the early phases of the development of Old Babylonian mathematics. It is perhaps against this background that one has to look for an explanation of the appearance of the strangely formulated first problem of the Seleucid combined problem text BM 34568 [Neugebauer 1937, 14-22]:

Flank 4, front 3, what is the diagonal? I do not know its number.
Add one half of your flank to your front, and that is it:
The flank 4 times 30 is 2; 2 plus 3 is 5; 5 is the diagonal.
Add one third of your front to your flank, this is the diagonal:
The front 3 times 20 is 1; 1 plus 4 is 5; 5 is the diagonal.

The meaning of this "problem" is more or less clear [Gandz 1938, 456]. The author wished to remind his readers that in any triangle which is a multiple of the basic (1 , 45, 1 15) triangle, the diagonal is related to the front and the flank of the triangle through one of the two equations,

$$c = b + a/2, \quad \text{and} \quad c = a + b/3. \qquad (6.2)$$

In other words, for multiples of that basic triangle one can set up, say, cane-against-a-wall problems in which one of the two parameters is either $t = (c - b)/a = 1/2$ or $(c - a)/b = 1/3$. (The same combined problem text contains also many other problems concerned with Pythagorean triangles, in particular, the cane-against-a-wall problem BM 34568:12.) It is tempting to hypothesize that the problem BM 34568:1 contains a hint about how the first "triangle parameter problem" may have been discovered by some Old Babylonian mathematician. For, suppose that it was his intention to pose a geometric problem, involving the (1 , 45 , 1 15) triangle, in which the object was to compute the lengths of, say, v, h, x, and y (see Figure 6.1b) when the value of the parameter u was given. He would have easily found that $v = c - u$, $h = bu$ (from the similarity theorem for right triangles, since in the actual case $a = 1$, hence $\bar{b} = b/a = b$), $y = 1 - uc$ (from the area formula, which can be used to show that $2A = b = by + buc$), and, finally, $x = 1 - y = uc$. In particular, he would have found, in the simple special case when $u = \frac{1}{2} = 30$, that

$$v = c - \tfrac{1}{2} = 45 = b, \quad h = \tfrac{1}{2}b = 22\ 30,$$

$$y = 1 - \tfrac{1}{2}c = 22\ 30 = h, \quad x = \tfrac{1}{2}c = 37\ 30.$$

In other words, he would have been led "by accident" to consider relations such as $c = b + a/2$, etc., for the (1 , 45 , 1 15) triangle, and it would not have been a big step for him to discover that there are other right triangles for which similar relations hold. Although it seems to me to be very likely that it was through such *geometric and numerical experimentation* that formulas like (2.2) for the generation of Pythagorean triangles and perhaps also the Pythagorean theorem itself were first discovered, I will refrain from going into a more detailed discussion of the matter here.

APPENDIX. ALGORITHMS FOR THE COMPUTATION OF PRODUCTS OF SEXAGESIMAL NUMBERS

Multiplication of sexagesimal numbers having arbitrarily many places can be achieved with essentially the same multiplication algorithm as that in the case of decimal numbers (Fig. A.1a),

```
55 33 20 × 13 53 20 =
(12 51 36 17 46 40)
           6 40
       17 40
          46
    4 20
    4 37 46 40
       11 00
          46
     29 09
        57
   7 09
   7 42 57 46
     18 20
      1 17
  48 35
   1 36
11 55
12 51 36 17
```

					400	$a_1\times b_1$
				6	40	$q_1B+c''_0$
				1060		$a_1\times b_2$
				1066		$a_1\times b_2+q_1$
			17	46		$q_2B+c''_1$
			260			$a_1\times b_3$
			277			$a_1\times b_3+q_2$
		4	37	46	40	$c''_3c''_2c''_1c''_0 = a_1\times b_3b_2b_1$
				660		$a_2\times b_1$
				706		$a_2\times b_1+c'_1$
			11	46		$q_1B+c''_0$
			1749			$a_2\times b_2$
			1797			$a_2\times b_2+c'_2+q_1$
		29	57			$q_2B+c''_1$
		429				$a_2\times b_3$
		462				$a_2\times b_3+c'_3+q_2$
	7	42	57	46		$c''_3c''_2c''_1c''_0c'_0 = a_2a_1\times b_3b_2b_1$
			1100			$a_3\times b_1$
			1157			$a_3\times b_1+c'_1$
		19	17			$q_1B+c''_0$
		2915				$a_3\times b_2$
		2976				$a_3\times b_2+c'_2+q_1$
	49	36				$q_2B+c''_1$
	715					$a_3\times b_3$
	771					$a_3\times b_3+c'_3+q_2$
12	51	36	17			$c''_3c''_2c''_1c''_0c'_0c_0 = a_3a_2a_1\times b_3b_2b_1$

Figure A.1a. Algorithm for the multiplication of the sexagesimal numbers 55 33 20 = 2×10^5 and 13 53 20 = 5×10^4. (12 51 36 17 46 40 = 10^{10}.)

Figure A.1b. The same algorithm, but with the intermediate calculations carried out in the decimal system. The notations in the algebraic description of the algorithm are identical with the notations in the computer program on the next page.

provided one knows how to add and multiply one-place sexagesimal numbers. The algorithm can easily be transformed into a computer program, making it possible to execute the multiplication of numbers in a given base B (= 60, for instance) on a programmable pocket calculator. Such a program is presented in this appendix, adapted for use on a Hewlett-Packard 25. Simpler programs can be devised for calculators having more extensive memory capacities than that available on the small HP-25.

A PROGRAM FOR THE COMPUTATION OF THE PRODUCT $a \cdot b = c$ IN BASE B (01 < B < 100). (HP-25)

	---			STO ÷ 2	$q := q_j + c''_{j-1} B^{-1}$
→	RCL 0	$X := d$		RCL 2	$X := q_j + c''_{j-1} B^{-1}, Y := B, Z := q_j B + c''_{j-1}$
	STO 7	$d^{2-j} := d$ (i.e. $j := 1$)		INT	$X := q_j, Y := B, \ldots$
	RCL 3	$X := b' := b, Y := d$		STO 2	$q := q_j$
→	x⇄y	$X := d, Y := b'$		×	$X := q_j B, Y := q_j B + c''_{j-1}$
	STO × 1	$c' := c'd (= c'_n \ldots c'_j)$		-	$X := c''_{j-1}$
	×	$X := b'd (= b_n \ldots b_{j+1} . b_j)$		STO + 5	$c'' := c''_{j-1} + c'' = c''_{j-1} . c''_{j-2} \ldots c''_0$
	ENTER ↑	$X, Y := b'd$		RCL 0	$X := d, \ldots$
	INT	$X := [b'd], Y := b'd$		STO × 5	$c'' := .c''_{j-1} \ldots c''_0$
	STO 6	$b' := [b'd]$		STO ÷ 7	$d^{2-j} := d^{2-j-1}$ (i.e. $j := j+1$)
	-	$X := \langle b'd \rangle = b_j d$		RCL 6	$X := b', \ldots$
	RCL 4	$X := a_i, Y := b_j d$		x ≠ 0	$b' \neq 0$? (i.e. $j \leq n$?)
	×	$X := a_i b_j d$	←	GTO 04	iterate if $j \leq n$
	RCL 1	$X := c'd, Y := a_i b_j d$		RCL 2	$X := q_{n+1} := c''_{n+1}$
	ENTER ↑	$X, Y := c'd, Z := a_i b_j d$		STO + 5	$c'' := c''_{n+1} . c''_n \ldots c''_0$
	INT	$X := [c'd], Y := c'd, Z := a_i b_j d$		RCL 5	$X := c'', Y := c''_{n+1}, \ldots$
	STO 1	$c' := [c'd] (= c^i_n \ldots c^i_{j+1})$		RCL 7	$X := d^{1-n}, Y := c'', \ldots$
	-	$X := \langle c'd \rangle = c'_j d, Y := a_i b_j d$		×	$X := c''_{n+1} \ldots c''_2 . c''_1, \ldots$
	+	$X := (a_i b_j + c'_j) d$		ENTER ↑	$X, Y := c''_{n+1} \ldots c''_2 . c''_1, \ldots$
	RCL 0	$X := d, Y := (a_i b_j + c'_j) d$		INT	$X := c''_{n+1} \ldots c''_2, Y := c''_{n+1} \ldots c''_2 . c''_1$
	÷	$X := a_i b_j + c'_j$		STO 1	$c' := c''_{n+1} \ldots c''_2$
	STO + 2	$q := a_i b_j + c'_j + q_{j-1} := q_j B + c''_{j-1}$		CLX	$X := 0, Y := c''_{n+1} \ldots c''_2 . c''_1$
	RCL 2	$X := q_j B + c''_{j-1}$		STO 2	$q := 0$
	B_1	-		STO 5	$c'' := 0$
	B_0	$X := B_1 B_0 = B, Y := q_j B + c''_{j-1}$	←	R ↓	$X := c''_{n+1} \ldots c''_2 . c''_1$

Operating instructions.

1. Read program to memory, in particular the chosen value for $B = B_1 B_0$.
2. Switch over to RUN. Store the value $d = .01$ in R0 if $B > 10$, $d = .1$ if $B \leq 10$.
3. → Store 0 in R1, i.e. reset c' (if the program is being used more than once).
4. Store $b = b_n \ldots b_1$ in R3, for $n \leq 4$ if $B > 10$, for $n \leq 9$ if $B \leq 10$. Store a_1 in R4.
5. → Press R/S and wait until the number $c''_{n+1} \ldots c''_2 . c''_1$ appears. Copy externally $c''_1 = c_i$.
6. Store new value for a_i in R4 and repeat from 5. if necessary. Copy final $c''_{n+1} \ldots$.

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