ENTROPY OF TRANSITIVE TREE MAPS

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We obtain lower bounds for the topological entropy of transitive self-maps of trees, depending on the number of endpoints and on the number of edges of the tree. Copyright © 1996 Elsevier Science Ltd

1. INTRODUCTION

One of the central questions in the theory of dynamical systems is how to recognize chaos and how to see how large it is. One of the best methods of measuring chaoticity is by means of topological entropy of the system (see e.g. [1] for a definition). Then we can restate our question as: How can we get estimates for topological entropy from other properties of a system? The simplest kind of a dynamical system is formed by taking iterates of one map of a compact space into itself. It turns out that if this space is a tree then already such a weak property as transitivity (existence of a dense orbit) implies positive entropy (see [2]), and therefore chaoticity. Immediately, a problem arises: Are there any natural lower bounds for the entropy in this case?

The problem of obtaining lower bounds for the topological entropy of a transitive map has been considered by several authors for some special cases of one-dimensional spaces. Namely, Blokh in [3] proved that if the tree under consideration is a closed interval of the real line then \( h(f) \geq (\log 2)/2 \). In [4] the lower bounds for the topological entropy of transitive circle and star maps were obtained. In [5] the same problem has been considered for a class of transitive tree maps that arises naturally in the study of the homeomorphisms of the disk. For such class of maps the bound \( (\log 2)/n \) for the topological entropy has been obtained, where \( n \) is the number of ends of the tree. In [6] a similar problem for the tree maps obtained from pseudo-Anosov diffeomorphisms of a punctured disk has been considered. The bound of the topological entropy obtained in this case is \( (\log(1 + \sqrt{2})/k) \), where \( k \) is the number of punctures.

The aim of this paper is to obtain lower bounds for the topological entropy of transitive tree maps. To be more precise we have to introduce some notation.

By an interval we mean the closed interval \([0, 1]\) and any space homeomorphic to it. A tree is a connected space that is a union of finite number of intervals, but does not contain a subset homeomorphic to a circle.

A map \( f: I \rightarrow I \) is called transitive if for every non-empty open subsets \( U, V \subset I \) there is \( n \geq 1 \) such that \( f^n(U) \cap V \neq \emptyset \). This is equivalent to the existence of a point with a dense orbit (see for instance [1]).

The topological entropy of a continuous map \( f \) from a tree to itself will be denoted by \( h(f) \) (see e.g. [1] for a definition).

Let \( T \) be a tree. We define a number \( L(T) \) as the infimum of topological entropies of transitive maps from \( T \) to \( T \). Our aim is to give some reasonable estimates for \( L(T) \). Of
course, the most natural goal would be to find a general formula allowing us to compute \( L(T) \) for every tree \( T \). However, it seems that such a formula would be quite complicated. We believe that the following conjecture is true. To state it we have to introduce the appropriate notions.

If \( T \) is a tree and \( x \in T \) then the number of components (we mean by this connected components) of \( T \setminus \{x\} \) is called the \textit{valence} of \( x \) in \( T \) and will be denoted \( \text{Val}_T(x) \). A point of \( T \) of valence 1 is called an \textit{end} of \( T \), and a point of valence different from 2 is called a \textit{vertex} of \( T \).

Let \( T \) be a tree and let \( f: T \to T \) be a continuous map. Assume that \( P \) is an \( f \)-invariant finite set containing all vertices of \( T \). We say that \( f \) is \textit{\( P \)-monotone} if for each connected component \( K \) of \( T \setminus P \) the map \( f|_K \) is a homeomorphism onto its image.

\textbf{Conjecture A.} For every tree \( T \) there exists a transitive map \( f: T \to T \) and an \( f \)-invariant finite set \( P \) containing all vertices of \( T \) such that \( f \) is \( P \)-monotone and \( h(f) = L(T) \).

If the above conjecture is true then \( L(T) \) is the logarithm of the greatest real zero of some polynomial. However, this polynomial may depend on \( T \) in a very complicated way. Nonetheless, we think that the degree of this polynomial grows not faster than linearly with the number of ends of \( T \).

A natural estimate for \( L(T) \), generalizing the estimate for interval maps, is the following one. We denote the number of ends of \( T \) by \( \text{End}(T) \).

\textbf{Theorem B.} For every tree \( T \) we have \( L(T) \geq \lceil 1/\text{End}(T) \rceil \log 2 \).

To prove Theorem B, we have to start with some special cases. Namely, the following proposition holds. It is a generalization of the results from [5] and [3] we quoted above.

\textbf{Proposition C.} Let \( f: T \to T \) be a transitive tree map. Assume that there exists an \( f \)-invariant finite set \( P \) containing all vertices of \( T \) such that \( f \) is \( P \)-monotone. Then \( h(f) \geq \lceil 1/\text{End}(T) \rceil \log 2 \).

Clearly, the assumptions of Proposition C are too strong. In fact, the following result follows easily from Proposition C.

\textbf{Corollary D.} Let \( g: T \to T \) be a tree map. Assume that there exists a \( g \)-invariant finite set \( P \) containing all the vertices of \( T \) and a transitive \( P \)-monotone map \( f: T \to T \) such that \( f|_P = g|_P \). Then \( h(g) \geq \lceil 1/\text{End}(T) \rceil \log 2 \).

Using different methods, we are able to strengthen Theorem B for some classes of trees. A point of a tree will be called an \textit{interior point} if it is not an end. A tree \( T \) will be called a \textit{star of type} \( n \) (an \( n \)-\textit{od}) if there is an interior point \( b \) of \( T \) such that \( T \setminus \{b\} \) has \( n \) components and the closure of each of them is an interval. Notice that according to this definition an interval is a star of type 2.

A tree \( T \) will be called a \textit{superstar of type} \( (n_1, \ldots, n_k) \) if there is an interior point \( b \) of \( T \) such that \( T \setminus \{b\} \) has \( k \) components and the closure of the \( i \)-th component is a star of type
If $k = 2$ then we will call $T$ a bistar. Clearly, a star of type $k$ is a superstar of type $(1, \ldots, 1)$ and a bistar of type $(1, k - 1)$.

**Theorem E.** Let $T$ be a bistar of type $(k, n)$ with $1 < k \leq n$. Let $f: T \to T$ be a transitive map. Then $h(f) \geq \frac{1}{N} \log 2$, where $N = \max \{n + 1, 2k\}$.

Although the estimate $[1/\text{End}(T)] \log 2$ for $L(T)$ is elegant, it seems that the equality is rare. In Proposition 6.4 we prove that for stars the equality holds, but already for asymmetric bistars that are not stars we have $L(T) > \frac{1}{N} \log 2$.

The paper is organized as follows. In Section 2 we fix the basic notation and obtain some preliminary results. In Section 3 we prove Proposition C (and deduce Corollary D); then in Section 4 we prove Theorem B. In Section 5 we introduce marked trees and maps, that generalize transitive tree maps, along with some tools using those notions. We apply those tools in Section 6 to prove Theorem E.

2. PRELIMINARIES

We start by defining several notions. The word “interior” will be used in the meaning “not an end”; this is different from the topological interior. However, “closure” will be used in the topological meaning. The closure of a set $A$ will be denoted by $\bar{A}$. A union of finite number of disjoint trees is called a forest. A union of a forest and a finite set will be called a generalized forest. The terms subtree, subforest, etc. will be used in the meaning “a subset which is a tree, a forest, etc.” By a map we will mean a continuous map.

A collection $(J_1, \ldots, J_k)$ of pairwise disjoint (except perhaps ends) subintervals of $T$ is called a $k$-horseshoe if $J_i \subset f(J_j)$ for every $i, j \in \{1, \ldots, k\}$. It is well known (see e.g. [7]) that if $f$ has a $k$-horseshoe then $h(f) \geq \log k$. Since $h(f') = s \cdot h(f)$, if $f'$ has a $k$-horseshoe then $h(f) \geq \frac{1}{s} \log k$. A map having a 2-horseshoe is called turbulent. Generalizing this notion, we shall call a map having a 3-horseshoe, 3-turbulent.

If $S$ is a subtree of $T$ then we denote by $r_s$ the natural retraction from $T$ to $S$, that is, the map such that $r_s(x)$ is the point of $S$ closest to $x$ (so, in particular, $r_s(x) = x$, if $x \in S$). Here “the closest” means the closest along the tree. If $f: T \to T$ is a map then we shall denote by $f_s$ the composition $r_s \circ f$. We shall need an elementary lemma.

**Lemma 2.1.** Let $T$ be a tree, $S \subset T$ a subtree, and $g: T \to T$ a map. Assume that $(J_1, \ldots, J_k)$ is an $l$-horseshoe for $(g_s)^k$. Then it is also an $l$-horseshoe for $g^k$.

**Proof.** Let $A \subset S$. We claim that the difference $(g_s)^k(A) \setminus g^k(A)$ is finite. We will prove the claim by induction. For $k = 0$ this is obvious. Assume that it is true for some $k$ and prove it for $k + 1$ replacing $k$. We have $(g_s)^k(A) = g^k(A) \cup B$ for some finite set $B$. Then

\[(g_s)^{k+1}(A) = r_s(g^{k+1}(A)) \cup g_s(B) \subset g^{k+1}(A) \cup C \cup g_s(B),\]

where $C$ is the set of ends of $S$. Since $C \cup g_s(B)$ is finite, $(g_s)^{k+1}(A) \setminus g^{k+1}(A)$ is also finite. This proves the claim.

If $i, j \in \{1, \ldots, l\}$ then we have $J_i = (g_s)^k(J_j)$ and hence $J_i \setminus g^k(J_j)$ is finite. Since $J_i$ is an interval and $g^k(J_j)$ is compact, we get $J_i \subset g^k(J_j)$.

For $x, y \in T$ we shall use the notation $[x, y]$ for the smallest connected subset of $T$ containing $x$ and $y$. Then we will also write $[x, y)$ for $[x, y] \setminus \{y\}$, $(x, y]$ for $[x, y] \setminus \{x\}$, and $(x, y)$ for $[x, y] \setminus \{x, y\}$. 


3. P-MONOTONE MAPS

The goal of this section is to prove Proposition C.

Assume that $z$ is an interior point of $T$. A sequence $(x_1, \ldots, x_n)$ will be called $z$-independent if there are no $i, j \in \{1, \ldots, n\}$ such that $i \neq j$ and $x_i \in [z, x_j]$.

**Lemma 3.1.** If $z$ is an interior point of $T$ then any $z$-independent sequence has length less than or is equal to $\text{End}(T)$.

**Proof.** Let $(x_1, \ldots, x_n)$ be a sequence of points of $T$ and let $\{y_1, \ldots, y_s\}$ be the set of all ends of $T$ ($s = \text{End}(T)$). Let $z$ be an interior point of $T$ and assume that $n > s$. Since $\bigcup_{i=1}^{s} [z, y_i] = T$, there are $i, j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, s\}$ such that $i \neq j$ and $x_i, x_j \in [z, y_k]$. Then either $x_i \in [z, x_j]$ or $x_j \in [z, x_i]$. Therefore, $(x_1, \ldots, x_n)$ is not $z$-independent.

The following proposition is a generalization of a result from [5].

**Proposition 3.2.** Let $f: T \to T$ be a transitive map and let $s = \text{End}(T)$. Then all points of $T$, except perhaps finitely many of them, have at least two inverse images under $f^s$.

**Proof.** We claim that $f$ has a fixed point $z$ which is not an end of $T$. Indeed, we start at some interior point $y$ of $T$ and perform the standard "a point chases its image" construction (unless $y$ is already a fixed point). That is, initially we have $x = y$ and then $x$ moves towards $f(x)$ (which then moves, too). After sliding along some interval, $x$ has to catch its image, and this is a fixed point $z$ of $f$. Notice that with this construction, for every $x \in [y, z]$ we have $x \in [y, f(x)]$. Therefore, $z$ cannot be an end of $T$, since then any sufficiently small neighborhood of $z$ would be mapped into itself, and this is impossible by transitivity of $f$.

Denote by $A$ the set consisting of $z$ and all the ends of $T$. Set $B = \bigcup_{i=0}^{s} f^i(A)$. The set $B$ is finite. We will prove that every point of $T \setminus B$ has at least two inverse images under $f^s$. Since $f$ is transitive, it is onto, so every point has at least one inverse image under $f$ (and hence at least one inverse image under $f^s$). Suppose that there is a point of $T \setminus B$ with only one inverse image under $f^s$. Call this inverse image $x$. Then for $i = 1, 2, \ldots, s$, $f^i(x)$ has only $f^{i-1}(x)$ as an inverse image under $f$. Moreover, none of the points $x, f(x), \ldots, f^s(x)$ lies in $A$.

By Lemma 3.1, the sequence $(x, f(x), \ldots, f^s(x))$ is not $z$-independent. Therefore there exist $i, j \in \{0, 1, \ldots, s\}$ such that $i < j$ and either $f^i(x) \in [z, f^j(x)]$ or $f^j(x) \in [z, f^i(x)]$. We may assume additionally that among the pairs with this property our one is such that $j - i$ is minimal possible. Then the sequences $(f^i(x), \ldots, f^{j-1}(x))$ and $(f^{i+1}(x), \ldots, f^j(x))$ are $z$-independent.

Let $U$ be the component of $T \setminus \{f^s(x)\}$ containing $z$. By our assumptions, $U$ and $T \setminus U$ are proper subtrees of $T$. Since $f$ is transitive, none of them is $f$-invariant. Assume first that $f^s(x) \in [z, f^j(x)]$. Then $f^j(x) \in T \setminus U$. Since $T \setminus U$ is not $f$-invariant, there is $y \in T \setminus U$ with $f(y) \in U$. All the points $f^{i+1}(x), \ldots, f^j(x)$ belong to $T \setminus U$, so $y \not \in \{f^i(x), \ldots, f^{j-1}(x)\}$. Therefore there is $k \in \{i, \ldots, j - 1\}$ such that $f^k(x) \in (z, y)$. Let $V$ be the component of $T \setminus \{f^k(x)\}$ containing $z$. Since $f^k(x) \notin A$, both $V$ and $T \setminus V$ are subtrees of $T$ with $f^k(x)$ as an end. There is a point $u \in T$ such that $f(u)$ belongs to the component of $T \setminus \{f^{k+1}(x)\}$ that does not contain $z$ (see Fig. 1). If $u \in V$ then $f^{k+1}(x) \in (f(z), f(u))$, so there is a point $v \in (z, u)$ with $f(v) = f^{k+1}(x)$. If $u \in T \setminus V$ then $f^{k+1}(x) \in (f(y), f(u))$, so there is a point $v \in (y, u)$ with
f(v) = f^{k+1}(x). In both cases v ≠ f^k(x), so f^{k+1}(x) has at least two inverse images under f, a contradiction.

Assume now that f^j(x) ∈ [z, f^j(x)]. Since $\overline{U}$ is not invariant, there is y ∈ $\overline{U}$ with f(y) ∈ T \ U. Therefore there is k ∈ {i, ..., j - 1} such that f^k(x) ∈ (z, f(y)). Notice that if k = i then also f^j(x) ∈ (z, f(y)), so we may assume that k ∈ {i + 1, ..., j}. There is u ∈ (z, y) with f(u) = f^j(x). Since y ∈ $\overline{U}$ and (f^j(x), ..., f^{j-1}(x)) is z-independent, we have u ∉ {f^i(x), ..., f^{j-1}(x)}. In particular, u ≠ f^{k-1}(x). Hence, f^k(x) has at least two inverse images under f, a contradiction.

Let f: T → T be a transitive tree map and assume that there exists an f-invariant finite set P containing all vertices of T. Then we can associate to f an n × n matrix $M = (m_{ij})$, where n is the number of components of T \ P. We do it in the following way. Denote the closures of those components by I_1, I_2, ..., I_n. Set m_{ij} = 1 if I_i ⊂ f(I_j). Otherwise, set m_{ij} = 0. The matrix M will be called the P-transition matrix of f. From [1] and [8], it follows that the topological entropy of f is larger than or equal to the logarithm of the spectral radius of M. Moreover, if f is P-monotone then the equality holds.

Now we are ready to prove

**Proposition C.** Let f: T → T be a transitive tree map. Assume that there exists an f-invariant finite set P containing all vertices of T such that f is P-monotone. Then h(f) ≥ [1/End(T)] log 2.

**Proof.** Let M be the P-transition matrix of f. Since f is P-monotone, from Proposition 3.2 it follows that in each column of $M^s$ there are at least two 1's (where s = End(T)). Let v be the vector of the same size as the number of rows (columns) of M, with all components equal to 1. Then, v$M^s$ is a vector with all components 2 or larger, that is v$M^s$ ≥ 2v. By induction we get v$M^{sn}$ ≥ 2^n v for all n. Therefore, the sum of all entries of $M^{sn}$ is v$M^{sn}v^T$ ≥ 2^n v$v^T$ ≥ 2^n. Hence, the spectral radius of M is at least $\sqrt{2}$ (see for instance [9]), and therefore h(f) ≥ $\frac{1}{2}$ log 2.

The following result is an immediate consequence to Proposition C.

**Corollary D.** Let g: T → T be a tree map. Assume that there exists a g-invariant finite set P containing all the vertices of T and a transitive P-monotone map f: T → T such that f|_P = g|_P. Then h(g) ≥ [1/End(T)] log 2.
4. THE MAIN ESTIMATE IN A GENERAL CASE

The proof of Theorem B uses a technique developed in [10] to reduce the proof to Corollary D. A dendrite is a locally connected, uniquely arcwise connected, compact metric space. Although we are interested in results on trees here, dendrites which are not trees will sometimes appear in an intermediate stage of the main proof.

Let $D$ be a dendrite, let $f: D \to D$ be a continuous map, and let $P$ be a finite $f$-invariant subset of $D$. Let $S(D, P)$ be the partition of $D$ defined by $A \in S(D, P)$ if and only if one of the following occurs:

1. $A = \{x\}$, for some $x \in P$;
2. $A$ is a component of $D \setminus P$ whose closure intersected with $P$ has at least two elements;
3. for some $x \in P$, $A$ is the union of all components of $D \setminus P$ whose closure intersected with $P$ is $\{x\}$.

The last item on the list includes all components of $D \setminus P$ that miss the convex hull of $P$, and are grouped in the way indicated to make $S(D, P)$ finite. If $D$ were a tree, we could just use singletons from $P$ and components of $D \setminus P$, but for dendrites the above more complicated definition is needed to make $S(D, P)$ finite (think for instance of a star with infinitely many rays of lengths converging to 0, and the set $P$ consisting of the central point). For each $x \in D$, we then define the $P$-itinerary of $x$ to be the unique sequence $(S_0, S_1, \ldots)$ of elements of $S(D, P)$ such that $f^i(x) \in S_i$ for all $i > 0$. The map $f$ is called $P$-expansive if different points of $D$ have different itineraries.

The key point of the construction in [10] is that given any map on a dendrite and any finite invariant set $P$, there is a natural semiconjugacy $\pi$ to a $\pi(P)$-expansive dendrite map. The following description covers the part of these results which we need to use here.

If $f$ is a map on a dendrite $D$, and $P$ is a finite $f$-invariant set (which need not contain all vertices of $D$), the Markov graph for $P$ is defined as usual. Basic intervals are all intervals $[x, y]$ such that $[x, y] \cap P = \{x, y\}$, and we draw an arrow from a basic interval $[x, y]$ to another basic interval $[u, v]$ if and only if $[u, v] \subseteq [f(x), f(y)]$. The main difference between this and the simpler case where $P$ contains all vertices is that two basic intervals can have a nontrivial overlap. We say that a basic interval has order 0 if it has no out arrows (i.e., its endpoints map to the same point). We say that a loop in a Markov graph has order 1 if every basic interval in the loop has exactly one out arrow. Note that in a loop of order 1, all endpoints of the intervals from the loop are periodic points of $f$.

The following theorem is one of the main results from [10], slightly reworded to the form in which it will be used here. See [10] for a proof.

**Theorem 4.1.** Let $D$ be a dendrite, let $f: D \to D$ be a continuous map and let $P$ be a finite subset of $D$ such that $f(P) \subseteq P$. Then there is a dendrite $E$, a map $g: E \to E$, and a semiconjugacy $\pi: D \to E$ (i.e., $\pi \circ f = g \circ \pi$) such that

1. $g$ is $\pi(P)$-expansive.
2. If $x, y, z \in P$, and $y \in [x, z]$, then $\pi(y) \in [\pi(x), \pi(z)]$.

If, in addition, the Markov graph of $P$ has no basic intervals of order 0 and no loops of order 1, then $\pi|_P$ is one-to-one.
The following result covers a special case in the proof of Theorem B.

**Proposition 4.2.** Assume that $T$ is a tree, $f: T \to T$ is transitive, and $f^{-1}(x) = \{x\}$ for some $x \in T$. Then $h(f) \geq [1/End(T)] \log 3$.

**Proof.** Assume first that $x$ is an endpoint of $T$. We identify the edge containing $x$ with the unit interval $[0, 1]$, with $x = 0$, and use the usual ordering $<$ to describe the relative position of points on this edge. Points not on this edge (if any) will be unimportant.

Set $y_0 = \min \{f(y): y \in T \setminus [0, 1], f(y) \in [0, 1], y_1 = \min \{y \in [0, 1]: f(y) = 1\}$, and $y_2 = \min \{y_0, y_1\}$. We need from transitivity only that if $y \in [0, 1]$ then none of the sets $[0, y]$ and $T \setminus [0, y]$ is $f$-invariant (plus we use the assumption $f^{-1}(0) = \{0\}$), and we will mean that when we say "by transitivity". Let $A_i$ (respectively $A_i^r$) be the set of those points of $[0, y_2]$ that lead movement to the left (respectively right). That is, $y \in A_i$ if $y \in [0, y_2], f(y) \leq y$ and $f(z) \geq f(y)$ for all $z \in [y, 1]$; similarly $y \in A_i$ if $y \in [0, y_2], f(y) \geq y$ and $f(z) \leq f(y)$ for all $z \in [0, y]$. Clearly, $A_i$ and $A_i^r$ are closed. By transitivity, $A_i \cap A_i^r = \emptyset$. Also by transitivity and by the definitions of $A_i$ and $A_i^r$, one can easily show the following properties:

1. $A_i \cap [0, y_2] \neq \emptyset$ for each $y \leq y_2$.
2. $A_i \cap [0, y_2] \neq \emptyset$ for each $y \leq y_2$.
3. $[f(y), y] \cap A_i \neq \emptyset$ for each $y \in A_i$.
4. $(y, f(y)] \cap A_i^r \neq \emptyset$ for each $y \in A_i$ such that $f(y) \leq y_2$.

Since $A_i$ and $A_i^r$ are closed and by properties (1) and (2) above, there are points $w < t$ with $w \in A_i, f(w) \leq y_2, t \in A_i$ and no points of $A_i \cup A_i^r$ in $(w, t)$. By properties (3) and (4) above, there are points $v \in (f(t), t) \cap A_i$ and $u \in (f(w), f(u)] \cap A_i^r$. Hence, we get $v < w < t < u$ such that $f(v), f(t) < u$ and $f(u) \geq u$. This means that $([v, w], [w, t], [t, u])$ is a 3-horseshoe, and consequently $h(f) \geq \log 3$.

Let us consider now the case when $x$ is not an endpoint. Then, since $x$ has no preimages other than itself, the components of $T \setminus \{x\}$ must map to each other cyclically. Let $k$ be the number of components of $T \setminus \{x\}$. By the first part of the proof, applied to $f^k$ and a component of $T \setminus \{x\}$, we get $h(f) = \frac{1}{k} h(f^k) \geq \frac{1}{k} \log 3 \geq [1/End(T)] \log 3$.

**Theorem B.** For every tree $T$ we have $L(T) \geq [1/End(T)] \log 2$.

**Proof.** By Proposition 4.2, we may assume that $f$ has a fixed point $x_0$ which has a preimage $x_1$ other than $x_0$. Since $f$ is onto, we may inductively pick points $x_i$ so that $f(x_i) = x_{i-1}$, continuing until we reach a point $x_k$ which is not an endpoint and is not in the forward orbit of any endpoint. Since $x_k$ is not an endpoint, transitivity of $f$ implies that $x_i \in \bigcup_i f^i([x_0, y])$ for all $y \neq x_0$. Thus, we may continue the induction, defining points $x_{k+1}, \ldots, x_n$ (with $f(x_k) = x_{k-1}$ as before), such that $x_k$ is between $x_0$ and $x_1$, and no $x_j$ is between $x_0$ and $x_k$. Let $P = \{x_0, \ldots, x_n\}$. It is easy to see that the Markov graph of $P$ has no intervals of order 0 (since $[x_0, x_1]$ is not a basic interval) and no loops of order 1 (since $x_0$ is the only periodic point in $P$). Thus, let $E$ be a dendrite, with $\pi: T \to E$ and $g: E \to E$ as in Theorem 4.1. Then $h(g) \leq h(f)$, since $\pi$ is a semiconjugacy. Let $y_i = \pi(x_i)$ for $i = 0, 1, \ldots, n$, $Q = \{y_0, \ldots, y_n\}$, and let $T'$ be the smallest tree containing $Q$. Then $g(T') \subset T'$, and $T'$ is a closed set, so $h(g|_{T'}) \leq h(g)$. If we let $Q'$ consist of $Q$ plus all vertices of $T'$, then $g(Q') \subset Q'$ and $g$ is $Q'$-expansive (and hence $Q'$-monotone).

We claim that $g|_{T'}$ is transitive. We have $T' = \bigcup_{i=1}^n [y_0, y_i] = \bigcup_{i=1}^n g^i([y_0, y_1])$, so it is enough to show that for every open non-empty subset $U \subset T'$ there is $m$ such that $[y_0, y_1] \subset g^m(U)$. Let $U$ be an open non-empty subset of $T'$. Since $g$ is $Q$-expansive, there is
\[ z \in U \text{ and } k \text{ such that } g^k(z) \in Q. \text{ Again by } Q \text{-expansiveness of } g, \text{ there is } z' \in U \text{ such that } g^j(z') \in Q \setminus \{y_0\} \text{ for some } j \geq k + n. \text{ Hence, } g^i(z') = y_i \text{ for some } i > 0, \text{ so } g^{i+1}(z') = v_i \text{ and } g^{i+1}(z) = y_0. \text{ This proves the claim. Thus, by Corollary D, } h(f) \geq h(g^k) > \frac{1}{\text{End}(T')} \log 2. \text{ Let } T'_P \text{ be the smallest tree (in } T) \text{ containing } P. \text{ Then, by Theorem 4.1(ii), we see that } \text{End}(T') \leq \text{End}(T'_P) \leq \text{End}(T). \text{ Hence, } h(f) \geq \frac{1}{\text{End}(T')} \log 2. \]

5. MARKED TREES AND MAPS

A subtree (subforest, generalized subforest) of a tree \( T \) is proper if it contains an interval and is not equal to \( T \). If \( f: T \to T \) is transitive and \( S \) is a proper subtree of \( T \) then \( f \) is not transitive. Since in our constructions we will often replace \( f \) by \( f^k \), we need a notion weaker than transitivity that is preserved under this operation. For this we introduce marked trees and marked maps.

Assume that \( z \) is a fixed point of \( f: T \to T \) which is an end of \( T \). We say that \( z \) is repelling if there is a neighborhood \( U \) of \( z \) such that \( x \in (z, f(x)) \) for every \( x \in U \setminus \{z\} \). A set \( A \subset T \) is \( f \)-invariant if \( f(A) \subset A \).

A pair \((T, E)\) will be called a marked tree if \( T \) is a tree and \( E \) is a subset of the set of ends of \( T \). The elements of \( E \) will be called marked ends of \( T \) and the ends of \( T \) which do not belong to \( E \) will be called free ends of \( T \). A map \( f: T \to T \) will be called a marked map of \((T, E)\) if each element of \( E \) is a repelling fixed point of \( f \) and there is no \( f \)-invariant proper generalized subforest of \( T \) disjoint from \( E \).

Notice that if \( E = \emptyset \) then the notion of a marked map is close to the notion of a transitive map. Clearly, any transitive map is marked with \( E = \emptyset \). On the other hand, if \( f \) is transitive and we choose up to a countable number of points which are neither periodic nor pre-periodic and blow up their full backward trajectories, in a similar way as in the Denjoy example on a circle (see [11]), then we get from \( f \) a map which is marked (with \( E = \emptyset \)) but not transitive.

Whenever we have an interior fixed point of a marked map, we can perform at it a certain construction, that reduces the complexity of the tree under consideration. It is described in the next proposition, but first we need a new definition.

If \((T, E)\) is a marked tree and \( z \) is an interior point of \( T \) then we shall call a family \( \{(T_i, E'_i): i = 0, 1, \ldots, s-1\} \) of marked trees a z-family if the following conditions are satisfied:

(a) each \( T_i \) is contained in the closure \( S_i \) of some component of \( T \setminus \{z\} \),
(b) if \( i \neq j \) then \( S_i \neq S_j \),
(c) either \( T_i = S_i \setminus [z, z_i) \) for some point \( z_i \) in the interior of the edge of \( S_i \) containing \( z \), or \( T_i = S_i \) (and then we set \( z_i = z \)),
(d) \( E' = (T_i \cap E) \cup \{z_i\} \).

**Proposition 5.1.** Let \( f \) be a marked map of a marked tree \((T, E)\). Let \( z \) be a fixed point of \( f \). Assume that \( z \) is an interior point of \( T \). Then there exists a z-family \( \{(T_i, E'_i): i = 0, 1, \ldots, s-1\} \) of marked trees such that for each \( i \) the map \( (f^s)^{T_i} = r_{T_i}, f^s \) is a marked map of \((T_i, E'_i)\).

**Proof.** Take a small connected neighborhood \( U \) of \( z \) and another one \( V \), much smaller. Since \( f \) is marked, there is \( k \) such that \( f^k(V) \not\subset U \). Then we get a sequence of points \( x_i, i = 0, 1, \ldots, t \), such that \( f(x_i) = x_{i+1} \) for \( i < t \), \( x_0 \in V \), \( x_t \in U \) for \( i < t \), and \( x_t \not\in U \).
Since $f(z) = z$ and $U$ is small, we get a sequence of intervals $[z, x_i]$, $i = 0, 1, \ldots, t$, such that each interval is contained in one edge, $[z, x_{i+1}] \subset f([z, x_i])$ for $i < t$. If $[z, x_0] \subset V$, $[z, x_i] \subset U$ for $i < t$, and $[z, x_i] \neq U$. If for some $i < j$ we have $[z, x_i] \cap [z, x_j]$ then we remove $[z, x_i], \ldots, [z, x_{j-1}]$ from our sequence, and the new sequence has the same properties, but is shorter. We may assume that our sequence is the shortest one with those properties. Then $[z, x_i] \subset [z, x_j]$ for $i < j$ cannot happen. We can assume that $V$ is so small that $f^i(V) \subset U$ for $i < \text{Val}_T(z)$. Therefore there are $k$ and $l$ such that $0 \leq k < l \leq t$, $x_k$ and $x_l$ lie in the same component of $T \setminus \{z\}$, and if $k \leq i < j \leq l$ then $x_i$ and $x_j$ lie in different components of $T \setminus \{z\}$. We cannot have $[z, x_i] \subset [z, x_k]$, so we have $x_k \in (z, x_l)$. Set $s = l - k$. If $0 < i < s - 1$ then $[z, x_{k+i}] = f^i([z, x_k])$, so there is a point $y_i \in (z, x_k)$ such that $f^i(y_i) = x_{k+i}$. Moreover, $[z, x_k] \subset f^{i-1}([z, x_{k+i}])$ and $y_i \in (z, x_k) \subset (z, x_l)$, so there is a point $u_i \in (z, x_{k+i})$ such that $f^{i-1}(u_i) = y_i$. Thus, $f^i(u_i) = x_{k+i}$ and $x_{k+i}$ lies on the same edge as $u_i$, but further from $z$. Let $z_i$ be the closest to $u_i$ fixed point of $f^i$ in $[z, u_i]$. Then for any $x \in (z_i, u_i)$ we have $x \in (z_i, f^i(x))$. Let $\{T_i, E'_i\}^s_{i=1}$ be the $z$-family of marked trees corresponding to our choice of $z_0, z_1, \ldots, z_{s-1}$. It remains to prove that for every $i$ the map $f_{T_i}$ is a marked map of $(T, E')$. The ends of $T_i$ which are marked in $T$ are fixed and repelling for $f$, so they are fixed and repelling for $(f^i)_{T_i}$. The additional marked end of $T_i$, namely $z_i$, is fixed and repelling for $(f^i)_{T_i}$ by our construction.

Suppose that $F$ is an $(f^i)_{T_{z_{i}}}$-invariant proper generalized subforest of $T_i$ and it does not contain any point of $E_i$. Then it is an $f^i$-invariant generalized subforest of $T$ and therefore $G = \bigcup_{i=0}^{s-1} f^i(F)$ is an $f$-invariant generalized subforest of $T$. Moreover, $G$ contains an interval. If $G \neq T$ then $G$ is proper. In this case, since $f$ is marked, there exists a point $d$ that belongs to $G \cap E$. We have $f(d) = d$ and $d \in f^i(F)$ for some $i < s$. Therefore, $d \in f^{i+u-\theta}(F) = f^{i}(F) \subset F$, so $d \in E \cap F \subset E \cap T_i \subset E_i$, and hence $E_i \cap F \neq \emptyset$, a contradiction. If $G = T$ then $z \in G$. We have $f(z) = z$ and $z \in f^i(F)$ for some $i < s$. Therefore, $z \in f^{i+u-\theta}(F) = f^{i}(F) \subset F$. This means that $z \in T_i$, so $z_i = z$. Since $z_i \in E_i$, we get $E_i \cap F \neq \emptyset$, a contradiction. This completes the proof.

Whenever we construct a $z$-family, we will use the same standard notations: $z_i, S_i, T_i, F_i$, and $s$ without mentioning this each time.

Remark 5.2. Suppose that there is $x \neq z$ such that for every $y \in [x, z)$ either $f(y) \in (y, z]$ or $z \in (y, f(y))$. Then the choice of $U$ sufficiently small in the proof of Proposition 5.1 rules out the possibility of $s = 1$ with $x \in S_1$.

Moreover, in any case the choice of $U$ sufficiently small gives us $z_i$ as close to $z$ as we wish.

The next lemma is the main tool allowing us to look for turbulence. We distinguish four cases and give them names resembling the pictures illustrating them (see Fig. 2).

**Lemma 5.3.** Let $f$ be a marked map of a marked tree $(T, E)$. Let $a, b$ be two distinct points of $T$ belonging to the same edge of $T$. Then the following properties hold

1. (ool) If $f(a) = a, f(b) = b$, and there are points $x, y \in (a, b)$ such that $x \in (a, y), x \in (a, f(x)),$ and $y \in (f(y), b)$, then $f$ is turbulent.
2. (olf) If $f(a) = a, b$ is a free end of $T$, and there is $x \in (a, b)$ such that $x \in (a, f(x))$, then $f$ is turbulent.
(llo) If \( f(b) = b \), there is \( y \in (a, b) \) such that \( y \in (f(y), b) \), and either \( f(a) \in (a, b] \) or \( b \in (a, f(a)) \), then \( f^2 \) is turbulent.

(lf) If \( b \) is a free end of \( T \) and \( f(a) \in (a, b] \), then \( f^2 \) is turbulent.

Proof. We prove (ollo) and (olf) simultaneously. We may assume that there are no fixed points in \((a, x)\), since otherwise we replace \( a \) by the fixed point from \((a, x)\) closest to \( x \). Similarly, in case (ollo) we may also assume that there are no fixed points in \((y, b)\). In case (ollo) we may also assume that \( x \) is so close to \( a \) that \( f(u) \in [u, y] \) for all \( u \in [a, x] \), and \( y \) is so close to \( b \) that \( f(u) \in [x, u] \) for all \( u \in [y, b] \).

Take \( u \in (a, x) \) in both cases, and \( w \in (y, b) \) in case (ollo) and set \( w = y = b \) in case (olf). The interval \([u, w]\) is not \( f\)-invariant, so there exists a point \( u \in [v, w] \) such that \( f(u) = u \) or \( w \) in case (ollo), and \( f(u) = v \) in case (olf). Because of our assumptions, we have \( u \in [x, y] \).

Letting \( u \to a \) and \( w \to b \) we get a sequence of such \( u \)'s, from which we can choose a subsequence convergent to some \( c \in [x, y] \). We have then \( f(c) = a \) or \( b \) in case (ollo), and \( f(c) = a \) in case (olf). Since in case (ollo) we had up to now a completely symmetric situation, we may assume that \( f(c) = a \) also in this case. We may also assume (in both cases) that \( c \) is the closest to a preimage of \( a \) in \((a, b)\).

We claim that there is \( d \in (a, c) \) with \( f(d) = c \). If not, then \( f([a, c]) = [a, e] \) for some \( e \in (a, c) \) (see Fig. 3). Since \( f([x, e]) \) does not contain \( a \), there is \( q \in (a, x) \) such that \( f([x, e]) \subset [q, e] \). Then the interval \([q, e]\) is \( f\)-invariant, a contradiction. This proves our claim. Now \([a, d]\) and \([d, c]\) form a 2-horseshoe for \( f \), so the proof of (ollo) and (olf) is complete.

Now we prove (llo) and (lf). In case (llo) there is a fixed point \( z \in (a, y) \) since \( a \) and \( y \) are mapped in different directions. In case (lf), if there is no fixed point in \((a, b)\) then all points of \((a, b)\) are mapped in the direction of \( b \), and \([x, b]\) is \( f\)-invariant for any \( x \in (a, b) \), a contradiction. Therefore there is a fixed point \( z \in (a, b) \).

We may assume (in both cases) that \( z \) is the fixed point closest to \( a \) in \((a, b)\). We construct a \( z \)-family at \( z \) (see Proposition 5.1). By Remark 5.2 we may assume that \( b \in T_1 \). For \( f^s \), where \( s = 1 \) or \( 2 \), we get a situation of type (ollo) in case (llo) and of type (olf) in case (lf).
Therefore either \( f \) or \( f^2 \) is turbulent. If \( f \) is turbulent then so is \( f^2 \). Thus, \( f^2 \) is turbulent in all the cases.

6. ENTROPY ESTIMATES FOR BISTARS

In order to get entropy estimates for bistars we need the following proposition.

**Proposition 6.1.** Let \( T \) be a star of type \( n + 1 \) for some \( n \geq 2 \). Let \( E \) be a set consisting of one end of \( T \). Let \( f \) be a marked map of \((T, E)\). Then either \( f^s \) is turbulent for some \( s < n \) or \( f^s \) is \( 3 \)-turbulent for some \( s < n + 1 \).

**Proof.** Denote by \( a \) the element of \( E \) and by \( b \) the center (the vertex which is not an end) of \( T \). Assume first that \( f(b) = b \). We perform a construction similar to that from Proposition 5.1. Take a small connected neighborhood \( V \) of \( b \). Since \( f \) is marked, there is \( k \) such that \( a \in f^k(V) \). Then we get a sequence of intervals \([b, x_i] \), \( i = 0, 1, \ldots, t \), such that \([b, x_{i+1}] \subset f([b, x_i]) \) for \( i < t \), \( x_0 \in V \), \( x_i \neq a \) for \( i < t \) and \( x_t = a \). If for some \( i < j \) we have \([b, x_j] \subset [b, x_i] \) then we remove \([b, x_i], \ldots, [b, x_{j-1}] \) from our sequence, and the new sequence has the same properties, but is shorter. We may assume that our sequence is the shortest one with those properties. Then \([b, x_j] \subset [b, x_i] \) for \( i < j \) cannot happen. We can assume that \( V \) is so small that \( a \not \in f^i(V) \) for \( 0 < i < n + 1 \). Then \( t > n + 1 \).

Now there are two possibilities. The first one is similar to the situation occurring in Proposition 5.1. Namely, it can happen that there exist \( k \) and \( l \) such that \( k < l \), \( x_k \) and \( x_l \) lie in the same component of \( T \setminus \{b\} \), if \( k < i < j \leq l \) then \( x_i \) and \( x_j \) lie in different components of \( T \setminus \{b\} \), and none of \( x_k, \ldots, x_l \) lies in \([a, b] \). Then, as in Proposition 5.1, we get subtrees \((T_i, E') \), \( i = 0, 1, \ldots, s - 1 \), where \( s = l - k \), with \((f^s)_{T_i} \), a marked map of \((T_i, E') \). Here each \( T_i \) is an interval and only its end that is closer to \( b \) is marked. Since \( x_k, \ldots, x_{l-1} \) lie in different components of \( T \setminus \{b\} \), and none of them lies in \([a, b] \), we have \( s \leq n \). By Lemma 5.3 \((olf), (f^s)_{T_i} \) is turbulent (for any \( i \)), so by Lemma 2.1, \( f^s \) is turbulent.

The second possibility differs more from the situation occurring in Proposition 5.1. Namely, it can happen that, \( k, l \) as above do not exist. Then it is \( k \leq t - (n + 1) \) such that \( x_k \in [a, b] \) and \( x_l \not \in [a, b] \) for \( i = k + 1, \ldots, t - 1 \). If \( k = t - 1 \) then by Lemma 5.3 \((ol) \) \( f \) is turbulent. Assume that \( k < t - 1 \). Then there is a point \( c \in [x_k, b] \) such that \( f(c) = x_{k+1} \). Since \( f(a) = a \), there is a point \( d \in [c, a] \) such that \( f(d) = b \). Thus, \([a, b] \subset f([a, d]), [b, x_{k+1}] \subset f([d, c]), \) and \([b, x_{k+1}] \subset f([d, c]) \). Therefore, if \( s = t - k \) then \([a, b] \subset f^s([a, d]), [a, b] \subset f^s([d, c]), \) and \([a, b] \subset f^s([c, b]) \). This means that \([a, d], [d, c] \) and \([c, b] \) form a 3-horseshoe for \( f^s \). Since \( k \geq t - (n + 1) \), we get \( s \leq n + 1 \).

Assume now that \( f(b) \neq b \). If \( f(b) \in [a, b] \) then \( f^2 \) is turbulent by Lemma 5.3 \((l) \). If \( f(b) \not \in [a, b] \) then \( f^2 \) is turbulent by Lemma 5.3 \((lf) \). This completes the proof.

**Theorem 6.2.** Let \( T \) be a bistar of type \((k, n)\) with \( k \leq n - 1 \). Let \( f \) be a marked map of \((T, \emptyset)\). Then either \( f^s \) is turbulent for some \( s \leq \max\{n + 1, 2k\} \) or \( f^s \) is \( 3 \)-turbulent for some \( s \leq \max\{n + 1, 2k + 2\} \).

**Proof.** If \( k = 1 \) then \( T \) is a star of type \( n + 1 \). Then by Theorem B, \( f^s \) is turbulent for some \( s \leq n + 1 \).

Assume that \( k > 1 \). Let us denote by \( b_1 \) and \( b_2 \) the two vertices of \( T \) which are not ends of \( T \), where \( \text{Val}_T(b_1) = k + 1 \) and \( \text{Val}_T(b_2) = n + 1 \).
Assume first that at least one of the points $b_j$ is fixed (call it $z$). We construct a $z$-family (see Proposition 5.1). If there is $i$ such that $T_i$ is an interval then by Lemma 5.3 (olf) and Lemma 2.1, $f^s$ is turbulent for some $s \leq \max \{k + 1, n + 1\} = n + 1$. Otherwise, by Proposition 6.1 and Lemma 2.1, either $f^s$ is turbulent for some $s \leq \max \{k, n\} = n$ or $f^s$ is 3-turbulent for some $s \leq \max \{k + 1, n + 1\} = n + 1$.

Suppose now that none of $b_j$ is fixed. Starting from $b_1$, we perform the “point chases its image” construction, as in the proof of Proposition 3.2. As before, the fixed point $z$ obtained in such a way is an interior point (this time since $f$ is marked). We construct a $z$-family (see Proposition 3.1). Assume first that $z \notin [b_1, b_2]$. By Remark 5.2, if $s = 1$ then $b_1 \notin S_1$. Thus, if $s = 1$ then $T_1$ is an interval and by Lemma 5.3 (olf) and Lemma 2.1, $f$ is turbulent. If $s \neq 1$ then $s = 2$ and either $T_1$ or $T_2$ is an interval. Again we use Lemma 5.3 (olf) and Lemma 2.1 and we see that $f^2$ is turbulent.

Assume now that $z \in (b_1, b_2)$. Then $s = 1$ or $2$, $T_1$ is a star of type $k + 1$ or $n + 1$, and $E^1$ consists of one point. Moreover, if $s = 2$ then we may assume that $T_1$ is a star of type $k + 1$. If $s = 1$ then by Proposition 6.1 and Lemma 2.1, either $f^s$ is turbulent for some $l \leq \max \{k, n\} = n$, or $f^s$ is 3-turbulent for some $l \leq \max \{k + 1, n + 1\} = n + 1$. If $s = 2$ then by Proposition 6.1 and Lemma 2.1, either $f^s$ is turbulent for some $l \leq 2k$, or $f^s$ is 3-turbulent for some $l \leq 2(k + 1)$.

We have considered all possible cases and in each of them either $f^s$ is turbulent for some $s \leq \max \{n + 1, 2k\}$ or $f^s$ is 3-turbulent for some $s \leq \max \{n + 1, 2k + 2\}$.

To be able to compare entropies of maps whose various iterates are turbulent or 3-turbulent, we make the following comparison.

**Lemma 6.3** If $k \geq 2$ then $(1/k) \log 2 < [1/(k + 1)] \log 3$.

**Proof.** This inequality is equivalent to $2^{k+1} < 3^k$ that is equivalent to $2 < (1.5)^k$. Since $(1.5)^2 = 2.25 > 2$, it holds for all $k \geq 2$. ■

**Theorem E.** Let $T$ be a bistar of type $(k, n)$ with $1 < k < n$. Let $f: T \to T$ be a transitive map. Then $h(f) \geq (1/N) \log 2$, where $N = \max\{n + 1, 2k\}$.

**Proof.** By Theorem 6.2

$$h(f) \geq \min \left\{ \frac{1}{\max \{n + 1, 2k\}} \log 2, \frac{1}{\max \{n + 1, 2k + 2\}} \log 3 \right\}$$

$$= \min \left\{ \frac{1}{n + 1} \log 2, \frac{1}{2k} \log 2, \frac{1}{n + 1} \log 3, \frac{1}{2k + 2} \log 3 \right\}.$$ 

Clearly, $[1/(n + 1)] \log 2 < [1/(n + 1)] \log 3$. By Lemma 6.3, $(1/2k) \log 2 < [1/(2k + 2)] \log 3$. Therefore, $h(f) \geq \min \left\{ [1/(n + 1)] \log 2, (1/2k) \log 2 \right\}$.

In the case when $T$ is an arbitrary superstar and $f$ a marked map of $(T, \emptyset)$, we could get results similar to Theorems 6.2 and E, but their statements would be too complicated to be meaningful. The reader interested in such results for particular superstars can get them easily by the methods of this section.

As we pointed out in the introduction, $[1/\text{End}(T)] \log 2$ is only a lower estimate for $L(T)$. The next result shows that for stars we have indeed the equality and that for bistars it
can happen that $L(T) > \frac{1}{\text{End}(T)} \log 2$. This fact seems to suggest that the equality $L(T) = \frac{1}{\text{End}(T)} \log 2$ holds only for stars.

**Proposition 6.4.** The following statements hold:

(a) If $T$ is a star then $L(T) = \frac{1}{\text{End}(T)} \log 2$.

(b) If $T$ is a bistar of type $(k, n)$ with $1 < k < n$ then $L(T) > \frac{1}{\text{End}(T)} \log 2$.

**Proof.** To prove (a) it suffices to look at the following example. Let $T$ be a star of type $n$ and let $f$ be the map from $T$ into itself such that it maps cyclically each edge to the next one, all but one linearly, and the remaining one as in the tent map, piecewise linearly with two pieces. A simple computation shows that $h(f) = \frac{1}{\text{End}(T)} \log 2$. Statement (b) follows from Theorem E and from the fact that if $1 < k < n$ then $\max\{n + 1, 2k\} < n + k = \text{End}(T)$.

Finally, we should point out that one cannot generalize in a satisfactory way the entropy estimate from Proposition 6.1 with Lemma 6.3 to marked maps of trees with more marked ends. The natural guess would be that $h(f) \geq (1/N) \log 2$, where $N$ is the number of free ends of $T$ (unless $N = 0$). However, the following simple example shows that this is false. Let $f$ be the $P$-monotone map where the points of $P$ and their images are shown in Fig. 4. Using the rome method [12], one can easily check that $h(f)$ is the logarithm of the largest zero of the polynomial $P(x) = x^4 - x^3 - 2x^2 + 2x - 2$. We have $P(x) = x^2(x - 2)(x + 1) + 2(x - 1)$, so $P(x) > 0$ for $x \geq 2$. Therefore, $h(f) < \log 2$. Looking at the Markov graph of $f$ one can also easily check that $f$ is marked.

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