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Forbidden subgraphs implying the MIN-algorithm gives a maximum independent set ☆

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Abstract

The well-known greedy algorithm MIN for finding a maximal independent set in a graph G is based on recursively removing the closed neighborhood of a vertex which has (in the currently existing graph) minimum degree. We give a forbidden induced subgraph condition under which algorithm MIN always results in finding a maximum independent set of G, and hence yields the exact value of the independence number of G in polynomial time.

1. Introduction

Throughout the paper, we consider only finite undirected graphs $G=(V(G),E(G))$ without loops and multiple edges. By $N_G(x)$ we denote the *neighborhood* of a vertex $x\in V(G)$, i.e., the set of all neighbors of x. We further denote by $N_G[x]=N_G(x)\cup\{x\}$ the *closed neighborhood* of x in G, by $d_G(x)=|N_G(x)|$ the *degree* of x in G and by $\delta(G)=\min\{d_G(x)\,|\,x\in V(G)\}$ the *minimum degree* of G. For a set $M\subset V(G)$, we

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denote by $\langle M\rangle_G$ the induced subgraph of G on M and we set $G-M=\langle V(G)\setminus M\rangle_G$. By $\alpha(G)$ we denote the *independence number* of G, i.e., the size of a maximum (i.e. largest) independent set in G. If $F_1,\dots,F_k$ are graphs, then we say that G is $\{F_1,\dots,F_k\}$-*free* if G does not contain a copy of any of the graphs $F_1,\dots,F_k$ as an induced subgraph. For other terminology and notation not defined here, we refer to [1].

The well-known greedy algorithm MIN for finding a maximal independent set in a graph G [4] can be stated as follows:

Algorithm MIN (Minimum degree).

1. $H_1:=G$; $i:=1$; $S_{\mathrm{MIN}}:=\emptyset$.
2. *Choose a vertex* $v_i\in V(H_i)$ *such that* $d_{H_i}(v_i)=\delta(H_i)$ *and set* $S_{\mathrm{MIN}}:=S_{\mathrm{MIN}}\cup\{v_i\}$; $H_{i+1}:=H_i-N_{H_i}[v_i]$.
3. *If* $V(H_{i+1})\neq\emptyset$ *then* $i:=i+1$ *and go to* 2.
4. *STOP*.

Obviously, the set S_{MIN}, generated by Algorithm MIN, is a maximal (but not necessarily maximum) independent set in G, and hence $\alpha(G)\geqslant|S_{\mathrm{MIN}}|$.

Mahadev and Reed [3] considered the following (also greedy) algorithm for finding a maximal independent set in G, based on an ordering of the vertices of G according to their degrees in G. This algorithm can be equivalently formulated as follows.

Algorithm VO (Vertex order).

1. *Order the vertices of* G *into a sequence* $v_1,\dots,v_n$ *such that* $d_G(v_j)\leqslant d_G(v_k)$ *for any* j,k, $1\leqslant j<k\leqslant n$.
2. $G_1:=G$; $i:=1$; $S_{\mathrm{VO}}:=\emptyset$.
3. *For* $i:=1$ *to* n *do*:
   If $N_G(v_i)\cap S_{\mathrm{VO}}=\emptyset$, *then* $S_{\mathrm{VO}}:=S_{\mathrm{VO}}\cup\{v_i\}$.
4. *STOP*.

It is clear that the set S_{VO}, generated by Algorithm VO, is a maximal independent set in G, and hence also $\alpha(G)\geqslant|S_{\mathrm{VO}}|$.

Note that both Algorithm MIN and Algorithm VO have polynomial time complexity whereas the determination of $\alpha(G)$ is difficult since the corresponding decision problem INDEPENDENT SET is a well-known NP-complete problem [2].

Denote by $k_{\mathrm{MIN}}(G)$ and $k_{\mathrm{VO}}(G)$ the smallest cardinality of an independent set of G that Algorithm MIN and Algorithm VO can create, respectively. Let $F_1,\dots,F_6$ be the graphs in Fig. 1 and let $\mathscr{F}_A=\{F_1,F_2,F_3,F_4,F_5,F_6\}$.

The following theorem, which forms the essential part of the main result of [3], shows that in the class of $\mathscr{F}_A$-free graphs, Algorithm VO always yields a maximum independent set.

Theorem A (Mahadev and Reed [3]). *Let* G *be an* $\mathscr{F}_A$-*free graph. Then*

$$k_{\mathrm{VO}}(G)=\alpha(G).$$

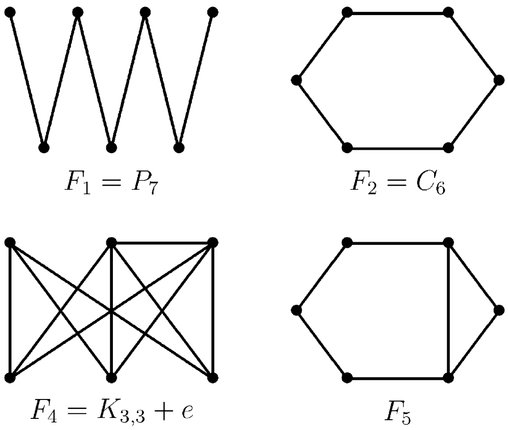

Fig. 1.

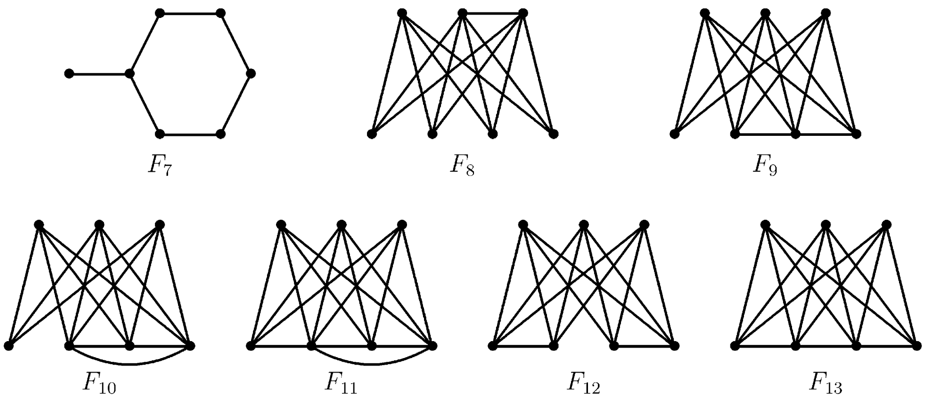

Fig. 2.

2. Main result

Let $F_7, \ldots, F_{13}$ be the graphs shown in Fig. 2 and let $\mathscr{F}_1 = \{F_1, F_3, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}\}$.

Since F_2 is an induced subgraph of F_7, and F_4 is an induced subgraph of each of the graphs $F_8, \ldots, F_{13}$, the class of $\mathscr{F}_A$-free graphs is a proper subclass of the class of $\mathscr{F}_1$-free graphs. Thus, the following theorem, which is the main result of this paper, extends Theorem A in the sense that even for $\mathscr{F}_1$-free graphs the independence number can be calculated in polynomial time.

Theorem 1. *Let G be an $\mathscr{F}_1$-free graph of order $n \geqslant 7$. Then*

$$k_{\mathrm{MIN}}(G) = \alpha(G).$$

Equivalently, Theorem 1 gives a collection of forbidden induced subgraphs which imply that Algorithm MIN always yields a maximum independent set. The proof of Theorem 1 is postponed to Section 3.

As already noted, $\mathscr{F}_A$-free $\Rightarrow \mathscr{F}_1$-free. However, the price for a more general result is paid here in larger number of forbidden subgraphs. The following corollary of Theorem 1 avoids this drawback and still extends Theorem A.

Let $\mathscr{F}_2 = \{F_1, F_3, F_4, F_5, F_6, F_7\}$. Note that, since F_7 contains an induced F_2 and each of the graphs $F_8, \ldots, F_{13}$ contains an induced F_4, we have $\mathscr{F}_A$-free $\Rightarrow \mathscr{F}_2$-free $\Rightarrow \mathscr{F}_1$-free.

Corollary 2. *Let G be an $\mathscr{F}_2$-free graph of order $n \geqslant 7$. Then*

$$k_{\mathrm{MIN}}(G) = \alpha(G).$$

The following statement shows that Corollary 2 (and hence also Theorem 1) is considerably stronger than Theorem A. More specifically, it says that under the assumptions of Corollary 2, the difference between the output of Algorithm MIN and that of Algorithm VO can be arbitrarily large.

Theorem 3. *For every integer k there is an $\mathscr{F}_2$-free graph G such that*

$$k_{\mathrm{MIN}}(G) - k_{\mathrm{VO}}(G) \geqslant k.$$

Proof. Let $\mathscr{G}$ be the class of graphs defined recursively as follows:

(i) $F_2 \in \mathscr{G}$,
(ii) for any $G_1, G_2 \in \mathscr{G}$, let also $(G_1 + G_2) \vee K_1 \in \mathscr{G}$ and $(G_1 + G_2) \vee \overline{K_2} \in \mathscr{G}$.

(Following [1], we denote by "+" the disjoint union and by "$\vee$" the join of two graphs, respectively.)

We show that every graph $G \in \mathscr{G}$ is $\mathscr{F}_2$-free. We first have the following observation, the proof of which is obvious. □

Claim. *Let $F \in \mathscr{F}_2$ with $|V(F)| = r$. Then $d_F(x) \leqslant r - 2$ for every $x \in V(F)$ and $\min\{d_F(x), d_F(y)\} \leqslant r - 3$ for any pair of independent vertices $x, y \in V(F)$.*

Since $F_2 \notin \mathscr{F}_2$, the graph F_2 is $\mathscr{F}_2$-free. Suppose now that G_1, G_2 are $\mathscr{F}_2$-free. If $(G_1 + G_2) \vee K_1$ or $(G_1 + G_2) \vee \overline{K_2}$ contains an induced $F \in \mathscr{F}_2$, then, since F is connected, $V(F)$ contains at least one vertex outside $V(G_1) \cup V(G_2)$, but then we have a contradiction with the claim. Hence, every graph in $\mathscr{G}$ is $\mathscr{F}_2$-free.

If we now set $G_1' = F_2$ and $G_{i+1}' = (G_i' + G_i') \vee K_1$ for $i \geqslant 1$, then $G_i' \in \mathscr{G}$ for any $i \geqslant 1$ and it is apparent that $k_{\mathrm{MIN}}(G_i') = \alpha(G_i') = 3 \cdot 2^{i-1}$, but $k_{\mathrm{VO}}(G_i') = 2^i$.

Remark. By Theorem 1, in the class of $\mathscr{F}_1$-free graphs, Algorithm MIN is always at least as good as Algorithm VO and by Theorem 3 the difference can be arbitrarily large. The following construction shows that without the assumption of $\mathscr{F}_1$-freeness

Algorithm VO can be better than Algorithm MIN (i.e., for all graphs, the two algorithms are incomparable).

Let $p \geqslant 3$ be an arbitrary integer, let $G^1 \simeq G^2 \simeq K_p$, $G^3 \simeq K_1$ and $G^4 \simeq \overline{K_p}$ be vertex-disjoint, let G'_p be the graph obtained by joining by an edge all pairs of vertices x, y for $x \in V(G^i)$, $y \in V(G^{i+1}) \pmod 4$, and let G_p be the graph obtained by adding one new vertex to G'_p and joining it to all vertices of G^2. Then clearly $k_{\mathrm{MIN}}(G_p) = 3$, while $k_{\mathrm{VO}}(G_p) = p + 1$.

Since Algorithm MIN is (clearly) polynomial, we further have the following consequence of Theorem 1.

Corollary 4. *In the class of $\mathscr{F}_1$-free graphs, the independence number can be computed in polynomial time.*

Note that it is obvious that $\mathscr{F}_1$-free graphs are recognizable in polynomial time.

3. Proof of Theorem 1

We basically follow the general idea of the proof of Theorem A in [3], by replacing Algorithm VO with Algorithm MIN and the set $\mathscr{F}_A$ by the set $\mathscr{F}_1$. For the sake of clarity, whenever we list vertices of some induced subgraph F, we always order the vertices of the list such that their degrees (in F) form a nonincreasing sequence (with the exception of $F_1 \simeq P_7$, where the ordering follows the path).

Let G be a (without loss of generality) connected graph satisfying the assumptions of Theorem 1 and suppose that Algorithm MIN creates a maximal independent set S in G such that $|S| = m < \alpha(G)$, i.e., such that S is not maximum. Let the notation of v_i, H_i be chosen in accordance with the description of Algorithm MIN in Section 1, i.e., such that $S = \{v_1, \ldots, v_m\}$, $H_1 = G$, $d_{H_i}(v_i) = \delta(H_i)$ and $H_{i+1} = H_i - N[v_i]$, and set $S_j = S \cap V(H_j) = \{v_j, \ldots, v_m\}$, $j = 1, \ldots, m$. Choose a maximum independent set $T = \{t_1, \ldots, t_\alpha\}$ in G such that $|S \cap T|$ is maximum, and set $T_j = T \cap V(H_j)$, $j = 1, \ldots, m$. Since both S and T are independent, $\langle S \cup T \rangle_G$ is bipartite with all its isolated vertices in $S \cap T$. Let R be a component of $\langle S \cup T \rangle_G$ with $|R \cap S| < |R \cap T|$ (such an R always exists since $|S| < |T|$) and set $k = \min\{i \in \{1, \ldots, m\} \mid v_i \in R \cap S\}$ (with a slight abuse of notation, we will use R for both the component and its vertex set).

We have the following observations.

Claim 1. *S_j is a dominating set in H_j, $j = 1, \ldots, m$.*

Proof. If $x \in V(H_j) \backslash S_j$, then $N_G(x) \cap \{v_1, \ldots, v_{j-1}\} = \emptyset$, since otherwise $x \notin V(H_j)$ by the definition of H_j. Since S is a dominating set in G, necessarily $N_G(x) \cap S_j \neq \emptyset$, implying $N_{H_j}(x) \cap S_j \neq \emptyset$. □

Claim 2. *$d_{H_j}(x) \geqslant d_{H_j}(v_j)$ for every $x \in V(H_j)$ and for every $j = 1, \ldots, m$.*

Proof. Follows immediately from the definition of Algorithm MIN. □

Claim 3. $R \subset V(H_k)$.

Proof. Obviously, $R \cap S \subset V(H_k)$. If $y \in (R \cap T) \setminus V(H_k)$, then $y \in N_{H_j}(v_j)$ for some $j < k$ and hence $v_j \in R \cap S$, contradicting the choice of k. Hence also $R \cap T \subset V(H_k)$. □

The following simple observation will be often used implicitly throughout the proof.

Claim 4. *If F is a subgraph of H_j for some $j \in \{1, \ldots, m\}$, then F is induced in H_j if and only if F is induced in G.*

In the sequel, we will use the following notation: $|R \cap S| = p$, $|R \cap T| = q$, $R \cap S = \{v_{i_1}, \ldots, v_{i_p}\}$, $R \cap T = \{t_1, \ldots, t_q\}$, and we suppose the notation of the vertices in $R \cap S$ is chosen such that $i_1 = k$ and $i_{j_1} < i_{j_2}$ for $j_1 < j_2$.

Case 1: R contains a cycle.

If R contains an induced cycle of length $\ell \geqslant 8$, then R contains also an induced F_1, a contradiction.

Suppose that R contains an induced cycle C of length 6, and let $C = s_1 t_1 s_2 t_2 s_3 t_3 s_1$, where $s_i \in R \cap S$, $t_i \in R \cap T$, $i = 1, 2, 3$. Since $|R \cap S| < |R \cap T|$, there is a $t_4 \in R \cap T$, adjacent to (say) s_1. If $s_2 t_4 \in E(G)$, then (since C is induced and T is independent), $\langle \{s_2, t_4, s_1, t_1, t_2\} \rangle_G \simeq F_3$, a contradiction. Hence $s_2 t_4 \notin E(G)$, and similarly $s_3 t_4 \notin E(G)$. But then $\langle \{s_1, t_1, s_2, t_2, s_3, t_3, t_4\} \rangle_G \simeq F_7$, a contradiction. Hence every induced cycle in R has length exactly 4. Since R is bipartite and F_3-free, it follows easily (by induction, starting with a C_4) that R is a complete bipartite graph with $2 \leqslant |R \cap S| < |R \cap T|$.

Consider the vertex $v_k \in R \cap S$. We have $d_R(v_k) > d_R(y)$ for every $y \in R \cap T$ (since $|R \cap S| < |R \cap T|$), but, on the other hand, by the choice of v_k and by Claim 2, $d_{H_k}(v_k) \leqslant d_{H_k}(y)$ for every $y \in R \cap T$. It follows that there are vertices $z \in V(H_k) \setminus R$ and $y \in R \cap T$ such that $zy \in E(G)$, but $zv_k \notin E(G)$.

Claim 5. *Let $z \in V(H_k) \setminus R$ be such that $zv_k \notin E(G)$ and $N_{R \cap T}(z) \neq \emptyset$. Then $N_{R \cap S}(z) \neq \emptyset$ and $N_{R \cap T}(z) = \{t_1, \ldots, t_q\}$.*

Proof. Let (without loss of generality) $zt_1 \in E(G)$. Suppose first that $N_{R \cap S}(z) = \emptyset$. Then $N_R(z) = R \cap T$, since otherwise $\langle R \cup \{z\} \rangle_{H_k}$ contains an induced F_3. Since S is dominating, $zs \in E(G)$ for some $s \in S \setminus R$. Then $N_{R \cap S}(s) = \emptyset$ (since S is independent) and $N_{R \cap T}(s) = \emptyset$ (otherwise $s \in R$), implying $N_R(s) = \emptyset$ and $\langle \{z, t_1, v_k, t_2, s\} \rangle_G \simeq F_3$. Hence $N_{R \cap S}(z) \neq \emptyset$.

Let (without loss of generality) $zv_{i_2} \in E(G)$. Recall that $i_1 = k$, i.e., $v_{i_1} = v_k$. If $zt_a, zt_b \notin E(G)$ for some $a, b \in \{2, \ldots, q\}$, then $\langle \{v_{i_2}, t_a, v_{i_1}, t_b, z\} \rangle_G \simeq F_3$, and if $zt_a \notin E(G)$ and $zt_b \in E(G)$ for some $a, b \in \{2, \ldots, q\}$, then $\langle \{v_{i_1}, t_1, z, t_b, t_a\} \rangle_G \simeq F_3$. Hence $zt_i \in E(G)$ for every $i = 1, \ldots, q$. □

Now, by Claim 5, $q \geqslant 4$ implies $\langle \{z, v_{i_2}, v_{i_1}, t_1, t_2, t_3, t_4\} \rangle_G \simeq F_8$. Hence $q = 3$ and, consequently, $p = 2$.

Denote $H=\langle\{z,v_{i_2},v_{i_1},t_1,t_2,t_3\}\rangle_G$ (note that $H\simeq F_4$). Since $|V(G)|\geqslant 7$, there is a vertex $y\in V(G)\backslash V(H)$ with $N_H(y)\neq\emptyset$.

Suppose first that $yv_{i_1}\in E(G)$. If $y\in V(H_k)$ and $yt_i\in E(G)$ for $i=1,2,3$, then $\langle\{y,z,v_{i_1},v_{i_2},t_1,t_2,t_3\}\rangle_G$ is isomorphic to one of the graphs F_{11}, F_{12} or F_{13}, depending on the existence of the edges yv_{i_2}, yz. If $y\in V(H_k)$ and (say) $yt_1\notin E(G)$, then, by the choice of v_k (as a vertex of minimum degree in H_k) and by Claim 2, there is a $z'\in V(H_k)\backslash V(H)$ such that $z'v_{i_1}\notin E(G)$, but $z't_1\in E(G)$. By Claim 5, $\{v_{i_2},t_1,t_2,t_3\}\subset N_H(z')$, i.e., $\langle V(H)\backslash\{z\}\cup\{z'\}\rangle_G\simeq F_4$. Then $\langle V(H)\cup\{z'\}\rangle_G$ induces F_{10} or F_9, depending on whether $zz'\in E(G)$ or not.

If $y\notin V(H_k)$, then $yv_{i_0}\in E(G)$ for some i_0, $1\leqslant i_0<k$. Note that $N_H(v_{i_0})=\emptyset$ (since $i_0<k$). Then either $N_H(y)=V(H)$, implying $\langle V(H)\cup\{y\}\rangle_G\simeq F_{11}$, or y is nonadjacent to some vertex of H, and then it is easy to see that $\langle V(H)\cup\{y,v_{i_0}\}\rangle_G$ contains an induced F_3 for any possible structure of $N_H(y)$. This contradiction proves that $yv_{i_1}\notin E(G)$.

If $N_{R\cap T}(y)=\emptyset$, then $yz\in E(G)$ or $yv_{i_2}\in E(G)$, but in both cases we have an induced F_3. Hence, $N_{R\cap T}(y)\neq\emptyset$. By Claim 5, $yv_{i_2}\in E(G)$ and $yt_i\in E(G)$ for $i=1,2,3$. Then again $\langle V(H)\cup\{y\}\rangle$ induces an F_{10} or F_9, depending on whether $yz\in E(G)$ or not. This contradiction completes the proof in Case 1.

Case 2: R is a tree.

Claim 6. *All leaves of R are in T.*

Proof. If $s\in S$ is a leaf of R and $t\in T$ is the (only) neighbor of s in R, then $T\backslash\{t\}\cup\{s\}$ is also a maximum independent set, contradicting the maximality of $|S\cap T|$. □

Claim 6 immediately implies that every longest path in R has an odd number of vertices. Since G is F_1-free, a longest path in R can be only a P_3 or a P_5.

Subcase 2.1: R contains a P_3, both endvertices of which are leaves of R.

By Claim 6, let $t_av_{i_\ell}t_b$ (where $1\leqslant\ell\leqslant p$ and $1\leqslant a,b\leqslant q$) be the vertices of the P_3. First observe that $t_a,t_b\in V(H_{i_\ell})$ (since otherwise e.g. $t_a\notin V(H_{i_\ell})$ would imply $t_av_c\in E(G)$ for some c, $1\leqslant c<i_\ell$, but then for $1\leqslant c<k$ the vertex t_a would not be in H_k, and for $k\leqslant c<i_\ell$ the vertex t_a would not be a leaf of R). By Claim 2, there are vertices $x_a,x_b\in V(H_{i_\ell})\backslash R$ such that $x_at_a\in E(G)$ and $x_bt_b\in E(G)$, but $x_av_{i_\ell},x_bv_{i_\ell}\notin E(G)$. By Claim 1, each of x_a, x_b has a neighbor (say, $v_{a'}$ and $v_{b'}$) in S_{i_ℓ}. Note that $v_{a'},v_{b'}$ are nonadjacent to t_a and t_b (otherwise t_a, t_b are not leaves). Now we have $x_a\neq x_b$ (otherwise $\langle\{x_a,t_a,v_{i_\ell},t_b,v_{a'}\}\rangle_G\simeq F_3$), $x_at_b\notin E(G)$ and $x_bt_a\notin E(G)$ (otherwise $\langle\{x_a,t_b,v_{i_\ell},t_a,v_{a'}\}\rangle_G\simeq F_3$ or $\langle\{x_b,t_a,v_{i_\ell},t_b,v_{b'}\}\rangle_G\simeq F_3$) and, finally, $x_ax_b\notin E(G)$ and $v_{a'}=v_{b'}$ (otherwise $\langle\{x_a,x_b,t_a,t_b,v_{i_\ell},v_{a'},v_{b'}\}\rangle_G$ induces F_1, F_6 or F_5).

Since the vertex $v_{a'}$ $(=v_{b'})$ is in S_{i_ℓ} (but not necessarily in R), we have $v_{a'}=v_{\ell'}$ for some ℓ', $i_\ell<\ell'\leqslant m$. Suppose that, among all common neighbors of x_a, x_b in S_{i_ℓ}, $v_{a'}$ is chosen such that ℓ' is minimum. Then $x_a,x_b\in H_{\ell'}$, but $t_a,t_b\notin H_{\ell'}$. By Claim 2 (for $j=\ell'$), there are $z_a,z_b\in V(H_{\ell'})$ such that $z_ax_a\in E(G)$ and $z_bx_b\in E(G)$, but $z_a,z_b\notin N_G(v_{\ell'})$ (and also $z_a,z_b\notin N_G(v_{i_\ell})$).

Suppose first that $z_a=z_b$. Since $\langle\{x_a,z_a,x_b,v_{\ell'},t_a\}\rangle_G\not\simeq F_3$, we have $t_az_a\in E(G)$. Symmetrically, $\langle\{x_b,z_a,x_a,v_{\ell'},t_b\}\rangle_G\not\simeq F_3$ implies $t_bz_a\in E(G)$. Then $z_a\notin S_{\ell'}$ (otherwise t_a,t_b

are not leaves). By Claim 1, z_a has a neighbor s in S, but then $\langle\{z_a,t_a,v_{i_\ell},t_b,s\}\rangle_G\simeq F_3$. Hence, $z_a\neq z_b$ (implying $z_ax_b\notin E(G)$ and $z_bx_a\notin E(G)$).

We show that $z_at_a\notin E(G)$. Let $z_at_a\in E(G)$. If $z_at_b\in E(G)$, then $\langle\{t_b,v_{i_\ell},t_a,z_a,x_b\}\rangle_G\simeq F_3$; hence $z_at_b\notin E(G)$. Clearly $z_a\notin S$ (otherwise t_a is not a leaf) and hence, by Claim 1, z_a has a neighbor s_a in S. Obviously, s_a is not adjacent to any of $v_{i_\ell},v_{\ell'},t_a,t_b$. If $s_ax_b\notin E(G)$, then $\langle\{s_a,z_a,t_a,v_{i_\ell},t_b,x_b,v_{\ell'}\}\rangle_G\simeq F_1$; hence $s_ax_b\in E(G)$, but then for $s_ax_a\in E(G)$ we have $\langle\{x_b,v_{\ell'},x_a,s_a,t_b\}\rangle_G\simeq F_3$, and for $s_ax_a\notin E(G)$ we have $\langle\{x_a,z_a,s_a,x_b,v_{\ell'},t_a\}\rangle_G\simeq F_5$. Hence $z_at_a\notin E(G)$.

Since $\langle\{x_a,t_a,v_{i_\ell},t_b,x_b,v_{\ell'},z_a\}\rangle_G\not\simeq F_7$, we obtain $z_at_b\in E(G)$. Symmetrically, $z_bt_b\notin E(G)$ and $z_bt_a\in E(G)$. This also implies that $z_a,z_b\notin S$ (otherwise t_a or t_b is not a leaf). By Claim 1, there are vertices $s_a,s_b\in S_{\ell'}$ such that $z_as_a\in E(G)$ and $z_bs_b\in E(G)$ (possibly $s_a=s_b$). Obviously, s_a and s_b are not adjacent to any of $t_a,t_b,v_{i_\ell},v_{\ell'}$. If $s_ax_a\in E(G)$, then for $s_a=s_b$ and $x_bs_b\in E(G)$ we have $\langle\{x_a,v_{\ell'},x_b,s_a,t_a\}\rangle_G\simeq F_3$, otherwise $\langle\{x_a,t_a,v_{i_\ell},t_b,x_b,v_{\ell'},s_a\}\rangle_G\simeq F_7$. Hence $s_ax_a\notin E(G)$ and, similarly, $s_ax_b\notin E(G)$. But then $\langle\{x_a,z_a,t_b,x_b,v_{\ell'},t_a,s_a\}\rangle_G\simeq F_6$. This contradiction completes the proof in Subcase 2.1.

Subcase 2.2: R contains no P_3 both endvertices of which are leaves of R.

In this subcase, R contains a P_5 (but no P_7). Using Claim 6, it is easy to show (by induction, starting with a P_5) that R is isomorphic to the subdivision of a star with center and leaves in $R\cap T$ and with vertices of degree 2 in $R\cap S$. Choose the notation such that v_{i_j} is adjacent to the center t_0 and to the leaf t_j, $j=1,\dots,p$. By Claim 2, there is a vertex $z_1\in V(H_{i_1})$ such that $z_1t_1\in E(G)$, but $z_1v_{i_1}\notin E(G)$. Clearly, $z_1\notin S$; thus, by Claim 1, $z_1v_\ell\in E(G)$ for some $v_\ell\in S$ with $i_1<\ell\leqslant m$. Note that v_ℓ is not adjacent to any of v_{i_1}, t_1, but possibly $t_0v_\ell\in E(G)$.

Suppose first that $z_1t_j\in E(G)$ for some j, $2\leqslant j\leqslant p$. Then clearly also $v_\ell v_{i_j}$, $v_\ell t_j\notin E(G)$. We further have $t_0z_1\notin E(G)$ (since otherwise $\langle\{z_1,t_1,v_{i_1},t_0,t_j\}\rangle_G\simeq F_3$) and $v_{i_j}z_1\notin E(G)$ (otherwise $\langle\{v_{i_j},z_1,t_1,v_{i_1},t_0,t_j\}\rangle_G\simeq F_5$). Now we have $t_0v_\ell\in E(G)$, since otherwise $\langle\{z_1,t_1,v_{i_1},t_0,v_{i_j},t_j,v_\ell\}\rangle_G\simeq F_7$. This implies $v_\ell\in R\cap S$ and, by the structure of R, v_ℓ has a (unique) neighbor t_ℓ in $R\cap T$. But now $\langle\{t_\ell,v_\ell,z_1,t_1,v_{i_1},t_0,v_{i_j}\}\rangle_G\simeq F_6$ if $t_\ell z_1\notin E(G)$, or $\langle\{v_\ell,z_1,t_1,v_{i_1},t_0,t_\ell\}\rangle_G\simeq F_5$, if $t_\ell z_1\in E(G)$, respectively. This contradiction proves that z_1 is not adjacent to any of $t_2,\dots,t_p$.

Now suppose that $z_1v_{i_a}\notin E(G)$ for some a, $2\leqslant a\leqslant p$. Then $t_0z_1\notin E(G)$ (otherwise $\langle\{t_0,v_{i_1},t_1,z_1,v_{i_a}\}\rangle_G\simeq F_3$) and $t_0v_\ell\in E(G)$ (otherwise $\langle\{v_\ell,z_1,t_1,v_{i_1},t_0,v_{i_a},t_a\}\rangle_G\simeq F_1$). This implies, as before, that v_ℓ is in $R\cap S$ and has a (unique) neighbor t_ℓ in $R\cap T$, but then $\langle\{v_\ell,t_0,v_{i_1},t_1,z_1,v_{i_a},t_\ell\}\rangle_G\simeq F_6$. Hence, z_1 is adjacent to all vertices in $(R\cap S)\setminus\{v_{i_1}\}$. This immediately implies $p=|R\cap S|=2$, for otherwise $\langle\{t_0,v_{i_2},z_1,v_{i_3},v_{i_1}\}\rangle_G\simeq F_3$.

Summarizing, it remains to consider the case when $R\cap S=\{v_{i_1},v_{i_2}\}$, $R\cap T=\{t_0,t_1,t_2\}$ and $N_R(z_1)=\{t_1,v_{i_2}\}$. We consider the graph H_{i_2}. Since $i_1<i_2$, $\{v_{i_1},t_0,t_1\}\cap V(H_{i_2})=\emptyset$, and since $z_1v_{i_1},t_2v_{i_1}\notin E(G)$, we have $z_1,t_2\in V(H_{i_2})$. By Claim 2 (for $j=i_2$), there is a vertex $z_1'\in V(H_{i_2})$ such that $z_1'z_1\in E(G)$ but $z_1'v_{i_2}\notin E(G)$ (and, of course, $z_1'v_{i_1}\notin E(G)$). If $t_0z_1'\notin E(G)$, then for $t_1z_1'\in E(G)$ we have $\langle\{t_1,z_1,v_{i_1},t_0,v_{i_2},z_1'\}\rangle_G\simeq F_5$, and for $t_1z_1'\notin E(G)$ we have $\langle\{z_1,t_1,v_{i_1},t_0,v_{i_2},z_1',t_2\}\rangle_G\simeq F_6$ if $z_1't_2\notin E(G)$ and $\langle\{v_{i_2},t_2,z_1',z_1,t_0\}\rangle_G\simeq F_3$ if $z_1't_2\in E(G)$. Hence $t_0z_1'\in E(G)$, but this implies $\langle\{t_0,v_{i_2},z_1,z_1',v_{i_1}\}\rangle_G\simeq F_3$. This final contradiction completes the proof. □

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