JOURNAL OF ALGEBRA 144, 371–389 (1991)

Actions of Divided Power Hopf Algebras

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Communicated by Nathan Jacobson

Received November 10, 1989

We study prime ideals in enveloping algebra smash products and use a duality construction to obtain results on prime ideals in rings on which divided power Hopf algebras act. These actions correspond to higher derivations. First, we consider chains of prime ideals in an enveloping algebra smash product over an arbitrary ring, where the Lie algebra is assumed to be finite dimensional abelian over a field of positive characteristic. We give a bound on the length of such a chain where the ideals all have the same intersection with the coefficient ring. Then using an explicit construction of a duality theorem of Blattner and Montgomery in this context, we are able to apply results on enveloping algebra smash products to study the invariant ideals of prime ideals in a ring, under a locally nilpotent divided power Hopf algebra action.

INTRODUCTION

In this paper we study prime ideals in enveloping algebra smash products and use a duality construction to obtain results on prime ideals in rings with divided power Hopf algebra actions. These Hopf algebras have a divided power coalgebra structure as well as a divided power commutative algebra structure. Over a perfect field, they are a special case of general pointed irreducible cocommutative Hopf algebras which are always divided power coalgebras [17]. In characteristic zero, these Hopf algebras are just enveloping algebras of abelian Lie algebras, but in positive characteristic p, these objects may be finite or infinite dimensional, and their actions correspond to higher derivations.

A crucial observation for using "duality" is that in any characteristic the

0021-8693/91 $3.00

dual H' of a divided power Hopf algebra H is a tensor product of an enveloping algebra and a restricted enveloping algebra. We use this fact to apply results concerning prime ideals for actions of these two types of Hopf algebras, along with duality, to obtain results concerning prime ideals in smash products and invariant ideals. While we limit the applications in this paper to the study of primes, it seems that the duality methods employed should yield other applications in this setting.

Divided power Hopf algebra actions arise, for example, in the study of rings of differential operators of affine varieties [15] and in the study of purely inseparable field extensions [18]. In fact the simple algebra $A_1^p(k)$ used in this paper is precisely the ring of differential operators of the affine line in positive characteristic.

The first section of the paper focuses on bounding lengths of chains of prime ideals in smash products of enveloping algebras of abelian Lie algebras over a field of nonzero characteristic. The main result in this section is Theorem 1.10. The proof depends on results for restricted Lie algebras along with the observation that the enveloping algebra here is actually a restricted enveloping algebra of a certain infinite dimensional restricted Lie algebra. The characteristic zero version of Theorem 1.10 was proved in [3]. If R is Noetherian and k has characteristic zero, Passman [12] has obtained the same result without assuming that L is abelian.

The second section deals with duality for divided power Hopf algebras H acting "locally nilpotently" on an algebra R, describing the duality construction via explicit embeddings in certain simple rings of differential operators which are analogs of the Weyl algebras. This approach is a special case of the duality theorem of Blattner and Montgomery [2, Theorem 2.1]. This duality, begun by Cohen and Montgomery [6], and continued by Blattner and Montgomery, has motivated similar constructions dealing with various Hopf algebra actions on rings. Cohen and Montgomery used their construction to apply known theorems about crossed products of finite groups to prove results on group-graded rings. For example that paper includes a generalization of Incomparability for crossed products of finite groups. Later, a similar but more concrete construction for infinite group-graded rings was employed to use results concerning crossed products of infinite groups to obtain results for rings graded by infinite groups [13, 5]. A special case of duality for certain enveloping algebra actions was given in [11]. This has been used to study certain smash products of enveloping algebras in characteristic zero [10, 14]. What is done here is to show that this latter approach can be employed, via the Blattner–Montgomery theorem, to study prime ideals in positive characteristic divided power Hopf algebra smash products, as well as in rings upon which such Hopf algebras act.

In the third section we record some results on finite dimensional irreducible Hopf algebra actions which we need for our applications.

Section 4 shows how the machinery of Section 2 along with results from Sections 1 and 3, as well as known results on enveloping algebra smash products, can be applied. Let R be an algebra on which the divided power algebra H acts locally nilpotently. Our results focus on chains of prime and H-prime ideals of R, and chains of prime ideals of the smash product $R \# H$. We obtain a bound on the lengths of chains of primes of R each having the same H-invariant ideal. Theorem 4.5 gives upper and lower bounds for the Krull dimension of $R \# U(L)$ in characteristic zero when L is a finite dimensional and abelian Lie algebra.

1. Smash Products of Positive Characteristic Enveloping Algebras

The main result of this section determines a sharp bound on the lengths of chains of prime ideals in a cocycle twisted smash product (crossed product) $R * U(L)$ where L is an abelian Lie algebra over a field k of positive characteristic p and R is an algebra over k.

Let $\mathrm{Spec}_0(R * H)$ denote the set of prime ideals of $R * H$ having zero intersection with R; H-$\mathrm{Spec}(R)$ denotes the set of H-prime ideals of R.

Let L be a Lie algebra acting as derivations on R. Let $U(L)$ and $u(L)$ denote the universal enveloping algebra and restricted enveloping algebra (if L is a restricted Lie algebra) of L. In characteristic $p > 0$, we set

$$L^{\wedge} = \sum_{\substack{x \in L \\ i \geqslant 0}} kx^{p^i}$$

the closure of L in $U(L)$ under pth powers. This makes $L^{\wedge}$ into a restricted Lie algebra in with the obvious pth power map. Also notice that the elements of $L^{\wedge}$ act as restricted derivations on L. In Hopf algebraic language this means $L^{\wedge}$ consists of primitive elements. In fact we have

Lemma 1.1. *With L and $L^{\wedge}$ as above, we have $U(L) = u(L^{\wedge})$.*

Proof. This is easily verified directly by considering PBW bases or one can note that $U(L)$ is a pointed irreducible cocommutative Hopf algebra which is generated by its Lie algebra of primitives (which is precisely $L^{\wedge}$). The result then follows from [16]. ∎

If R is an L-prime ring, S shall denote the symmetric quotient ring R. Twisted (restricted) enveloping algebra smash products are constructed in [3, 4] (denoted there simply by $R * L$) and it is shown in those papers that

$R * U(L)$ extends uniquely to $S * U(L)$ and $R * u(L)$ extends uniquely to $S * u(L)$.

We briefly describe the construction here. Let L be a Lie algebra where each element of L corresponds to a derivation of R. $R * U(L)$ has underlying k-space $R \otimes U(L)$. The multiplication is determined by the formulas

$$\bar{x}r - r\bar{x} = x \cdot r,$$

$$\bar{x}\bar{y} - \bar{y}\bar{x} = \overline{[x, y]} - t(x, y),$$

where $\bar{x}, \bar{y} \in 1 \otimes L$, $r \in R$, and $t: L \times L \to R$ is a bilinear twisting. $R * u(L)$ is defined by using $u(L)$ in place of $U(L)$ and adding the relation $\bar{x}^p = \overline{x^p} + \pi(x)$ where $\pi: L \to R$ is an additional twisting of the restricted Lie algebra L.

Given a smash product of this kind we say that L "acts" on R, although here the action $L \to \mathrm{Der}_k R$ need not be a (restricted) Lie homomorphism but is "cocycle" twisted so that the resulting smash product is associative. We shall say that L is *R-inner* or *L is inner on* R if for all $x \in L$, there exists $d \in R$ such that $x \cdot r = dr - rd$, all $r \in R$; that is, L acts as inner derivations on R.

We are mainly interested in actions in the usual sense and in the ordinary smash product $R \# u(L)$, where L acts via a Lie homomorphism, but in dealing with these we are compelled to deal with the more general Lie-cocycle twisted construction. For example if K is a restricted ideal of L then $R \# u(L) = (R \# u(K)) * u(L/K)$, an iterated twisted smash product. These remarks also apply for ordinary enveloping algebras.

More about the twisted smash product constructions and extensions to the quotient ring may be found in [3, 4].

Our basic strategy for dealing with $U(L) = u(L^\wedge)$ is to use the *Ideal Intersection Property* (nonzero L-invariant ideals of $R * u(K)$ have nonzero intersection with R where K is a certain restricted ideal of L) to deal with the case where $L^\wedge$ has no S-inner restricted Lie subalgebra of finite codimension. This allows us to reduce the problem to dealing with finite dimensional restricted enveloping algebra smash products and inner smash products of universal enveloping algebras. The following three theorems and their corollaries form the basic ingredients in this approach.

Let F denote $Z(S)^L$, the subring of L-constants of the extended center of the L-prime ring R. F is a field by [4, Lemma 7]. Below we use the fact that $F \otimes L$ acts as F-linear derivations on the F-algebra S. The following extends [4, Theorem 11].

THEOREM 1.2. *Let L be a restricted Lie algebra with $K \lhd L$ a central restricted ideal. Let $R * u(L)$ be given and assume that R is an L-prime ring* *with quotient ring S. If no nonzero element of $F \otimes L$ is inner in S, then $R * u(K)$ has the ideal intersection property for L-invariant ideals.*

Proof. Let I be a nonzero L-invariant ideal of $R * u(K)$ and suppose that $I \cap R = 0$. Fix a basis $\{x_i\}$ for K. Let m be minimal among the total degrees of nonzero elements of I. Our assumption on I implies that $m > 0$. Let V denote the set of dim K-tuples v, over $\mathbb{Z}^+$ with finite support and let

$$|v| = \sum_i v_i = m.$$

Further, let W denote a subset of V of minimal size subject to the condition that there is a nonzero element $\alpha \in I$ with $|\alpha| = m$, and $\mathrm{Supp}_m(\alpha) = \bar{x}^W = \{\bar{x}^v \mid v \in W\}$.

Define, for dim K-tuples v,

$$A_v = \left\{ r \in R \mid \text{there exists } \alpha = \sum \alpha_\xi \bar{x}^\xi \in I \text{ with} \right.$$

$$\left. \alpha_v = r,\ \mathrm{Supp}_m(\alpha) \subset \bar{x}^W, \text{ and } |\alpha| \leqslant m \right\}.$$

Observe that for $y \in L$, α as in the definition of A_ω, and $\omega \in W$,

$$[\bar{y}, \alpha] = \sum_{\mu \in W} \delta_y(\alpha_\mu)\, \bar{x}^\mu + \alpha_- \in I, \qquad |\alpha_-| < m,$$

using the fact that $[K, L] = 0$. It follows that $\delta_y(\alpha) \in A_\omega$ for $\alpha \in A_\omega$ and thus A_ω is a nonzero L-invariant ideal of R.

Fix $\omega \in W$ and let $A = A_\omega$. We may assume that our basis was chosen so that $\omega = (\omega_1, \omega_2, \ldots)$ with $\omega_1 > 0$. Define maps $f_v \colon A \to A_v$ as follows. Let $a \in A$ and let α be as in the definition of $A = A_\omega$. It follows from the minimality of W that this α depends only on a. Write

$$\alpha = \sum a_v \bar{x}^v$$

and define $af_v = a_v$. The f_ζ are easily seen to be left R-module maps. Furthermore, if $\zeta \in W$, f_ζ is actually an R–R bimodule map. Note that f_ω is the identity map.

Let $\zeta \in W$ and let c_ζ be the element of the extended center of R represented by f_ζ. Note that $c_\omega = 1$. Fix $\zeta \in W$ and set $c = c_\zeta$, $f = f_\zeta$. We claim that $c \in F$, the L-extended center of R. To see this, let $y \in L$ and note that, by the formula for $[\bar{y}, \alpha]$ above and the definition of f, we have

$$\delta_y(a)\, f = \delta_y(a_\zeta) = \delta_y(af).$$

Therefore,

$$\delta_y(a)c = \delta_y(ac).$$

Thus we conclude that $a\delta_y(c) = 0$, and so $A\delta_y(c) = 0$. Therefore $\delta_y(c) = 0$, showing that $c \in F$.

Set $\omega' = (\omega_1 - 1, \omega_2, ...)$. (Recall that $\omega_1 > 0$.) Using $r \in R$, $a \in A$, and $\alpha \in I$ as above, compute

$$\begin{aligned} \alpha r &= \sum_{\mu \in W} a_\mu \bar{x}^\mu r + a_{\omega'} \bar{x}^{\omega'} r + \cdots \\ &= \sum_{\mu \in W} ((a_\mu r)\, \bar{x}^\mu + a_\mu [\bar{x}^\mu, r]) + a_{\omega'} r \bar{x}^{\omega'} + \cdots, \end{aligned}$$

where we omit terms of degree less than m except for the $\bar{x}^{\omega'}$ term. Given $i > 0$, let $\mu = \mu(i)$ be the element of V with $\mu_i = \omega'_i + 1$. In particular $\mu(1) = \omega$. Observe that the coefficient of $\bar{x}^{\omega'}$ in αr depends on these i's and in fact, this coefficient is

$$a_{\omega'} r + \sum_i a_{\mu(i)} \mu(i)_i\, \delta_i(r).$$

Now by the definition of $f_{\omega'}$ we have

$$(ar)\, f_{\omega'} = (af_{\omega'})r + \sum_i (af_{\mu(i)})\, \mu(i)_i \delta_i(r).$$

Furthermore letting s denote the element of the left quotient of R represented by $f_{\omega'}$, we see that

$$ars = a\left(sr + \sum_i c_{\mu(i)} \mu(i)_i \delta_i(r) \right).$$

Thus

$$A\left(rs - sr - \sum_i (c_{\mu(i)} \mu(i)_i)\, \delta_i(r) \right) = 0,$$

so we see that

$$[s, \;] = \sum_i \mu(i)_i c_{\mu(i)}\, \delta_i$$

as a derivation of R (and hence of S). Thus

$$\sum_i c_{\mu(i)} \mu(i)_i \delta_i$$

is inner in the left quotient ring of R. But now [3, Lemma 1.1] applies to show that $s \in S$.

We saw above that $c_\mu \in F$ since $\mu \in W$, so

$$\sum_i (c_{\mu(i)}\mu(i)_i)\, x_i$$

is an F-linear combination of $x_i \in K$ which is inner in S. This linear combination is nonzero because when $i=1$, we have $\mu=\omega$, $c_\mu=1$, and $0<\mu_1<p$, so the proof is complete. ∎

COROLLARY 1.3. *Let $R * u(K)$ be as in the theorem and consider the extension to $S * u(L)$. Then $S * u(K)$ has the ideal intersection property for L-invariant ideals.*

Proof. Observe that every nonzero ideal of $S * u(K)$ intersects $R * u(K)$ in a nonzero ideal. By the ideal intersection property, this ideal has nonzero intersection with R and hence with S. ∎

Let K_{inn} denote the restricted Lie ideal of K^F $(=F \otimes K)$ consisting of the F-linear combinations of elements of K whose action on S is inner in S. In particular if $K_{\text{inn}}=0$, then the hypotheses of the theorem are satisfied. Hence we have

COROLLARY 1.4. *Let L be a restricted Lie algebra with $K \lhd L$ a central restricted ideal and consider $R * u(K) \subset S * u(K)$; if $K_{\text{inn}}=0$, then both $R * u(K)$ and $S * u(K)$ have the ideal intersection property for L-invariant ideals.*

The following is the main result of [4].

THEOREM 1.5 (Incomparability). *Let $R * u(L)$ be given where L is a finite dimensional abelian restricted Lie algebra. If $P_1 < P_2$ are prime ideals of $R * u(L)$, then $P_1 \cap R < P_2 \cap R$.*

The next result follows immediately from the previous theorem and [4, Lemma 18].

LEMMA 1.6. *Let $R * u(L)$ be given and assume that L has a restricted ideal K with L/K finite dimensional abelian. Let $Q_1 < Q_2$ be prime ideals of $R * u(L)$ having equal intersections with R. Then there exist prime ideals $Q_1' < Q_2'$* of *$R * u(K)$ such that Q_i' is the unique minimal covering prime of $Q_i \cap (R * u(K))$ $(i=1,2)$ and $Q_1' \cap R = Q_2' \cap R$.*

THEOREM 1.7. (a) [3, Lemma 2.8]. *Let $R * U(L)$ be given with R being an L-prime ring. Then* $\operatorname{Spec}_0(R * U(L))$ *embeds in* $\operatorname{Spec}_0(S * U(L))$ *via an inclusion preserving map.*

(b) [4, Theorem 17]. *Let S be a centrally closed prime ring with center C. Let E be a C-algebra. Then* $\mathrm{Spec}_0(S \otimes_C E)$ *is in bijection with* Spec E *via the inclusion-preserving maps*

$$P \to P \cap E, \qquad P \in \mathrm{Spec}_0(S \otimes E)$$

$$L \to S \otimes L, \qquad L \in \mathrm{Spec}(E).$$

COROLLARY 1.8. *Let $R * U(L)$ be given. Assume that R is a prime ring and the action of L is R-inner. Then* $\mathrm{Spec}_0(R * U(L))$ *embeds in* $\mathrm{Spec}(E)$ *via an inclusion-preserving map, where E is some twisted product $C_t[U(L)]$ over a field C.*

Proof. Let S denote the symmetric quotient ring of R, and note that, since L is R-inner, L is certainly also S-inner. By [3, Theorem 1.4], S is a centrally closed prime ring. There it is also shown that $S * U(L)$ is isomorphic to $S \otimes_C E$ where E is of the stated form and C, the center of S, is a field. Thus the result follows from parts (a) and (b) of the previous theorem. ∎

Chains of Prime Ideals

We need the following lemma after which we conclude with the main result of this section.

LEMMA 1.9. *Let L be a finite dimensional abelian Lie algebra over a field F of positive characteristic p. Let K be a restricted ideal of $L^\wedge$ with the property that for each nonzero element $x \in L$, $(Fx)^\wedge \cap K$ is nonzero. Then K is of finite codimension in $L^\wedge$.*

Proof. Fix a basis $x_1, x_2, ..., x_d$ for L over F and set $x = x_a$. Using the hypothesis concerning x, multiplying by a scalar if necessary, we obtain

$$x^{p^m} - \sum_{i=0}^{m-1} c_i x^{p^i} \in K,$$

and thus

$$x^{p^{s+m}} \equiv \sum_{i=0}^{m-1} c_i^{p^s} x^{p^{i+s}} \qquad (\mathrm{mod}\ K),$$

where $m = m(a)$ depends on the basis element x_a. It follows by induction on j, that for all $j \geqslant 0$, x^{p^j} can be expressed modulo K as a linear combination of x^{p^t}, $0 \leqslant t < m$.

Thus $L^\wedge/K$ is spanned by the images of the finite set

$$\{x_i^{p^t} \mid i=1, ..., d;\ t=1, ..., m(i)\}. \quad \blacksquare$$

The characteristic zero version of the final result of this section was proved in [3]. Also in characteristic zero, D. Passman [12] has obtained the same conclusion when R is Noėtherian, without assuming that L is abelian.

THEOREM 1.10. *Let* $R * U(L)$ *be a twisted smash product with* L *finite dimensional abelian and* $\operatorname{char} k > 0$. *If*

$$P_0 < P_1 < \cdots < P_n$$

is a chain of prime ideals of $R * U(L)$, *each having the same intersection with* R, *then* $n \leqslant \dim L$.

Proof. We begin by making some reductions. By passing to the L-prime factor ring $R/(P_i \cap R)$ we may assume that the P_i are in $\operatorname{Spec}_0(R * U(L))$ and that R is L-prime. Let S be the symmetric quotient ring of R and let $S * U(L)$ be the unique extension of $R * U(L)$. In view of Theorem 1.7(a) it suffices to prove the result for $S * U(L)$. Thus we may assume that R is the symmetric quotient ring of some L-prime ring. Furthermore, tensoring up to the L-(extended) center $F = Z(R)^L$ of R, we may replace L with $F \otimes L$ and assume that L is a Lie algebra over the field F. Let I denote the restricted Lie ideal of $L^\wedge$ consisting of elements whose action is R-inner.

First suppose that there exists some nonzero $x \in L$ such that $(Fx)^\wedge \cap I = 0$. Now, with $K = Fx$,

$$\begin{aligned} R * U(L) &= R * u(L^\wedge) \\ &= (R * U(K)) * U(L/K) \\ &= (R * u(K^\wedge)) * U(L/K). \end{aligned}$$

To conclude this case, note that $P_i \cap ((R * u(K^\wedge))$ is an L-invariant ideal of $R * u(K^\wedge)$ and that $K^\wedge$ is central in $L^\wedge$. Consequently Corollary 1.4 implies (together with the fact that $P_i \cap R = 0$ for all i) that P_i has zero intersection with $R * u(K^\wedge)$ for all i. Thus by induction, with $R * u(K^\wedge)$ in the role of R, we obtain $n \leqslant \dim L/K = \dim L - 1$.

Next we consider the complementary case, where for each nonzero element $x \in L$, $(Fx)^\wedge \cap I$ is nonzero. Let $x_1, x_2, ..., x_{\dim L}$ be a basis for L over F. For each x_i, let y_i denote a fixed nonzero element of $(Fx_i)^\wedge$ which is S-inner. Further, let K denote the F-linear span of the y_i. The y_i are surely

F-linearly independent so they form a basis for K, and since $K \leqslant L^\wedge$, K is an abelian Lie algebra acting as derivations on S. Also observe that $K^\wedge$ is a restricted ideal of $L^\wedge$. Thus we may write

$$\begin{aligned} R * U(L) &= R * u(L^\wedge) \\ &= (R * u(K^\wedge)) * u(L^\wedge/K^\wedge) \\ &= (R * U(K)) * u(L^\wedge/K^\wedge), \end{aligned}$$

using Lemma 1.1.

By Lemma 1.9 $K^\wedge$ has finite codimension in $L^\wedge$, so $L^\wedge/K^\wedge$ is a finite dimensional restricted Lie algebra. Thus Theorem 1.5 allows us to intersect the chain of primes down to a chain of $L^\wedge/K^\wedge$-primes

$$Q_0 < Q_1 < \cdots < Q_n$$

of $R * U(K)$. By Lemma 1.6 each Q_i has a unique minimal covering prime $Q_i' \subset R * U(K)$, satisfying

$$Q_0' < Q_1' < \cdots < Q_n',$$

and

$$Q_0' \cap R = Q_1' \cap R = \cdots = Q_n' \cap R.$$

Note that $Q_0' \cap R$ is a K-prime ideal of R and hence prime, since K is R-inner. Also the action of K is certainly still inner in $R/Q_0' \cap R$. Thus by factoring out $Q_0' \cap R$ we may assume that $Q_0' \cap R = 0$ and that R is a prime ring with K still inner on R. Finally observe that the desired conclusion $n \leqslant \dim L$ now follows immediately from Corollary 1.8 and the fact that the twisted product $C_t[U(K)]$ there has classical Krull dimension at most $\dim K = \dim L$. ∎

2. Duality

We now wish to develop some machinery which will enable us to use the results of Section 1 to study invariant ideals for divided power Hopf algebra actions. This involves studying special cases of the duality theorem of Blattner and Montgomery [2, Theorem 2.1] in detail. In characteristic zero this is really an old result of Nouazé and Gabriel [11]. In positive characteristic we have a special case of the Blattner–Montgomery construction [2, Theorem 2.1].

We begin by recalling the characteristic zero methods. Let R be an algebra over a field of characteristic zero and let δ be a locally nilpotent derivation on R. We define $\eta: R \to R[Y] \subseteq A_1(R)$ by

$$\eta(r) = \sum_i \frac{\delta^i(r)}{i!} Y^i.$$

Here $A_1(R) = R \otimes A_1(k)$, where $A_1(k)$ is the Weyl algebra over k. Recall that $A_1(k)$ is generated as an algebra over k by X and Y where $[X, Y] = XY - YX = 1$.

It is easily checked that η is a ring monomorphism preserving the identity. We write $\tilde{r}$ for $\eta(r)$ and $\tilde{R}$ for the image of R under η. The following is easily deduced from [11].

THEOREM 2.1. *In the above situation $R[x; \delta]$ is isomorphic to $S = \tilde{R}[X]$ the subring of $A_1(R)$ generated by $\tilde{R}$ and X. Furthermore Y is transcendental over S, S is invariant under the derivation* ad_Y, *and $A_1(R)$ is generated as a ring by S and Y. In particular $A_1(R) = S[Y; \tau]$ is a differential operator ring over S, where τ is the restriction of* ad_Y *to S.*

It is well known that $A_1(k)$ is the ring of differential operators on $k[Y]$, where k is a field of characteristic zero. If k has positive characteristic p, we let $A_1^p(k)$ denote the ring of differential operators on $k[Y]$. This ring has been studied by S. P. Smith [15]. $A_1^p(k)$ contains the polynomial ring $k[Y]$, and it has a free (right or left) basis $1 = X_0, X_1, X_2, \ldots$ over $k[Y]$, where X_i is the operator which sends Y^j to $\binom{j}{i} Y^{j-i}$. Multiplication is given by the rules $X_i X_j = \binom{i+j}{i} X_{i+j}$ and

$$X_t Y^m = \sum_{j=0}^{t} \binom{m}{j} Y^{m-j} X_{t-j}.$$

By H_∞ we will mean the divided power Hopf algebra with basis $1 = x_0, x_1, x_2, \ldots$ where $x_i x_j = \binom{i+j}{i} x_{i+j}$,

$$\Delta x_n = \sum_{i=0}^{n} x_{n-i} \otimes x_i, \qquad \text{and} \qquad \varepsilon(x_i) = \delta_{i,0}.$$

If k has positive characteristic p and $n = p^t$ for some integer t, then H_n denotes the finite dimensional divided power Hopf algebra with basis $1 = x_0, x_1, \ldots, x_{n-1}$, with the same structure as H_∞, except that $x_i x_j = 0$ if $i + j \geq n$. Finally

$$H_{(n: n_1, n_2, \ldots, n_t)}$$

denotes the divided power Hopf algebra

$$H_\infty \otimes H_\infty \otimes \cdots \otimes H_\infty \otimes H_{n_1} \otimes \cdots \otimes H_{n_t},$$

with n factors of H_∞.

Let $A_n^p(k) = A_1^p(k) \otimes \cdots \otimes A_1^p(k)$ (n factors) and define $A_n^p(R)$ to be $R \otimes A_n^p(k)$. If we consider actions of $H_\infty^n = H_{(n;0,\ldots,0)}$, Theorem 2.1 still holds as long as $A_1(R)$ is replaced by $A_n^p(R)$.

If $\nu \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+ = (\mathbb{Z}^+)^n$, then X_ν denotes

$$X_{\nu_1} \otimes X_{\nu_2} \otimes \cdots \otimes X_{\nu_n} \in A_n^p(k)$$

and

$$Y^\nu = Y^{\nu_1} \otimes Y^{\nu_2} \otimes \cdots \otimes Y^{\nu_n} \in A_n^p(k).$$

Note that $\{X_\nu Y^\mu\}$ is k-basis for $A_n^p(k)$ where ν and μ range over $(\mathbb{Z}^+)^n$. We let x_ν denote

$$x_{\nu_1} \otimes x_{\nu_2} \otimes \cdots \otimes x_{v_n} \in H_n^\infty.$$

By a slight abuse of terminology we say H_∞^n acts locally nilpotently on the k-algebra R if, given $r \in R$, $x_\nu \cdot r = 0$ for

$$|\nu| = \sum_i \nu_i$$

sufficiently large.

If H is a Hopf algebra, H' is the dual Hopf algebra consisting of those linear functionals on H which vanish on a power of the augmentation ideal. See [17] for details. This Hopf algebra H' acts in a natural way on H via

$$f \rightharpoonup h = \sum_{(h)} h_{(1)} \langle f, h_{(2)} \rangle$$

so we can form the smash product $H \# H'$. (See [2] for details.)

Although it is not explicitly needed for our applications, the following lemma is a key observation in motivating the consideration of $A_n^p(k)$ in this context.

LEMMA 2.2. *Let H be the divided power Hopf algebra H_∞^n. Then $H \# H' \cong A_n^p(k)$.*

Proof. If $H = L \otimes K$ as Hopf algebras, then $H \# H' \cong (L \# L') \otimes (K \# K')$. Thus it suffices to prove that $H_\infty \# H'_\infty$ is isomorphic $A_1^p(k)$.

Let $y^i \in H'$ be given by $\langle y^i, x_j \rangle = \delta_{i,j}$. Then

$$\langle y^i y^j, x_m \rangle = \sum_t \langle y^i, x_t \rangle \langle y^j, x_{m-t} \rangle = \delta_{i+j,m}.$$

Hence we have $y^i y^j = y^{i+j}$. Similarly $\Delta y = 1 \otimes y + y \otimes 1$, so that $H' = k[y]$ is the enveloping algebra of the one dimensional Lie algebra ky. Viewing H and H' as contained in $H \# H'$ via their canonical images, we get

$$\begin{aligned}
[y, x_i] &= yx_i - x_i y = (1 \# y)(x_i \# 1) - (x_i \# 1)(1 \# y) \\
&= (y \rightharpoonup x_i) \# 1 + (x_i \# y) - (x_i \# y) \\
&= x_{i-1} \# 1 = x_{i-1}.
\end{aligned}$$

It is now clear that the map sending x_i to X_i and y to $-Y$ induces an isomorphism from $H \# H'$ to $A_1^p(k)$. ∎

In Theorem 2.4 we will give an explicit description of duality for H_∞^n actions. A detailed knowledge of the duality isomorphism is necessary for our applications.

Suppose $H = H_\infty^n$ acts locally nilpotently on the H-module algebra R. We define

$$\eta\colon R \to A_n^p(R) \text{ by } \eta(r) = \sum_v (x_v \cdot r)\, Y^v.$$

LEMMA 2.3 [2]. *η is a ring monomorphism with $\eta(1) = 1$.* ∎

Let T denote the subring of $A_n^p(R)$ generated by $\tilde{R} = \eta(R)$ and $\{X_v \mid v \in (\mathbb{Z}^+)^n\}$. Also let $e_i \in (\mathbb{Z}^+)^n$ be the tuple with 1 in the ith position and zeroes elsewhere, and let Y_i denote Y^{e_i}.

We now state the special case of [2, Theorem 2.1]. We remark that this result is a generalization of Theorem 2.1 above. By replacing $x^i/i!$ by x_i in characteristic zero one sees that $k[x] = H_\infty$. Thus we allow $p = 0$ from here forward.

THEOREM 2.4. *Let $H = H_{(n;0,\ldots,0)}$ act locally nilpotently on the H-module algebra R. Then $R \# H$ is isomorphic to T, the subring of $A_n^p(R)$ generated by $\tilde{R}$ and $\{X_v\}$, via the map sending r to $\eta(r)$ and x_v to X_v. Furthermore T is invariant under each of the derivations ad_{Y_i}, $Y_1, Y_2, \ldots, Y_n$ are transcendental over T, and $A_n^p(R)$ is generated as an algebra by T and $Y_1, \ldots, Y_n$. Thus $A_n^p(R)$ is isomorphic to $T \# U(L)$, a smash product of the enveloping algebra of the abelian Lie algebra $L = kY_1 + \cdots + kY_n$ over T.*

We note that the hypotheses of [2, Theorem 2.1] are satisfied: Since H is cocommutative, the RL-condition is automatic, and since the action of H on A is locally nilpotent, it is "H'-locally finite." Thus, using the dual bases for H and H' as in the proof of Lemma 2.2, we see that our formula for η coincides with the embedding on [2, p. 164].

We now give some results relating prime, H-prime, and H-invariant ideals for $H = H_\infty^n$ actions which are used in Section 4.

If R is an H-module algebra for any Hopf algebra A and A is an ideal of R, we write $(A:H)$ for the largest H-invariant ideal of R contained in A. When $H=k\{y\}$, the enveloping algebra of the one dimensional Lie algebra, we simply write $(A:\delta)$ instead of $(A:H)$, where y acts on R as the derivation δ. It is easily checked that if A is an ideal of the ring R and H acts on R, then $(A:H)=\{r\in R \mid h\cdot r\in A, \text{ for all } h\in H\}$.

If I is an ideal of R, then $\bar{I}=I\otimes A_n^p(k)$ is an ideal of $A_n^p(R)$. Since $A_n^p(k)$ is simple with center k, all ideals of $A_n^p(R)$ are of this form. We will need the following

LEMMA 2.5. *Let $H=H_\infty^n$ act locally nilpotently on R and let I be an ideal of R. Then $\bar{I}\cap T$ is generated as an ideal of T by $\bar{I}\cap\tilde{R}=(I:H)^\sim$.*

Proof. Let $J=\bar{I}\cap T$. Since $\bar{I}$ and T are both invariant under ad_{Y_i}, so also is J. Recall that $X^vY_i-Y_iX^v=X^{v-e_i}$ if $v_i>0$ and is zero otherwise. It follows that $J=(J\cap\tilde{R})T$. Finally $J\cap\tilde{R}=\bar{I}\cap\tilde{R}=\{\tilde{r}\mid x^v\cdot r\in I, \text{ for all } v\}=(I:H)^\sim$. ∎

The following lemma extends [9, Proposition 1.2] and the proof is an easy adaptation of that proof.

LEMMA 2.6. *Let $H=H_\infty^n$ act on R and let P be a prime ideal of R. Then $(P:H)$ is again a prime ideal of R.*

Proof. If $H=L\otimes K$ where H, L, and K are Hopf algebras, and A is an ideal of R, then $(A:H)=((A:L):K)$. Thus it suffices to prove the result for H_∞.

Suppose $(P:H)$ is not prime so that we can find $a, b\in R\backslash(P:H)$ with $aRb\subseteq(P:H)$. Choose $s, t\in\mathbb{Z}^+$ minimal so that $x_s\cdot a$ and $x_t\cdot b$ are not in P. Then for any $r\in R$, $x_{s+t}(arb)\in P$ since $aRb\subseteq(P:H)$. But $x_{s+t}(arb)\equiv x_s(a)\,rx_t(b) \pmod{P}$ so that $x_s(a)\,Rx_t(b)\subseteq P$, which contradicts our assumption that P is a prime ideal. ∎

The following proposition is generalized in Section 4.

PROPOSITION 2.7. *Let R be an H-module algebra where $H=H_\infty^n$ acts locally nilpotently and suppose $P_0<P_1<\cdots<P_{n+1}$ is a chain of prime ideals of R. Then $(P_0:H)<(P_{n+1}:H)$.*

Proof. Clearly $A_n^p(P_0)<A_n^p(P_1)<\cdots<A_n^p(P_{n+1})$ is a chain of ideals of $A_n^p(R)$, and these ideals are prime, since every ideal of $A_n^p(R)$ is of the form $A_n^p(I)$ for some ideal I of R. Now we can apply Theorem 1.10 in positive characteristic, or [3, Theorem 2.11] in characteristic zero, to the smash product $A_n^p(R)=T\,\#\,U(L)$ given by Theorem 2.4, to conclude that $A_n^p(P_0)\cap T<A_n^p(P_{n+1})\cap T$ is a strict inclusion. Finally, by Lemma 2.5,

we know that $A_n^p(P_i) \cap T$ is generated by $(P_i : H)^{\sim}$ so that $(P_0 : H)^{\sim} < (P_{n+1} : H)^{\sim}$, and hence $(P_0 : H) < (P_{n+1} : H)$. ∎

3. Finite Dimensional Actions

In this section we record some results about actions of finite dimensional, irreducible Hopf algebras. A finite dimensional, irreducible Hopf algebra H has a coradical filtration $k = H_0 < H_1 < \cdots < H_m = H$, where $\Delta H_i \subset \sum_{j=0}^{i} H_{i-j} \otimes H_j$. We call m the length of this filtration. (See [16] for details.) The following two results are partial generalizations of [4, Lemma 1].

Lemma 3.1. *Let H act on R, where H is finite dimensional and irreducible. Suppose R is H-prime and m is the length of the coradical filtration of H. If N is an ideal of R, then*

(i) *N is nilpotent if and only if $(N : H) = 0$*

(ii) *when N is nilpotent, $N^{m+1} = 0$.*

Proof. First we show, by induction on j, that $H_j(N^n) \subset N$, if $j < n$. This is clear if $j = 0$. Suppose $j > 0$. Note that

$$\begin{aligned} H_j(N^n) &= H_j(NN^{n-1}) \\ &\subseteq H_0(N)\, H_j(N^{n-1}) + H_j(N)\, H_0(N^{n-1}) + H_{j-1}(N)\, H_{j-1}(N^{n-1}) \\ &\subset N, \end{aligned}$$

by induction. Thus $H(N^{m+1}) = H_m(N^{m+1}) \subset N$, so that $N^{m+1} \subset (N : H)$. It follows now that (i) holds.

To prove (ii), suppose N is nilpotent. Then $(N : H) \subset N$ is also nilpotent and hence zero, since R is H-prime. Now we get that $N^{m+1} \subset (N : H) = 0$. ∎

Proposition 3.2. *Let H be a finite dimensional, irreducible Hopf algebra acting on R. Let Q be an H-prime ideal of R and P a prime ideal of R. Then the maps $P \to (P : H)$, $Q \to N(Q)$ are inclusion preserving inverses, providing a bijection between* Spec R and H-Spec R, *where $N(Q)$ is the unique largest ideal which is nilpotent modulo Q.*

Proof. Let $Q \in H$-Spec R. Applying Lemma 3.1 to R/Q, we see that $N = N(Q)$ exists and is maximal subject to $(N : H) = Q$. Suppose A, B are ideals of R strictly containing N. Then $(A : H) > Q$ and $(B : H) > Q$, so that $Q < (A : H)(B : H) \subset (AB : H)$. Thus N does not contain AB and hence is a prime ideal of R.

Similarly, if $P \in \operatorname{Spec} R$, $(P : H)$ is an H-prime ideal of R and P is nilpotent modulo $(P : H)$. Being prime, P must be the unique prime ideal nilpotent modulo $(P : H)$, so that $P = N((P : H))$. ∎

Our next result is Incomparability for smash products over $H_{(0; n_1, \ldots, n_t)}$.

THEOREM 3.3. *Let $H = H_{(0; n_1, \ldots, n_t)}$ act on R and let $P_0 < P_1$ be prime ideals of $R \# H$. Then $P_0 \cap R < P_1 \cap R$.*

Proof. Let H^* be the dual Hopf algebra of H. We know H^* acts on $R \# H$ (see [2]). In fact if we let $\{y^\nu\}$ be the dual basis to $\{x_\nu\}$, it is easily checked that $y^{e_i} \rightharpoonup (r \# x_\nu) = r \# x_{\nu - e_i}$. From this it follows that ideals of $R \# H$ which are invariant under the action of H^* are precisely those which are generated, as right ideals, by their intersection with R.

H^* is a finite dimensional irreducible Hopf algebra. (In fact H^* is a restricted enveloping algebra of a finite dimensional restricted Lie algebra.) Thus Proposition 3.2 applies to give $(P_0 : H^*) < (P_1 : H^*)$. Since $(P_i : H^*)$ is generated by $P_i \cap R$, it follows that $(P_0 \cap R) < (P_1 \cap R)$. ∎

4. APPLICATIONS

The main result of this section is

THEOREM 4.1. *Let R be an H-module algebra where $H = H_{(n; n_1, \ldots, n_t)}$ acts locally nilpotently and suppose $P_0 < P_1 < \cdots < P_{n+1}$ is a chain of prime ideals of R. Then $(P_0 : H) < (P_{n+1} : H)$.*

Proof. We can write H as $L \otimes M$, where $L = H_\infty^n$ and $M = H_{(0; n_1, \ldots, n_t)}$. By Proposition 2.7, $(P_0 : L) < (P_{n+1} : L)$ is a strict inequality and it follows from Lemma 2.6 that these are prime ideals of R. $M = H_{(0; n_1, \ldots, n_t)}$ is a finite dimensional irreducible Hopf algebra so we can apply Proposition 3.2 to get that $((P_0 : L) : M) < ((P_{n+1} : L) : M)$, which is the desired result since $(P_i : H) = ((P_i : L) : M)$. ∎

COROLLARY 4.2. *Let R be an algebra over a field of characteristic zero and let δ be a locally nilpotent derivation on R. Assume $P_0 < P_1 < P_2$ is a chain of prime ideals of R. Then $(P_0 : \delta) < (P_2 : \delta)$.*

Proof. Let $H = k[x]$ be the universal enveloping algebra of the one dimensional Lie algebra kx and let H act on R by letting x act as δ. Letting $x_i = x^i/i!$ we have that $H = H_\infty$ acts locally nilpotently on R. Note that if $A \lhd R$, then $(A : H) = (A : \delta)$. Thus $(P_0 : \delta) < (P_2 : \delta)$ follows from the previous result. ∎

K. Goodearl has pointed out an example which shows that Corollary 4.2 fails if δ is not locally nilpotent: Let k be a field of characteristic zero and let $R=k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$. Define δ by $\delta(x_i)=x_{i-1}^{-1}$, for $i>1$ and $\delta(x_1)=1$. Then R is δ-simple, so that $(P:\delta)=0$ for all prime ideals P.

Let $d(T)$ denote the classical Krull dimension of a ring T. Another immediate corollary is

COROLLARY 4.3. *Let R be an algebra over a field of characteristic zero and let δ be a locally nilpotent derivation. If $R[x;\delta]$ is simple, then $d(R)\leqslant 1$.* ∎

We now give a slight generalization of [8, Proposition 3.3(a)] which we need for our final result.

PROPOSITION 4.4. *Let k be a field of characteristic zero and let L be a finite dimensional solvable Lie algebra acting locally finitely on the k-algebra R. Then L-prime ideals of R are prime.*

Proof. We first prove the result for L one dimensional and k algebraically closed. Let $L=ky$, where y acts on R as the locally finite derivation δ. It suffices to show that if R is δ-prime, then R is prime.

Suppose $aRb=0$, with $a, b\in R\backslash 0$. Let A and B be the δ-subspaces of R generated by a and b, respectively. Since δ is locally finite and k is algebraically closed, A and B are finite dimensional triangularizable δ-modules. Let $P\subset A\otimes B$ be the subspace of elements $\sum_i a_i\otimes b_i$ such that $\sum_i a_i r b_i=0$ for all $r\in R$. It is easily seen that P is invariant under $\delta\otimes 1+1\otimes\delta$ and that P contains the pure tensor $a\otimes b$. It now follows from [7, Lemma 3.7.1] that P contains an element $\tilde{a}\otimes\tilde{b}$ where $\tilde{a}$ and $\tilde{b}$ are nonzero δ-eigenvectors. If $\tilde{A}$ and $\tilde{B}$ are the ideals of R generated by $\tilde{a}$ and $\tilde{b}$, respectively, then $\tilde{A}$ and $\tilde{B}$ are δ-invariant and $\tilde{A}\tilde{B}=0$, which contradicts the assumption that R is δ-prime. Thus R is prime.

If k is not algebraically closed, let K denote its algebraic closure. Again assume that R is δ-prime. If R is not prime, choose I and J, nonzero ideals of R, with $IJ=0$. Let $\tilde{\delta}=\delta\otimes 1$ be the natural extension of δ to $R\otimes K$, which is again a locally finite K-derivation. We can use Zorn's Lemma to find Q, and ideal of $R\otimes K$, which is maximal with respect to being $\tilde{\delta}$-invariant and having zero intersection with R. It follows easily that Q is a $\tilde{\delta}$-prime of R, and hence a prime ideal, by the argument above. Now $I\otimes K$, $J\otimes K$ are ideals of $R\otimes K$, not contained in Q (since $Q\cap R=0$), and $(I\otimes K)(J\otimes K)=IJ\otimes K=0$. This implies that Q is not prime. This contradiction finishes the proof in the one dimensional case.

Now let L be a finite dimensional solvable, which acts locally finitely on R. To complete the proof we need to show if R is L-prime, then R is prime. Since R is L-prime it is proved in [3, Theorem 2.6] that $R \# U(L)$ is a prime ring. $R \# U(L)$ can be written as an interated differential operator ring $R[x_1;\delta_1][x_2;\delta_2]\cdots[x_n;\delta_n]$ where δ_i is locally finite on $R[x_1;\delta_1]\cdots[x_{i-1};\delta_{i-1}]$ for each i. Since 0 is a prime ideal of $R[x_1;\delta_1]\cdots[x_n;\delta_n]$, it follows that 0 is a δ_n-prime ideal of $R[x_1;\delta_1][x_2;\delta_2]\cdots[x_{n-1};\delta_{n-1}]$. Now, by the one dimensional case we get that $R[x_1;\delta_1]\cdots[x_{n-1};\delta_{n-1}]$ is a prime ring. Repeating this argument for $\delta_{n-1}, \delta_{n-2}, ..., \delta_1$ in turn, we see that R is a prime ring. ▮

We conclude by giving upper and lower bounds for $d(R \# U(L))$, where L is a finite dimensional abelian Lie algebra over a field of characteristic zero, which acts locally nilpotently on R. Let [ ] denote the greatest integer function.

THEOREM 4.5. *Let L be a finite dimensional abelian Lie algebra over a field of characteristic zero and assume L acts locally nilpotently on R. Then* $[d(R)/(n+1)] \leqslant d(R \# U(L)) < (n+1)(d(R)+1)$.

Proof. Let $P_0 < P_1 < \cdots < P_m$ be a chain of prime ideals in $R \# U(L)$. Each $P_i \cap R$ is an L-prime ideal of R and by Proposition 4.4, we have that $P_i \cap R$ is a prime ideal of R. Now applying [3, Theorem 2.11] we get that

$$P_0 \cap R < P_{n+1} \cap R < P_{2(n+1)} \cap R < \cdots$$

is a chain of prime ideals of R so that $m < (n+1)(d(R)+1)$; thus $d(R \# U(L)) < (n+1)(d(R)+1)$.

Now let $Q_0 < Q_1 < \cdots < Q_t$ be a chain of prime ideals of R. Since $U(L) = H_{(n;0,...,0)}$, we map apply Theorem 4.1 to get

$$(Q_0 : L) < (Q_{n+1} : L) < (Q_{2(n+1)} : L) < \cdots$$

and each $(Q_i : L)$ is an L-prime ideal of R. (In fact by Lemma 2.6 $(Q_i : L)$ is actually a prime ideal.)

From [3, Theorem 2.6] we get that

$$(Q_0 : L) \# U(L) < (Q_{n+1} : L) \# U(L) < \cdots$$

is a chain of prime ideals of $R \# U(L)$ so that $t < (n+1)(d(R \# U(L))+1)$.

Thus $t/(n+1) < d(R \# U(L)) + 1$ which yields $[t/(n+1)] \leqslant d(R \# U(L))$. ▮

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