ELSEVIER Journal of Pure and Applied Algebra 100 (1995) 93-102

JOURNAL OF PURE AND APPLIED ALGEBRA

Local systems over complements of hyperplanes and the Kac–Kazhdan conditions for singular vectors

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Received 11 November 1994; revised 12 March 1995

Introduction

In this paper we strengthen a theorem by Esnault, Schechtman and Viehweg [3], which states that one can compute the cohomology of a complement of hyperplanes in a complex affine space with coefficients in a local system using only logarithmic global differential forms, provided certain "Aomoto non-resonance conditions" for monodromies are fulfilled at some "edges" (intersections of hyperplanes). We prove that it is enough to check these conditions on a smaller subset of edges, see Theorem 9.

We show that for certain known one-dimensional local systems over configuration spaces of points in a projective line defined by a root system and a finite set of affine weights (these local systems arise in the geometric study of Knizhnik–Zamolodchikov differential equations, cf. [8]), the Aomoto resonance conditions at non-diagonal edges coincide with the Kac–Kazhdan conditions of reducibility of Verma modules over affine Lie algebras, see Theorem 18.

1. Quasi-isomorphisms

Let $\{H_i\}_{i\in I}$ be an affine arrangement of hyperplanes, i.e., $\{H_i\}_{i\in I}$ is a finite collection of (distinct) hyperplanes in the affine complex space $\mathbb{C}^n$. Define $U = \mathbb{C}^n - \bigcup_{i\in I} H_i$. We denote by Ω_U^p the sheaves of holomorphic forms on U for $0 \le p \le n$. We set $\mathcal{O}_U = \Omega_U^0$.

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SSDI 0022-2049(95)00014-3

For any $i \in I$, choose a degree-one polynomial function f_i on $\mathbb{C}^n$ whose zero locus is equal to H_i. Define $\omega_i = d\log f_i = df_i/f_i \in \Gamma(U, \Omega_U^1)$. For a given $r \in \mathbb{N} - \{0\}$ we choose matrices $P_i \in \operatorname{End}\mathbb{C}^s$, $i \in I$. Define

$$\omega = \sum_{i \in I} \omega_i \otimes P_i \in \Gamma(U, \Omega_U^1) \otimes \operatorname{End}\mathbb{C}^s.$$

The form ω defines the connection $d + \omega$ on the trivial bundle $\mathcal{E} = \mathcal{O}_U^s$. We suppose that $(d + \omega)$ is *integrable* which is equivalent to the condition $\omega \wedge \omega = 0$ as $d\omega = 0$. Let $\Omega_U^\bullet(\mathcal{E}) = \Omega_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{E}$ be the de Rham complex with the differential $d + \omega$.

Define finite-dimensional subspaces

$$A^p \subset \Gamma(U, \Omega_U^p(\mathcal{E})) = \Gamma(U, \Omega_U^p) \otimes_{\mathbb{C}} \mathbb{C}^s$$

as the $\mathbb{C}$-linear subspaces generated by all forms $\omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \otimes v$, $v \in \mathbb{C}^s$. Then the exterior product by ω defines

$$A^\bullet : 0 \longrightarrow A^0 \xrightarrow{\omega} A^1 \xrightarrow{\omega} \cdots \xrightarrow{\omega} A^n \longrightarrow 0$$

as a subcomplex of $\Gamma(U, \Omega_U^\bullet(\mathcal{E}))$.

Let $\overline{\mathbb{C}}^n$ be any smooth compactification of $\mathbb{C}^n$ such that H_∞ is a divisor. Write $H = H_\infty \cup \left(\bigcup_{\in I} H_i\right)$. Then $U = \overline{\mathbb{C}}^n - H$. (Typical examples for $\overline{\mathbb{C}}^n$ include the complex projective space $\mathbf{P}^n$, $(\mathbf{P}^1)^n$ and any toric manifold.) Note that $\omega \in \Gamma(U, \Omega_U^1) \otimes \operatorname{End}\mathbb{C}^s$ can be uniquely extended to be an $\operatorname{End}\mathbb{C}^s$-coefficient rational 1-form $\overline{\omega}$ on $\overline{\mathbb{C}}^n$.

Theorem 1. *Suppose $\pi : X \to \overline{\mathbb{C}}^n$ is a blowing up of $\overline{\mathbb{C}}^n$ with centers in H such that*
(1) *X is nonsingular,*
(2) *$\pi^{-1}H$ is a normal crossing divisor, and*
(3) *none of the eigenvalues of the residue of $\pi^{-1}\bar{\omega}$ along any component of $\pi^{-1}H$ lies in $\mathbb{N} - \{0\}$.*

Then the inclusion

$$A^\bullet \hookrightarrow \Gamma(U, \Omega_U^\bullet(\mathcal{E}))$$

is a quasi-isomorphism.

Proof. Same as the proof of the first theorem in [3]. $\square$

2. Decomposable arrangements

Let $\mathcal{A}$ be a central arrangement in V, i.e., a finite collection of hyperplanes with $\bigcap_{A \in \mathcal{A}} A \neq \emptyset$. Then $\mathcal{A}$ is called *decomposable* if there exist nonempty subarrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ (disjoint) and, after a certain linear coordinate change, defining equations for $\mathcal{A}_1$ and $\mathcal{A}_2$ have no common variables.

Let $\mathcal{A}$ be a nonempty central arrangement in $\mathbb{C}^n$. Let $T = \bigcap_{A \in \mathcal{A}} A \neq \emptyset$. Suppose $\operatorname{codim} T = k+1 > 0$. Then the points of $\mathbf{P}_T = \mathbf{P}^k$ parametrize the $(\dim X+1)$-dimensional linear subspaces of $\mathbb{C}^n$ which contain T. In particular, if H is a hyperplane containing T, it uniquely determines a hyperplane H' in $\mathbf{P}^k$. Define $P(\mathcal{A}) = \mathbf{P}^k - \bigcup_{H\in\mathcal{A}} H'$.

Definition 2. Define the *beta invariant* of a central arrangement $\mathcal{A}$ by

$$\beta(\mathcal{A}) = (-1)^r \chi(P(\mathcal{A}))$$

where χ denotes the Euler characteristic.

Let $L(\mathcal{A})$ be the set of all edges of $\mathcal{A}$. We regard $L(\mathcal{A})$ as a lattice with the reverse inclusion as its partial order. Then $\mathbb{C}^n$ itself is the minimum element of $L(\mathcal{A})$. Let μ be the Möbius function of $L(\mathcal{A})$.

Definition 3 (see [7, Definition 2.52]). Define the characteristic polynomial of $\mathcal{A}$ by

$$\chi(\mathcal{A},t) = \sum_{X\in L(\mathcal{A})} \mu(V,X)t^{\dim X}.$$

Proposition 4.

$$\beta(\mathcal{A}) = (-1)^k \frac{d}{dt}\chi(\mathcal{A},1).$$

Proof. Since $P(\mathcal{A})$ is homotopy equivalent to the complement of the decone $d\mathcal{A}$ [7, p.15] of $\mathcal{A}$ by [7, Proposition 2.51 and Theorem 5.93], one has

$$(1+t)\operatorname{Poin}(P(\mathcal{A}),t) = \operatorname{Poin}(U,t),$$

where U is the complement of $\mathcal{A}$ and Poin stands for the Poincaré polynomial. Thus, by [7, Definition 2.52],

$$\begin{aligned}(t-1)^{-1}\chi(\mathcal{A},t) &= (t-1)^{-1}t^{\ell}\operatorname{Poin}(U,-t^{-1})\\ &= (t-1)^{-1}t^{\ell}(1-t^{-1})\operatorname{Poin}(P(\mathcal{A}),-t^{-1})\\ &= t^{\ell-1}\operatorname{Poin}(P(\mathcal{A}),-t^{-1}).\end{aligned}$$

Take the limit as t approaches 1. (Note $\chi(\mathcal{A},1) = 0$.) □

Proposition 4 shows that the beta invariant for the matroid determined by $\mathcal{A}$. The beta invariant for a matroid was introduced by Crapo [2].

Theorem 5 (see [2, Theorem 2]). (1) *If $\mathcal{A}$ is not empty, then $\beta(\mathcal{A}) \geq 0$.*
(2) *$\beta(\mathcal{A}) = 0$ if and only if $\mathcal{A}$ is decomposable.* □

Let $\mathcal{A}$ be an affine arrangement of hyperplanes in $\mathbb{C}^n$. Let L be an edge of $\mathcal{A}$.

Definition 6. An edge L is called *dense* in $\mathcal{A}$ if and only if the central arrangement

$$\mathcal{A}_L = \{A \in \mathcal{A} \mid L \subseteq A\}$$

is not decomposable.

By Theorem 5, we have

Proposition 7. *Let $L \in L(\mathcal{A})$ with* $\operatorname{codim} L = r+1$. *Then the following conditions are equivalent*:

(1) *L is dense,*
(2) *$\mathcal{A}_L$ is not decomposable,*
(3) *$\chi(P(\mathcal{A}_L)) \neq 0$,*
(4) *$\beta(\mathcal{A}_L) = (-1)^r \chi(P(\mathcal{A}_L)) > 0$.* $\square$

3. Resolution of a hyperplanelike divisor

Let Y be a smooth complex compact manifold of dimension n, $\mathcal{D}$ a divisor. $\mathcal{D}$ is *hyperplanelike* if Y can be covered by coordinate charts such that in each chart $\mathcal{D}$ is a union of hyperplanes. Such charts will be called *linearizing*.

Let $\mathcal{D}$ be a hyperplanelike divisor, U a linearizing chart. A *local edge* of $\mathcal{D}$ in U is any nonempty irreducible intersection in U of hyperplanes of $\mathcal{D}$ in U. An *edge* of $\mathcal{D}$ is the maximal analytic continuation in Y of a local edge. Any edge is an immersed submanifold in Y. An edge is called *dense* if it is locally dense.

For $0 \le j \le n-2$, let $\mathcal{L}_j$ be the collection of all dense edges of $\mathcal{D}$ of dimension j. The following theorem is essentially in [10, 10.8].

Theorem 8. *Let $W_0 = Y$. Let $\pi_1 : W_1 \to W_0$ be the blow up along points in $\mathcal{L}_0$. In general, for $1 \le s \le \ell - 1$, let $\pi_s : W_s \to W_{s-1}$ be the blow up along the proper transforms of the $(s-1)$-dimensional dense edges in $\mathcal{L}_{s-1}$ under $\pi_1 \circ \cdots \circ \pi_{s-1}$. Let $\pi = \pi_1 \circ \cdots \circ \pi_{n-1}$. Then $W = W_{n-1}$ is nonsingular and $\pi^{-1}(\mathcal{D})$ normal crossing.* $\square$

4. Arrangements in $\mathbf{P}^n$

Let $\{H_i\}_{i\in I}$ be an affine arrangement of hyperplanes in $\mathbb{C}^n$. Recall U, f_i, ω_i, P_i, ω, $\mathcal{E}$, and $A^\bullet$ from Section 1. Choose $\mathbf{P}^n$ as the compactification of $\mathbb{C}^n$. Let $H_\infty = \mathbf{P}^n - \mathbb{C}^n$ and $\mathcal{A} = \{\overline{H}_i\}_{i\in I} \cup \{H_\infty\}$. ($\overline{H}_i$ is the closure of H_i in $\mathbf{P}^n$.) Obviously $(\bigcup_{i\in I} \overline{H}_i) \cup H_\infty$ is a hyperplanelike divisor. Suppose $(z_0 : \cdots : z_n)$ be a homogeneous coordinate system with $H_\infty : z_0 = 0$. Then each ω_i is uniquely extended to a rational form $\overline{\omega}_i$ on $\mathbf{P}^n$; $\overline{\omega}_i = \omega_i - (dz_0/z_0)$. Thus the form $\omega = \sum_{i\in I} \omega_i \otimes P_i \in \Gamma(U, \Omega_U^1) \otimes \operatorname{End} \mathbb{C}^s$. can be uniquely extended to $\overline{\omega}$:

$$\overline{\omega} = \sum_{i\in I} \overline{\omega}_i \otimes P_i = \sum_{i\in I} \omega_i \otimes P_i - (dz_0/z_0) \otimes \Big(\sum_{i\in I} P_i\Big).$$

Define $P_\infty = -\sum_{i\in I} P_i$. For any edge L of $\mathcal{A}$, let $I_L = \{i \in I \cup \{\infty\} \mid L \subseteq H_i\}$. Let $P_L = \sum_{i\in I_L} P_i$. By Theorems 1 and 8, we get

Theorem 9. *We set $\mathcal{L}$ be the set of all dense edges of $\mathcal{A}$. Suppose that*

(Mon)* *for all $L \in \mathcal{L}$, none of the eigenvalues of P_L lies in* $\mathbb{N} - \{0\}$.

Then the inclusion

$$A^\bullet \hookrightarrow \Gamma(U, \Omega_U^\bullet(\mathcal{E}))$$

is a quasi-isomorphism. □

Remark. Since "dense" implies "bad" [3], Theorem 9 improves the main theorem of [3].

Corollary 10. *Under the assumption of Theorem 9, one has*

$$H^p(U, \mathcal{S}) \cong H^p(A^\bullet) \quad \text{for } 0 \le p \le n,$$

where $\mathcal{S}$ is the local system of flat sections of $(\mathcal{E}, d + \omega)$ on U. □

Corollary 11. *Suppose that*

(Mon)** *for all $L \in \mathcal{L}$, none of the eigenvalues of P_L lies in* $\mathbb{N} \cup \{0\}$.

Also suppose that $P_i P_j = P_j P_i$ for all i, j. Then

$$H^p(U, \mathcal{S}) = 0 \quad \text{for } p \ne n.$$

Proof. By Theorem 9 and [11, 4.1]. □

5. Discriminantal arrangements in $(\mathbf{P}^1)^n$

See [8] for discriminantal arrangements.

Let Γ be a graph without loops with vertices $v_1, \dots, v_p$. Let $n_1, \dots, n_r$ be nonnegative integers, $n = n_1 + \cdots + n_r$, $X = \{(i, \ell) \mid \ell = 1, \dots, r,\ i = 1, \dots, n_\ell\}$, $Y = (\mathbf{P}^1)^n$. Label the factors of Y by elements of X and for every $(i, \ell) \in X$ fix an affine coordinate $t_i(\ell)$ on the (i, ℓ)-th factor.

For pairwise distinct $z_1, \dots, z_k \in \mathbb{C}$, $z_{k+1} = \infty$, introduce in Y a *discriminantal arrangement* $\mathcal{A}$ of "hyperplanes"

$$H_{(i,\ell),j}\colon\ t_i(\ell) = z_j \quad \text{for } (i, \ell) \in X,\ j = 1, \dots, k+1,$$

$$H_{(i,\ell),(j,\ell)}\colon\ t_i(\ell) = t_j(\ell) \quad \text{for } 1 \le i < j \le n_\ell,$$

and

$$H_{(i,\ell),(j,m)}\colon\ t_i(\ell) = t_j(m)$$

for ℓ, m such that v_ℓ and v_m are joined by an edge in the graph and $i = 1, \dots, n_\ell$, $j = 1, \dots, n_m$. The union of these "hyperplanes" is a hyperplanelike divisor. Let $\Delta \subseteq \Gamma$

be a connected subgraph with vertices labelled by $V \subseteq \{1,\dots,r\}$. For every $\ell \in V$ fix a nonempty subset $I_\ell \subseteq \{1,\dots,n_\ell\}$. Fix $j \in \{1,\dots,k+1\}$. Introduce edges

$$L(\{I_\ell\},j) = \{t \in Y \mid t_i(\ell) = z_j \text{ for } \ell \in V,\, i \in I_\ell\}.$$

Next assume that the graph Δ remains connected after any vertex $\ell \in V$ with $|I_\ell| = 1$ is removed. Under these assumptions, define edges

$$L(\{I_\ell\}) = \{t \in Y \mid t_i(\ell) = t_h(\ell),\, t_i(\ell) = t_g(m) \text{ for } \ell, m \in V;\, i,h \in I_\ell;\, g \in I_m\}.$$

Proposition 12. (1) *$L(\{I_\ell\},j)$, and $L(\{I_\ell\})$ are dense.*

(2) *Every dense edge has the form above.*

Proof. For any graph G with vertices $\{1,\dots,m\}$ and edges E, associate a central arrangement $\mathcal{A}_G$ in $\mathbb{C}^m$ consisting of $\{x_i = 0 \mid 1 \le i \le m\}$ and $\{x_i = x_j \mid \{i,j\} \in E\}$. Define a central arrangement $\mathcal{B}_G$ consisting of $\{x_i = x_j \mid \{i,j\} \in E\}$. (The arrangement $\mathcal{B}_G$ is called a graphic arrangement [7, 2.4].) In order to prove (1) and (2), it is enough to show the following lemma:

Lemma 13. (a) *$\mathcal{A}_G$ is not decomposable iff G is connected.*

(b) *$\mathcal{B}_G$ is not decomposable iff G is 2-connected, that is, G remains connected after any vertex is removed.*

Proof. (a) If G is disconnected, $\mathcal{A}_G$ is obviously decomposable. If G is connected, let T be a maximal tree inside G. Choose an edge $\{i,j\}$ such that j is a terminal point of T. Let $\mathcal{A}'$ and $\mathcal{A}''$ be the deletion and the restriction of $\mathcal{A}_T$ with respect to the hyperplane $\{x_i = x_j\}$. Since $\beta(\mathcal{A}') + \beta(\mathcal{A}'') = \beta(\mathcal{A}_T)$ [2, Theorem 1], we can prove $\beta(\mathcal{A}_T) = 1$ for any tree by induction on the number of edges. This shows $\beta(\mathcal{A}_G) \ge \beta(\mathcal{A}_T) = 1$.

(b) Note that the matroid associated with the arrangement $\mathcal{B}_G$ is the same as the matroid associated with the graph G. The matroid is connected if and only if G is 2-connected [9]. □

Let $\mathbb{C}^n = Y - \bigcup_{(i,\ell)\in X} H_{(i,\ell),k+1}$. Let U be the complement in Y to the union of "hyperplanes" of $\mathcal{A}$. Recall $f_i, \omega_i, P_i, \omega, \mathcal{E}$, and $A^\bullet$ from Section 1. ω can be uniquely extended to be an $\operatorname{End}\mathbb{C}^s$-coefficient rational 1-form $\overline{\omega}$ on Y. For $(i,\ell) \in X$ the residue of $\overline{\omega}$ at $H_{(i,\ell),k+1}$ is

$$P_{(i,\ell),k+1} = -\sum_{j=1}^{k} P_{(i,\ell),j} - \sum_{\substack{j=1\\ j\ne i}}^{n_\ell} P_{(j,\ell),(i,\ell)} - \sum P_{(i,\ell),(j,m)},$$

where the last sum is over all m such that v_ℓ and v_m are joined by an edge in Γ and $j = 1,\dots,n_m$.

For any edge L in $\mathcal{A}$, let P_L be the sum of residues of $\overline{\omega}$ at all "hyperplanes" of $\mathcal{A}$ containing L.

Theorem 14. *Let $\mathcal{L}$ be the set of dense edges of $\mathcal{A}$. Suppose that*

(Mon)* *for all $L \in \mathcal{L}$, none of the eigenvalues of P_L lies in* $\mathbb{N} - \{0\}$.

Then the inclusion

$$A^\bullet \hookrightarrow \Gamma(U, \Omega_U^\bullet(\mathcal{E}))$$

is a quasi-isomorphism. □

Corollary 15. *Suppose that*

(Mon)** *for all $L \in \mathcal{L}$, none of the eigenvalues of P_L lies in* $\mathbb{N} \cup \{0\}$.

Also suppose that $P_i P_j = P_j P_i$ for all i, j. Then

$$H^p(U, \mathcal{S}) = 0 \quad \text{for } p \neq n. \qquad \square$$

6. Kac–Kazhdan conditions

Let $\mathcal{G}$ be a finite-dimensional simple complex Lie algebra with Chevalley generators e_i, f_i, h_i, $i = 1, \dots, r$. Let $\mathcal{G} = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+$ be the corresponding Cartan decomposition; $\alpha_1, \dots, \alpha_r \in \mathcal{H}^*$ the simple roots, θ the highest root. Let $(-,-)$ be the symmetric non-degenerate bilinear form on $\mathcal{G}$ such that $(\theta, \theta) = 2$.

Let T be an independent variable, $\mathbb{C}[T]$ the ring of polynomials, $\mathbb{C}[T, T^{-1}]$ the ring of Laurent polynomials. For $f(T), g(T) \in \mathbb{C}[T, T^{-1}]$, set

$$\mathrm{res}_0(f(T)dg(T)) = \text{coefficient at } T^{-1} \text{ in } f(T)g'(T).$$

The space $\mathcal{G} \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$ is a Lie algebra with bracket

$$[b \otimes f(T), c \otimes g(T)] = [b, c] \otimes f(T)g(T)$$

for $b, c \in \mathcal{G}$. Define $\hat{\mathcal{G}}$ as a central extension of $\mathcal{G} \otimes_{\mathbb{C}} \mathbb{C}[T, T^{-1}]$,

$$\hat{\mathcal{G}} = \mathcal{G} \otimes \mathbb{C}[T, T^{-1}] \oplus \mathbb{C}K,$$

where K is a central element of $\hat{\mathcal{G}}$, and

$$[b \otimes f(T), c \otimes g(T)] = [b, c] \otimes f(T)g(T) + (b, c)\, \mathrm{res}_0(f(T)dg(T))K.$$

Set $\hat{\mathcal{G}}^+ = \mathcal{G} \otimes \mathbb{C}[T] \oplus \mathbb{C}K$; it is a Lie subalgebra of $\hat{\mathcal{G}}$.

Fix a complex number k. Set $\kappa = k + g$ where g is the dual Coxeter number of $\mathcal{G}$, cf. [5, 6.1].

For $\Lambda \in \mathcal{H}^*$, let $M(\Lambda)$ be the Verma module over $\mathcal{G}$ with highest weight Λ. Consider $M(\Lambda)$ as a $\hat{\mathcal{G}}^+$-module by setting $\mathcal{G} \otimes T\mathbb{C}[T]$ to act as zero and K as multiplication by k. Set

$$\hat{M}(\Lambda) = U(\hat{\mathcal{G}}) \otimes_{U(\hat{\mathcal{G}}^+)} M(\Lambda).$$

It is a Verma module over $\hat{\mathcal{G}}$.

Proposition 16 (Kac–Kazhdan conditions). *$\hat{M}(\Lambda)$ is reducible if and only if at least one of the following three conditions is satisfied.*

(1) $\kappa = 0$.

(2) *There exist a positive root α of $\mathcal{G}$ and natural numbers $p, s \in \mathbb{N} - \{0\}$ such that*

$$(\Lambda, \alpha) + (\rho, \alpha) = p\frac{(\alpha, \alpha)}{2} - (s-1)\kappa,$$

where ρ is half-sum of positive roots of $\mathcal{G}$.

(3) *There exist a positive root α of $\mathcal{G}$ and natural numbers $p, s \in \mathbb{N} - \{0\}$ such that*

$$(\Lambda, \alpha) + (\rho, \alpha) = -p\frac{(\alpha, \alpha)}{2} + s\kappa.$$

Proof. We use notations of [5, Chapters 6 and 7]. In these notations the Kac–Kazhdan reducibility condition, [6, Theorem 1], reads as

$$\langle \Lambda, \nu^{-1}(\beta)\rangle + \langle \hat{\rho}, \nu^{-1}(\beta)\rangle - p\frac{(\beta, \beta)}{2} = 0$$

for some positive root β of $\hat{\mathcal{G}}$ and a positive integer p. (Here we denoted by $\hat{\rho}$ an element denoted by ρ in [5], to distinguish it from our ρ.)

By [5, 6.3], every such β has one of the following forms: (1) $\beta = m\delta$, $m > 0$; (2) $\beta = \alpha + m\delta$, $m \geq 0$; (3) $\beta = -\alpha + m\delta$, $m > 0$, where α is a positive root of $\mathcal{G}$, m an integer. From [5] it follows easily that $\langle \Lambda, \nu^{-1}(\delta)\rangle = k$, $\langle \hat{\rho}, \nu^{-}(\delta)\rangle = g$ and $\langle \hat{\rho}, \nu^{-1}(\alpha)\rangle = (\rho, \alpha)$. This implies the proposition. □

Let w be the longest element of the Weyl group of $\mathcal{G}$. For $\Lambda \in \mathcal{H}^*$, set $\Lambda' = -w(\Lambda)$.

Proposition 17. *$\hat{M}(\Lambda')$ is reducible if and only if $\hat{M}(\Lambda)$ is reducible. The Kac–Kazhdan conditions for Λ' expressed in terms of Λ coincide with the Kac–Kazhdan conditions for Λ.*

Proof. For a positive root α, $-w(\alpha)$ is a positive root. This implies the proposition. □

7. Resonances of discriminantal arrangements

Let Γ be the Dynkin diagram of a complex simple Lie algebra $\mathcal{G}$. The vertices of the diagram are labelled by simple roots $\alpha_1, \ldots, \alpha_r$ of the algebra. Let $n_1, \ldots, n_r$ be nonnegative integers, $n = n_1 + \cdots + n_r$. For pairwise distinct $z_1, \ldots, z_k \in \mathbb{C}$, $z_{k+1} = \infty$, consider in $Y = (\mathbf{P}^1)^n$ the discriminantal arrangement $\mathcal{A}$ associated to these data.

Let $\Lambda_1, \ldots, \Lambda_k \in \mathcal{H}^*$. Set $\Lambda_{k+1} = -\omega(\Lambda_1 + \cdots + \Lambda_k - n_1\alpha_1 - \cdots - n_r\alpha_r)$. Fix a nonzero complex number κ. Introduce an integrable connection $d + \omega$ on the trivial bundle $\mathcal{E} = \mathcal{O}_U$ with

$$\omega = \sum_{(i,\ell)\in X} \sum_{j=1}^{k} P_{(i,\ell),j}\omega_{(i,\ell),j} + \sum_{\ell=1}^{r} \sum_{1\le i<j\le n_\ell} P_{(i,\ell),(j,\ell)}\omega_{(i,\ell),(j,\ell)}$$
$$+ \sum_{1\le \ell<m\le r} \sum_{i=1}^{n_\ell}\sum_{j=1}^{n_m} P_{(i,\ell),(j,m)}\omega_{(i,\ell),(j,m)},$$

where

$$\omega_{(i,\ell),j} = d(t_i(\ell) - z_j)/(t_i(\ell) - z_j),$$
$$\omega_{(i,\ell),(j,m)} = d(t_i(\ell) - t_j(m))/(t_i(\ell) - t_j(m)),$$
$$P_{(i,\ell),j} = -(\alpha_\ell, \Lambda_j)/\kappa, \qquad P_{(i,\ell),(j,m)} = -(\alpha_\ell, \alpha_m)/\kappa,$$

see [8] and [10]. ω extends to be a rational 1-form $\overline{\omega}$ on Y.

For any edge L in $\mathcal{A}$, let P_L be the sum of residues of $\overline{\omega}$ at all "hyperplanes" of $\mathcal{A}$ containing L. For $p \in \mathbb{N}\cup\{0\}$, we say that the connection $d+\omega$ has a *resonance at L of level p*, if $P_L = p$.

The following theorem connects resonances of $\mathcal{A}$ with the Kac–Kazhdan conditions for the Verma modules $\hat{M}(\Lambda_1),\dots,\hat{M}(\Lambda_{k+1})$ of the affine algebra $\hat{\mathcal{G}}$. Let $\alpha = \sum a_\ell\alpha_\ell$ be a positive root of $\mathcal{G}$, p a natural number. Assume that $a_\ell p \le n_\ell$ for all ℓ. For every ℓ, fix a subset $I_\ell \subseteq \{1,\dots,n_\ell\}$ consisting of $a_\ell p$ elements.

Theorem 18. (1) *For every $j = 1,\dots,k+1$, the edge $L_j = L(\{I_\ell\}, j)$ is dense.*

(2) *For $j = 1,\dots,k$ and every natural number s, the resonance condition at L_j of level ps, $P_{L_j} = ps$, coincides with the Kac–Kazhdan condition of type* (2) *for $\hat{M}(\Lambda_j)$,*

$$(\Lambda_j,\alpha) + (\rho,\alpha) = p\frac{(\alpha,\alpha)}{2} - s\kappa.$$

(3) *For $j = k+1$ and every natural number s, the resonance condition at L_{k+1} of level ps, $P_{L_{k+1}} = ps$, coincides with the Kac–Kazhdan condition of type* (3) *for $\hat{M}(\Lambda_{k+1})$,*

$$(\Lambda_{k+1},\alpha) + (\rho,\alpha) = -p\frac{(\alpha,\alpha)}{2} + s\kappa.$$

Remarks. (1) For resonance values of $\Lambda_1,\dots,\Lambda_k,\kappa$, nontrivial cohomological relations occur in the image of $A^\bullet \subset \Gamma(U,\Omega_U(\mathcal{E}))$. The theorem suggests that the relations correspond to singular vectors in the Verma modules $\hat{M}(\Lambda_1),\dots,\hat{M}(\Lambda_{k+1})$. In [4] this correspondence was established for the simplest singular vector in $\hat{M}(\Lambda_{k+1})$, the correspondence implied algebraic equations satisfied by conformal blocks in the WZW model of conformal field theory.

(2) For $j = 1,\dots,k$ and natural number p, the Kac–Kazhdan condition, $(\Lambda_j,\alpha) + (\rho,\alpha) = p\frac{(\alpha,\alpha)}{2}$, appears as a degeneration condition for a certain contravariant form of the arrangement $\mathcal{A}$, see [8, Sections 3 and 6].

Proof. (1) For a positive root $\alpha = \sum a_\ell \alpha_\ell$ consider the subset $\{\alpha_\ell \mid a_\ell > 0\}$ of the set of simple roots. The subset distinguishes a subgraph of the Dynkin diagram. The subgraph is connected [1, Chapter 7, Section 1]. Now L_j is dense by Proposition 12.

The following proves (2).

$$\begin{aligned} P_{L_j} - ps &= \frac{1}{\kappa}\Big[(-\Lambda_j,\alpha)p + \sum_{r=1}^{r} \frac{pa_\ell(pa_\ell - 1)}{2}(\alpha_\ell,\alpha_\ell) \\ &\quad + \sum_{1\le \ell<m\le r} pa_\ell pa_m(\alpha_\ell,\alpha_m)\Big] - ps \\ &= \frac{p}{\kappa}\Big[-(\Lambda_j,\alpha) + p\frac{(\alpha,\alpha)}{2} - \sum_{\ell=1}^{r} a_\ell \frac{(\alpha_\ell,\alpha_\ell)}{2} - s\kappa\Big] \\ &= \frac{p}{\kappa}\Big[-(\Lambda_j,\alpha) - (\rho,\alpha) + p\frac{(\alpha_\ell,\alpha_\ell)}{2} - s\kappa\Big]. \end{aligned}$$

Part (3) is proved by similar direct computations using Proposition 17. □

Acknowledgement

The authors thank Hélène Esnault for pointing out an error in an earlier version. The second author thanks G. Ziegler for directing his attention to the work of H. Crapo about the beta invariant. He is also thankful to A. Libgober and S. Yuzvinsky for useful discussions.

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