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Fuchsian bispectral operators

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Abstract

The aim of this paper is to classify the bispectral operators of any rank with regular singular points (the infinite point is the most important one). We characterise them in several ways. Probably the most important result is that they are all Darboux transformations of powers of generalised Bessel operators (in the terminology of [4]). For this reason they can be effectively parametrised by the points of a certain (infinite) family of algebraic manifolds as pointed out in [4].

0. Introduction

The present paper is devoted to the characterisation and the classification of bispectral operators of any rank and order with only regular singularities. Before stating our results and placing them properly amongst the other research we would like to give few definitions and to recall some of the fundamental results in the area.

An ordinary differential operator $L(x, \partial_x)$ is called bispectral if it has an eigen-function $\psi(x, z)$, depending also on the spectral parameter z, which is at the same time an eigenfunction of another differential operator $\Lambda(z, \partial_z)$ now in the spectral parameter z. In other words we look for operators L, Λ and a function $\psi(x, z)$ satisfying equations of the form:

$$L\psi = f(z)\psi, \tag{0.1}$$

$$\Lambda\psi = \theta(x)\psi. \tag{0.2}$$

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Initially the study of bispectral operators has been stimulated by certain problems of computer tomography (cf. [19,20]). Later it turned out that the bispectral operators are connected to several actively developing areas of mathematics and physics – the HP-hierarchy, infinite-dimensional Lie algebras and their representations, particle systems, automorphisms of algebras of differential operators, etc. (see, e.g., [4,7,8,12,17,26,31,32], as well as the papers in the proceedings volume of the conference in Montréal [10]). There are also indications for eventual connections with non-commutative algebraic geometry [33].

In the fundamental paper [17] Duistermaat and Grünbaum raised the problem to find all bispectral operators and completely solved it for operators L of order two. The complete list is as follows. If we present L as a Schrödinger operator

$$L = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + u(x),$$

the bispectral operators, apart from the obvious Airy $(u(x) = ax)$ and Bessel $(u(x) = cx^{-2})$ ones, are organised into two families of potentials $u(x)$, which can be obtained by finitely many "rational Darboux transformations"

(1) from $u(x) = 0$,
(2) from $u(x) = -(\frac{1}{4})x^{-2}$.

Thus the classification scheme prompted by the paper [17] is by the order of the operators. G. Wilson [31] introduced another classification scheme – by the rank of the bispectral operator L. We recall that *the rank* of the operator L is the dimension of the space of the joint eigenfunctions of all operators commuting with L. For example, all the operators of the family (1) have rank 1, while those of the family (2) have rank 2, In the above cited paper [31] (see also [32]) Wilson gave a complete description of all bispectral operators of rank 1 (and any order). In the terminology of Darboux transformations (see [4]) all bispectral operators of rank 1 are those obtained by rational Darboux transformations on the operators with constant coefficients, i.e. polynomials $p(\partial_x)$. Several beautiful connections of the bispectral operators with KdV- and KP-hierarchies, algebraic curves and Calogero–Moser particle systems have also been revealed in [17,31,32].

We will not touch upon all results in the papers [17,31] but we would like to point that in both of them the classification is split into two, more or less independent parts. First, there is an explicit construction of families of bispectral operators of a given class (order 2 in [17]; rank 1 in [31]) The construction can be given in terms of Darboux transformations of "canonical" operators. The second part is to give a proof that, if an operator (in the corresponding class) is a bispectral one, then it belongs to the constructed families.

In several other papers devoted to the bispectral problem (see [20,24,34]) the authors deal with an analog of the first part of the problem, i.e. they construct new families of bispectral operators. The most complete results in that direction have been obtained in [4, 7]. To the best of our knowledge, all known up to now families of bispectral operators can be constructed by the methods of the latter papers. A challenging problem is to prove that all the bispectral operators have already been found. A natural approach would be to divide the differential operators into suitable classes, e.g. – by order as in [17] or by rank and to try to isolate the bispectral ones amongst them. But having in mind the constructions of the fundamental papers [17,31], with their different and quite involved methods, the complete classification seems to be a difficult and lengthy project. One may try to consider the operators with a fixed type of singularity at infinity. Obviously, then there arises another difficult problem – to determine what restrictions on the kinds of singularities are imposed by the condition of bispectrality.

In the present paper we consider the class of operators with regular singularities at infinity. In fact the main results sound much stronger. To explain them we introduce some definitions and notations which will be used also throughout the paper. We are going to consider operators, normalized as follows:

$$L = \sum_{k=0}^{N} V_k(x)\partial_x^k, \tag{0.3}$$

where the coefficient at the highest derivative $V_N = 1$ and the next coefficient $V_{N-1} = 0$. Now our assumption is that

$$\lim V_j(x) = 0, \quad j = 0, \ldots, N-1 \text{ when } x \to \infty. \tag{0.4}$$

(It is well known that with the above normalization all coefficients of L are rational functions (see [17,31]) and hence (0.4) makes sense.)

Important examples of such operators are the generalized Bessel operators. As we are going to use them throughout the paper we recall the definition. Introduce the notation $D = x\partial_x$.

Definition 0.1. Generalized Bessel operators L_β are the operators

$$L_\beta = x^{-N}(D-\beta_1)\cdots(D-\beta_N), \quad (\beta_1, \ldots, \beta_N) \in \mathbb{C}^N. \tag{0.5}$$

In what follows we will call the above operators by abuse of terminology (but for simplicity) *Bessel operators*.

After this preparation we can formulate the result which is the core of the present paper.

Theorem 0.2. *Let L be a bispectral operator* (0.3) *with coefficients satisfying* (0.4). *Then L is a monomial Darboux transformation of a Bessel operator.*

The class defined by (0.3) and (0.4) includes essentially all the bispectral operators found in [17]: the Bessel operators and both of the families (1) and (2), the only exception being the Airy operator. On the other hand it includes one of the most interesting classes, found in [4]. These are the operators obtained by Darboux transformations on powers of the Bessel operators. This class was later characterized as follows. In [5] there have been constructed highest weight modules $\mathcal{M}_\beta$ with highest weight vectors – the corresponding to (0.5) τ-functions τ_β. Then in [8] it is shown that the τ-functions in the modules $\mathcal{M}_\beta$ are exactly the τ-functions of the operators which are monomial Darboux transformations.

In the course of performing the proof of Theorem 0.2 we show that the assumptions (0.3) and (0.4) for the bispectral operator L impose further restrictions on it, which justify partially the title.

Theorem 0.3. *If the bispectral operator* (0.3) *satisfies* (0.4)*, then the point* $x = \infty$ *is a regular singular point.*

The proof of this theorem is probably the most involved part of our constructions (see Section 3). The regularity of the finite points follows indirectly from Theorem 0.2.

In Section 4 we give another characterization of the bispectral operators (0.3) with the restriction (0.4).

Theorem 0.4. *Any rank* r *bispectral operator* L *is* $\mathbb{Z}_r$*-invariant.*

The result is interesting and natural by itself (cf. [4,17]) but in the present paper it is also the next step in our final goal.

Finally in Section 6, putting together the different pieces of our construction in the preceding sections and using the main results of [4,8] we obtain the following complete characterization of the Fuchsian bispectral operators.

Theorem 0.5. *The following conditions on the operator* L *in the form* (0.3) *are equivalent*:

(1) L *is bispectral and satisfies* (0.4);
(2) L *is bispectral and has only regular singular points* (*i.e.,* L *is Fuchsian*);
(3) L *is a monomial Darboux transformation of a Bessel operator* (0.5);
(4) *the corresponding to* L τ*-function belongs to one of the modules* $\mathcal{M}_\beta$.

In the case when the order of L is two the equivalence between (1) and (3) contains two of the most important (and difficult) theorems of [17], concerning the families (1) and (2) above. In that sense the present paper represents their direct generalization.

The methods which we utilize have some resemblance to the ones used in [17]. In particular the Darboux transformations constitute one of the main steps of our proof. But as a whole we use different ideas. First, we work essentially with the algebraic structure of different rings of differential or pseudo-differential operators. Essentially we do not use the wave function as in [17]. This we achieve by using the bispectral involutions on pseudo-differential operators in Section 2. In the same section we observe that a bispectral operator L (with the restrictions (0.3) and (0.4)) satisfies a variant of the so-called "string equation":

$$[L, Q] = NL^{n+1}, \tag{0.6}$$

where Q is an operator built out of L. Eq. (0.6) prepares us to use certain techniques from differential algebra in order to study the singular point of L at infinity. In particular we use the methods invented by J. Dixmier [16] in his studies on the Weyl algebra. Roughly speaking one associates with each differential operator L a quasi-homogeneous polynomial $p_L(X, Y)$ in such a way that it contains the information about the "worst" terms of L (in our case these are the most irregular ones). See [16] and Section 3 for more details. Then in the same section the analysis of $p_L(X, Y)$ shows that the assumption of irregularity of the point $x = \infty$ is incompatible with the string equation (0.6).

The techniques from Section 2 is used also in Section 4 to prove that the rank r of the operator L imposes its $\mathbb{Z}_r$-invariantness. Using it and the fact that the infinite point is regular it is easy to perform $\mathbb{Z}_r$-invariant Darboux transformations on L in order to reduce the number n in the string equation (0.6) to 0. This automatically gives that the operator obtained in this way is a Bessel operator.

At the end of the introduction we point out that our method treats all ranks and orders in one scheme. We expect that some of its components can be useful in other classification problems.

1. Preliminaries

In this section we have collected some terminology, notations and results relevant for the study of bispectral operators. Our main concern is to introduce unique notation which will be used throughout the paper and to make the paper self contained. There are also few results which cannot be found formally elsewhere, but in fact are reformulations (in a suitable for the present paper form) of statements from other sources.

1.1. In this subsection we recall some definitions, facts and notation from Sato's theory of KP-hierarchy [14,28,29] needed in the paper. For a complete presentation of the theory we recommend also [15,30]. We start with the notion of *the wave operator* $K(x, \partial_x)$. This is a pseudo-differential operator

$$K(x, \partial_x) = 1 + \sum_{j=1}^{\infty} a_j(x)\partial_x^{-1}, \tag{1.1}$$

with coefficients $a_j(x)$ which could be convergent or formal power (Laurent) series. In the present paper we will consider a_j most often as formal Laurent series in x^{-1}. The wave operator defines the (stationary) Baker–Akhiezer function $\psi(x, z)$:

$$\psi(x, z) = K(x, \partial_x)e^{xz}. \tag{1.2}$$

From (1.1) and (1.2) it follows that ψ has the following asymptotic expansion:

$$\psi(x, z) = e^{xz}\left(1 + \sum_{1}^{\infty} a_j(x)z^{-j}\right), \quad z \to \infty. \tag{1.3}$$

Introduce also the pseudo-differential operator P:

$$P(x, \partial_x) = K\partial_x K^{-1}. \tag{1.4}$$

The following spectral property of P, crucial in the theory of KP-hierarchy, is also very important for the bispectral problem:

$$P\psi(x, z) = z\psi(x, z). \tag{1.5}$$

When it happens that some power of P, say P^N, is a differential operator, we get that $\psi(x,z)$ is an eigenfunction of an ordinary differential operator $L = P^N$:

$$L\psi = z^N\psi. \tag{1.6}$$

It is possible to introduce the above objects in many different ways, starting with any of them (and with other, not introduced above). For us it would be important also to start with given *differential operator* L:

$$L(x,\partial_x) = \partial_x^N + V_{N-2}(x)\partial^{N-2} + \cdots + V_0(x). \tag{1.7}$$

One can define the pseudo-differential operator P as an Nth root of the operator L:

$$P = L^{1/N} = \partial + \cdots, \tag{1.8}$$

and the wave operator K as:

$$LK = L\partial^N. \tag{1.9}$$

An important notion, connected to an operator L is the algebra $\mathcal{A}_L$ of operators commuting with L (see [11,25]). This algebra is commutative one. The wave function $\psi(x,z)$ (defined in (1.2)) is a common wave function for all operators M from $\mathcal{A}_L$:

$$M\psi(x,z) = g_M(z)\psi(x,z). \tag{1.10}$$

We define also the algebra $\mathcal{A}_L$ of all functions $g_M(z)$ for which (1.10) holds for some $M \in \mathcal{A}_L$. Obviously the algebras A_L and $\mathcal{A}_L$ are isomorphic.

Following [25] we introduce *the rank of the algebra* $\mathcal{A}_L$ as the greatest common divisor of the orders of the operators in $\mathcal{A}_L$.

1.2. Here we shall briefly recall the definition of Bessel wave function and of monomial Darboux transformations from it. For more details see [4]. Let $\beta \in \mathbb{C}^N$ be such that

$$\sum_{i=1}^{N} \beta_i = \frac{N(N-1)}{2}. \tag{1.11}$$

Definition 1.1 ([4,18,34]). *Bessel wave function* is called the unique wave function $\Psi_\beta(x,z)$ depending only on xz and satisfying

$$L_\beta(x,\partial_x)\Psi_\beta(x,z) = z^N\Psi_\beta(x,z), \tag{1.12}$$

where the Bessel operator $L_\beta(x,\partial_x)$ is given by (0.5).

Because the Bessel wave function depends only on xz, (1.12) implies

$$D_x\Psi_\beta(x,z) = D_z\Psi_\beta(x,z), \tag{1.13}$$

$$L_\beta(z,\partial_z)\Psi_\beta(x,z) = x^N\Psi_\beta(x,z). \tag{1.14}$$

To introduce the monomial Darboux transformations of Bessel wave functions we first recall the definition of polynomial Darboux transformations given in [4].

Definition 1.2. We say that the wave function Ψ is a *Darboux transformation* of the Bessel wave function $\Psi_\beta(x,z)$ iff there exist polynomials $f(z)$, $g(z)$ and differential operators $P(x,\partial_x)$, $Q(x,\partial_x)$ such that

$$\Psi = \frac{1}{g(z)} P(x, \partial_x) \Psi_\beta(x, z), \tag{1.15}$$

$$\Psi_\beta(x, z) = \frac{1}{f(z)} Q(x, \partial_x) \Psi. \tag{1.16}$$

The Darboux transformation is called *polynomial* iff the operator $P(x, \partial_x)$ from (1.15) has the form

$$P(x, \partial_x) = x^{-n} \sum_{k=0}^{n} p_k\big(x^N\big) D_x^k, \tag{1.17}$$

where p_k are rational functions, $p_n \equiv 1$.

We will need the following two definitions of monomial Darboux transformations. Their equivalence is proved in [4].

Definition 1.3. We say that the wave function $\Psi(x, z)$ is a *monomial Darboux transformation* of the Bessel wave function $\Psi_\beta(x, z)$ iff it is a polynomial Darboux transformation of $\Psi_\beta(x, z)$ with $g(z) f(z) = z^{dN}$, $d \in \mathbb{N}$. Further the differential operator

$$L = \partial^M + V_{M-2}\partial^{M-2} + \cdots + V_0$$

is a monomial Darboux transformation of L_β if the wave function corresponding to L is a monomial Darboux transformation of the wave function corresponding to L_β.

Definition 1.4. The wave function $\Psi(x, z)$ is a *monomial Darboux transformation* of the Bessel wave function $\Psi_\beta(x, z)$ iff (1.17) holds with $g(z) = z^n$, $n = \operatorname{ord} P$ and the kernel of the operator $P(x, \partial_x)$ has a basis consisting of several groups of the form

$$\partial_y^l \left(\sum_{k=0}^{k_0} \sum_{j=0}^{\operatorname{mult}(\beta_i + kN) - 1} b_{kj} x^{\beta_i + kN} y^j \right) \Bigg|_{y = \ln x}, \quad 0 \leqslant l \leqslant j_0, \tag{1.18}$$

where $\operatorname{mult}(\beta_i + kN) :=$ multiplicity of $\beta_i + kN$ in $\bigcup_{j=1}^N \{\beta_j + N\mathbb{Z}_{\geqslant 0}\}$ and $j_0 = \max\{j \mid b_{kj} \neq 0 \text{ for some } k\}$.

From Definitions 1.2 and 1.3 one immediately obtains the following description of monomial Darboux transformations:

Lemma 1.5. *The differential operator L is a monomial Darboux transformation of the Bessel operator L_β iff there are differential operators $P = P(x, \partial_x)$, $Q = Q(x, \partial_x)$ and numbers d, d' such that*

$$Q(x, \partial_x) P(x, \partial_x) = L_\beta(x, \partial_x)^d, \tag{1.19}$$

$$P(x, \partial_x) Q(x, \partial_x) = L(x, \partial_x)^{d'}, \tag{1.20}$$

where the operator P satisfies (1.17).

We will also reformulate some results from [4]. In [4] one can find a proof of the following statement.

Lemma 1.6. *If L_β is a Bessel operator of order N and rank r, there exists a Bessel operator $L_{\beta'}$ of order r such that L_β is a monomial Darboux transformation of $L_{\beta'}$.*

For the proof of this lemma see the proof of Proposition 2.4 from [4] (although the statement there is formulated in a different way). We end this subsection by reformulating (in a weaker form) the main result, which we need from [4].

Theorem 1.7. *The monomial Darboux transformations of the Bessel operators are bispectral operators.*

1.3. Here we recall several simple properties of bispectral operators following [17, 31]. As we have already mentioned in the introduction we are going to study ordinary differential operators L of arbitrary order N which are normalised as in (0.3), i.e. with $V_N = 1$ and $V_{N-1} = 0$. Assuming that L is bispectral means that we have also another operator Λ, a wave function $\psi(x,z)$ and two other functions $f(z)$ and $\theta(x)$, such that Eqs. (0.1) and (0.2) hold. The following lemma, due to [17], has been fundamental for all studies of bispectral operators.

Lemma 1.8. *There exists a number m, such that*

$$(\operatorname{ad} L)^{m+1}\theta = 0. \tag{1.21}$$

For its simple proof, see [17,31]. We will consider that m is the minimal number with this property. An important corollary of the above lemma is the following result.

Lemma 1.9. *The functions $f(z)$ and $\theta(x)$ are polynomials.*

The next result is also contained in [17,31], but it is not formulated as a separate statement. We give its short proof following [31].

Lemma 1.10. *The coefficients α_j in the expansion* (1.1) *of the wave operator K are rational functions.*

Proof. From Eq. (1.21) it follows that

$$\left(\operatorname{ad} \partial_x^N\right)^{m+1}\left(K^{-1}\theta K\right) = 0.$$

On the other hand the kernel of the operator $(\operatorname{ad} \partial_x^N)^{m+1}$ consists of all pseudo-differential operators whose coefficients are polynomials in x of degree at most m. This gives that

$$\theta K = K\Theta, \tag{1.22}$$

with a pseudo-differential operator Θ:

$$\Theta = \Theta_0 + \sum_1^\infty \Theta_j \partial_x^{-j}. \tag{1.23}$$

whose coefficients Θ_j are polynomials of degree at most m. We have $\theta = \Theta_0$. Comparing the coefficients at ∂_x^{-j} we find that all the coefficients $\alpha_j(x)$ of K are rational functions. □

Remark 1.11. We notice that at least one of the coefficients of Θ_j has degree exactly m, where m from Lemma 1.8 is minimal. This fact will be used later.

The last lemma has as an obvious consequence one of the few general results, important in all studies of bispectral operators. Noticing that the coefficients of L are polynomials in the derivatives of $\alpha_j(x)$ we get

Lemma 1.12. *The coefficients of L are rational functions.*

2. Bispectral involutions and the string equation

The condition (0.4) for vanishing of the coefficients $V_j(x)$ of a bispectral operator L implies further restrictions on all objects connected to L – the wave function $\psi(x,z)$, the wave operator K and the coefficients of L itself. This gives us the opportunity to define two anti-isomorphisms b and b_1 ("bispectral involutions") between the algebras of pseudo-differential operators with coefficients – formal Laurent series in the variables x^{-1} and z^{-1}. In its turn using these anti-isomorphisms will allow us to continue our further constructions in the rest of the paper by purely algebraic analysis on the differential or pseudo-differential operators, avoiding the wave function.

2.1. Bispectral involutions

In the next lemma, following [17] we find the simplest restrictions on the coefficients of the wave operator K and on L.

Lemma 2.1. (i) *The coefficients $V_j(x)$, $j = N-2,\dots,0$, of L vanish at ∞ at least as x^{-2}.*
(ii) *The coefficients α_j, $j = 1,\dots,$ of the wave operator K vanish at least as x^{-1}.*

Proof. We are going to prove both statements simultaneously. We use the formula

$$LK = K\partial^N,$$

Lemmas 1.10 and 1.12. Comparing the coefficients at ∂^{N-2} at the both sides of the above identity we get:

$$V_{N-2} + N\alpha_1' = 0.$$

Having in mind that V_{N-2} is equal to the derivative of the rational function α_1 and that it vanishes at ∞ we see that it vanishes at least as x^{-2}. Continuing in the same manner we find

$$V_{N-3} + V_{N-2}\alpha_1 + \frac{N(N-1)}{2}\alpha_1'' + N\alpha_2' = 0.$$

We see that α_2' is vanishing (at least as x^{-2}) and that V_{N-3} vanishes again at least as x^{-2}, being a sum of such terms. By induction we get that α_{s-1}', $s = 1, \ldots, N-1$, vanishes at least as x^{-2} and the same holds for V_{N-s}, $s = 2, \ldots, N$, as it is a sum of products $V_j \alpha_m^{(k)}$, where $N-1 > j > s$, $m = 1, \ldots, N-1$ (here $\alpha^{(k)}$ denotes kth derivative), and also pure derivatives of α_m. Arguing as above we get the statement of the lemma. □

Following [7] we will introduce an anti-isomorphism b between the algebra $\mathcal{B}$ of pseudo-differential operators $P(x, \partial_x)$ in the variable x and the algebra $\mathcal{B}'$ of pseudo-differential operators $R(z, \partial_z)$ in the variable z. More precisely $\mathcal{B}$ consists of those pseudo-differential operators

$$P = \sum_k^{\infty} p_j\bigl(x^{-1}\bigr)\partial_x^{-j},$$

for which there is a number $n \in \mathbb{Z}$ (depending on P) such that $x^n p_j(x^{-1})$, $j = k, k+1, \ldots,$ are formal power series in x^{-1}. The involution

$$b : \mathcal{B} \to \mathcal{B}'$$

is defined by

$$b(P)\mathrm{e}^{xz} = P\mathrm{e}^{xz} = \sum_k^{\infty} z^{-j} p_j\bigl(\partial_z^{-1}\bigr)\mathrm{e}^{xz}, \quad \text{for } P \in \mathcal{B}, \tag{2.1}$$

i.e. b is just a continuation of the standard anti-isomorphism between two copies of the Weyl algebra. In what follows we will use also the anti-isomorphism

$$b_1 : \mathcal{B} \to \mathcal{B}', \quad b_1(P) = b(\mathrm{Ad}_K\, P). \tag{2.2}$$

Obviously b and b_1 can be considered as involutions of $\mathcal{B}$ and without any ambiguity we can denote the inverse isomorphisms b^{-1}, $b_1^{-1} : \mathcal{B}' \to \mathcal{B}$ by the same letters.

Remark 2.2. If we use relations (0.1) and (0.2) to define an involution b_1 on the subalgebra of $\mathcal{B}$ generated by L and θ, then we have

$$b_1(L) = b\bigl(K^{-1}LK\bigr) = b(\mathrm{Ad}_K\, L),$$
$$b_1(\theta) = b\bigl(K^{-1}\theta K\bigr) = b(\mathrm{Ad}_K\, \theta).$$

This prompts definition (2.2).

Since the operators K and Θ are from $\mathcal{B}$ we can define two operators S and Λ as follows:

$$S(z, \partial_z) = b\bigl(K(x, \partial_x)\bigr), \tag{2.3}$$

$$\Lambda(z, \partial_z) = b(\Theta). \tag{2.4}$$

Explicitely one has

$$S = \sum_{j=0}^{\infty} z^{-j}\alpha_j(\partial_z) = \sum_{j=0}^{\infty} a_j\bigl(z^{-1}\bigr)\partial_z^{-j}, \quad \alpha_0 = 1, \tag{2.5}$$

and also

$$\Lambda(z,\partial_z)=\sum_{j=0}^{\infty}z^{-j}\Theta_j(\partial_z)=\sum_{i=0}^{m}\Lambda_i\big(z^{-1}\big)\partial_z^i, \tag{2.6}$$

where $\Lambda_m \neq 0$ (see Remark 1.11) and the coefficients Λ_i and a_j should be viewed as formal power series. We are going to see that they are polynomials in z^{-1}.

Lemma 2.3. *The coefficients a_j of the operator S are polynomials in z^{-1}.*

Proof. Using that

$$LK=K\partial^N,$$

we can apply the involution b and to derive:

$$Sb(L)=z^N S.$$

Rewrite in details the last formula:

$$\left(\sum_0^{\infty}a_j\big(z^{-1}\big)\partial_z^{-j}\right)\big(z^N+z^{N-2}V_{N-2}(\partial_z)+\cdots\big)=z^N\left(\sum_0^{\infty}a_j\big(z^{-1}\big)\partial_z^{-j}\right).$$

Comparing the coefficients at ∂_z^{-j} for $j=2,3,\ldots$ and having in mind that according to Lemma 2.1:

$$V_k(\partial_z)=\sum_2^{\infty}V_{k,s}\partial_z^{-s},\quad k=0,\ldots,N-2,$$

we obtain relations for a_1 and a_2 in the form:

$$-Nz^{N-1}a_1+\sum_0^{N-2}z^kV_{k,2}=0,$$

$$-2Nz^{N-1}a_2+\left(\sum_0^{N-2}z^kV_{k,2}+N(N-1)z^{N-2}\right)a_1+\sum_0^{N-2}z^kV_{k,3}=0.$$

We see that a_1, a_2 are polynomials in z^{-1}. By induction we get that any a_s satisfies an equation of the form:

$$-sNz^{N-1}a_s+\sum_0^{N-2}z^kq_{k,s}\big(z^{-1}\big)=0,$$

where $q_{k,s}$ are already polynomials in z^{-1}. This proves the lemma. □

Now we are ready to show that the operator Λ has coefficients Λ_j, which are polynomials in z^{-1}. Denote temporarily by r the degree of the polynomial θ, i.e. if $\theta(x) = \theta_r z^r + \cdots$, then $\theta_r \neq 0$.

Lemma 2.4. *The coefficients Λ_i of the operator Λ are polynomials in z^{-1}. The degree of θ $r = m$ and*

$$\Lambda_m = \theta_m, \qquad \Lambda_{m-1} = \theta_{m-1}, \tag{2.7}$$

Proof. Using the definition (2.4) and applying the involution b to the relation $\theta(x)K = K\Theta$ we get:

$$\Lambda S = S\theta(\partial_x).$$

As the coefficients of Λ are expressed as differential polynomials of the coefficients a_j, of S we get that Λ_j are also polynomials in z^{-1}. Comparing the first two coefficients of the above equality we get also (2.7). □

2.2. The string equation

In this subsection we are going to show that for the bispectral operator L there exists another operator Q, for which the string equation (0.6) holds. This equation as well as other properties of the operator Q (with appropriate normalisation) would be crucial for our constructions.

In what follows we would assume that the number m is divisible by N. This is not a restriction since we can always replace Λ by Λ^N. We put $m = Nl$.

Lemma 2.5. *There is a natural number n such that*:

$$Q = K^{-1} x \partial_x^{nN+1} K, \tag{2.8}$$

is a differential operator. The operator Q is a solution to the string equation (0.6).

Proof. Using the bispectral property one can write

$$(\operatorname{ad} L)^{m-1}\theta = (-1)^{m-1} b_1\big(\big(\operatorname{ad} z^N\big)^{m-1}\Lambda\big).$$

Each application of the operator $\operatorname{ad} z^N$ to any differential operator P reduces its order by 1. Using the fact that the operator

$$\Lambda = \Lambda_m \partial_z^m + \Lambda_{m-2}\partial_z^{m-2} + \cdots,$$

where Λ_m is a nonzero constant, we get that the operator

$$\big(\operatorname{ad} z^N\big)^{m-1}\Lambda = \Lambda_m \big(\operatorname{ad} z^N\big)^{m-1}\partial_z^m$$

is an operator of order 1. Now prescribing weights to z and to ∂_z as follows: $\operatorname{wt}(z) = 1$, $\operatorname{wt}(\partial_z) = -1$ we obtain that the right-hand side of the above identity has weight equal to $(m-1)N - m$. This shows that the operator in the above equality has the form:

$$\big(\operatorname{ad} z^N\big)^{m-1}\Lambda = c z^{(m-1)(N-1)}\partial_z + c_1 z^{mN-m-N}, \quad c \neq 0.$$

In this way we get that

$$Q_1 := (\operatorname{ad} L)^{m-1}\theta = b_1\Big((-1)^{m-1}\big(cz^{(m-1)(N-1)}\partial_z + c_1 z^{mN-m-N}\big)\Big)$$

is a differential operator. Using the fact that $m = Nl$ and that $b_1(z) = L^{1/N}$ we obtain

$$\big((-1)^{m-1}Q_1 - c_1 L^{Nl-l-1}\big) = cb_1\big(z^{(m-1)(N-1)}\partial_z\big) = cb_1\big(z^{nN+1}\partial_z\big),$$

where we have put $n = l(N-1) - 1$. Now it is obvious that

$$Q := b_1\big(z^{nN+1}\partial_z\big) = \frac{1}{c}\big((-1)^{m-1}Q_1 - c_1 L^{Nl-l-1}\big)$$

is a differential operator. The identity (0.6) is obtained by applying the bispectral involution to

$$\big[z^{nN+1}\partial_z, z^N\big] = Nz^{N(n+1)}. \qquad \square$$

Corollary 2.6. *For any positive integer i the following formula holds*:

$$(\operatorname{ad} L)^i\big(Q^i\big) = i!N^i L^{i(n+1)}. \tag{2.9}$$

Proof. Assume that (2.9) is true for $1, 2, \ldots, i$. Then

$$(\operatorname{ad} L)^{i+1}\big(Q^i\big) = 0$$

Since $\operatorname{ad} L$ is a differentiation in the ring of differential operators with rational coefficients we can use the Leibnitz's rule:

$$(\operatorname{ad} L)^{i+1}\big(Q^{i+1}\big) = (\operatorname{ad} L)^{i+1}\big(Q^i \cdot Q\big) = \sum_{j=0}^{i+1}\binom{i+1}{j}(\operatorname{ad} L)^{i+1-j}\big(Q^i\big)(\operatorname{ad} L)^j(Q).$$

The only nonzero term in the above sum is the one for $j = 1$, hence

$$(\operatorname{ad} L)^{i+1}\big(Q^{i+1}\big) = (i+1)N.(\operatorname{ad} L)^i\big(Q^i\big)L^{n+1}.$$

Now (2.9) follows by induction on i. $\square$

3. The infinite point

The present section is divided into three subsections, corresponding to the basic results, which we shortly describe. In the first subsection we present the operator Q as a polynomial in L with coefficients – operators of lower order. The second result is a proof that the point ∞ is a regular singular point for the operator L (Theorem 0.3). In the last subsection we give an estimate of the degree n (of the string equation) in terms of the roots of the indicial equation at ∞. All results will be used in performing Darboux transformations. Now we will fix the situation in which we are going to work.

Definition 3.1. By $\mathcal{O}$ we denote the set of all functions that are holomorphic at ∞. If we write a function $V(x)$ from $\mathcal{O}$ as:

$$V(x) = x^{-\nu}\left(a_0 + a_1\frac{1}{x} + a_2\frac{1}{x^2} + \cdots\right), \quad a_0 \neq 0,\ \nu \geqslant 0.$$

then the number

$$\operatorname{ord}(V) := -\nu$$

will be called order of V at ∞ and will be denoted by $\operatorname{ord}(V)$.

We introduce also the ring $\mathcal{O}[\partial]$ of all differential operators with coefficients from $\mathcal{O}$. Obviously the bispectral operator L is from $\mathcal{O}[\partial]$. The only properties of L and Q relevant for our purposes in the present section are summed up in terms of the wave operator as follows:

Definition 3.2. We say that the operator $L \in \mathcal{O}[\partial]$ solves the string equation iff the following conditions are satisfied:

(1) There is a wave operator $K = 1 + \alpha_1\partial^{-1} + \cdots$ with coefficients from $\mathcal{O}$ for which $LK = K\partial^N$ and $\operatorname{ord}(\alpha_i) \leqslant -1$.
(2) There is an integer $n \geqslant 0$ such that $Q = Kx\partial_x^{nN+1}K^{-1}$.

We will call the pair (L, Q) a string pair. The minimal number n in (2) will be called the string number of L.

3.1. Q as a polynomial in L

For convenience denote by $\mathcal{R}$ the differential extension of $\mathbb{C}[x]$ by adjoining the differential indeterminates $y_1, y_2, \ldots$ (see [23] for details). We endow the differential ring $\mathcal{R}$ with graduation which will be useful in the sequel: for a monomial $\tau = x^{n_0}y_{i_1}^{(n_i)}\cdots y_{i_s}^{(n_s)}$ set $\operatorname{wt}(\tau) = n_0 - (n_1 + i_1) - \cdots - (n_s + i_s)$. This weight provides $\mathcal{R}$, with the structure of a $\mathbb{Z}$-graded ring:

$$\mathcal{R} = \bigoplus_{n\in\mathbb{Z}} \mathcal{R}_n,$$

where $\mathcal{R}_n$ is spanned over $\mathbb{C}$ by all monomials $\tau \in \mathcal{R}$ for which $\operatorname{wt}(\tau) = n$. This graduation can be extended in a natural way to graduation of the ring of all pseudo-differential operators with coefficients from the ring $\mathcal{R}$, by prescribing to the symbol of differentiation ∂ weight $\operatorname{wt}(\partial) = -1$. For convenience the last mentioned ring will be denoted by $\operatorname{Psd}\mathcal{R}$. In this way a pseudo-differential operator

$$P = \sum_{j\leqslant m} a_j\partial^j, \quad a_j \in \mathcal{R},$$

is homogeneous of weight n if for every j the coefficient a_j is a homogeneous element from $\mathcal{R}_{n+j}$. If P is homogeneous then by $\operatorname{wt}(P)$ we will denote it's weight. We will need two lemmas. The proof of the first one being trivial will be omitted.

Lemma 3.3. (i) *Assume that $P = \partial^N + \cdots$ is a homogeneous pseudo-differential operator from $\operatorname{Psd}\mathcal{R}$. Then P is invertible in $\operatorname{Psd}\mathcal{R}$, and P^{-1} is homogeneous with weight $-N$.*

(ii) *For every two homogeneous operators P_1 and P_2 from* $\operatorname{Psd}\mathcal{R}$ *with weights respectively n_1 and n_2 their product $P_1.P_2$ is also homogeneous and its weight is n_1+n_2.*

Lemma 3.4. *Assume that $L_y=\partial^N+\cdots$ and Q_y are arbitrary homogeneous pseudo-differential operators from* $\operatorname{Psd}\mathcal{R}$. *Then one can find an integer number n and homogeneous differential operators $\tilde q_0,\tilde q_1,\ldots$ from $\mathcal{R}[\partial]$ of orders $\leqslant N-1$ such that*:

$$Q_y=\tilde q_0L_y^n+\tilde q_1L_y^{n-1}+\cdots. \tag{3.1}$$

More precisely, for any $i=0,1,\ldots,$ such that $\tilde q_i\neq 0$ the weight of $\tilde q_i$ is: $\operatorname{wt}(Q_y)-(n-i)\operatorname{wt}(L_y)$.

Proof. For given Q_y we will show that $\tilde q_0$ and n can be determined uniquely and that they satisfy the properties stated in the lemma. After $\tilde q_0$ is determined we move the term $\tilde q_0L^n$ to the left-hand side of (3.1) and then in the same manner we can determine $\tilde q_1$. Now it is clear that all $\tilde q_i$ can be found successively and the lemma will be proved.

Denote by m the order of Q_y and divide m by N: $m=nN+r$, $0\leqslant r\leqslant N-1$. Multiply both sides of (3.1) by L^{-n} and compare the differential parts of the two pseudo-differential operators:

$$\tilde q_0=\left(Q_yL^{-n}\right)_+.$$

Combining this equality with Lemma 3.3 we obtain the statement of the lemma. □

Denote by $\operatorname{Psd}\mathcal{O}$ the ring of all pseudo-differential operators with coefficients from $\mathcal{O}$. To use the result of Lemma 3.4 we need a ring homomorphism

$$\pi:\operatorname{Psd}\mathcal{R}\to\operatorname{Psd}\mathcal{O}$$

defined as follows: take the unique differential homomorphism between $\mathcal{R}$ and $\mathcal{O}$ that maps y_j into a_i, $i=1,2,\ldots,$ where α_i are the coefficients of the wave operator K and then extend this homomorphism to homomorphism between $\operatorname{Psd}\mathcal{R}$ and $\operatorname{Psd}\mathcal{O}$ by leaving ∂ fixed. Now from the representation in Lemma 3.4 we can derive a similar one for the operators L and Q.

Lemma 3.5. *Let L and Q form a string pair and n is the corresponding string number. Then one can find differential polynomials $\tilde q_0,\tilde q_1,\ldots,\tilde q_n$ from* $\operatorname{Psd}\mathcal{R}$, *such that if we set $q_i=\pi(\tilde q_i)$ then*:

$$Q=q_0L^n+q_1L^{n-1}+\cdots+q_n. \tag{3.2}$$

The operators $\tilde q_i$ are homogeneous. More precisely: $\tilde q_0=x\partial_x$ *and if $\tilde q_i\neq 0$, then*

$$\operatorname{wt}(\tilde q_i)=-iN.$$

The differential operator q_n is not zero.

Proof. Introduce the pseudo-differential operator

$$K_y=1+y_1\partial^{-1}+\cdots\in\operatorname{Psd}\mathcal{R}.$$

It is easy to check that $L_y = K_y \partial_x^N K_y^{-1}$ and $Q_y = K_y x \partial_x^{Nn+1} K_y^{-1}$ are homogeneous elements from $\operatorname{Psd}\mathcal{R}$ with weights respectively: $\operatorname{wt}(L_y) = -N$ and $\operatorname{wt}(Q_y) = -nN$. The definition of $\mathcal{R}$ was given in such a way that $\pi(L_y) = L$ and $\pi(Q_y) = Q$. Applying Lemma 3.4 with L_y, Q_y we get:

$$Q_y = \tilde{q}_0 L_y^n + \tilde{q}_1 L_y^{n-1} + \cdots,$$

where each $\tilde{q}_i$ is homogeneous with weight $\operatorname{wt}(\tilde{q}_i) = \operatorname{wt}(Q_y) - (n+i)\operatorname{wt}(L_y) = -iN$. Map both sides of the last equality by π:

$$Q = \pi(\tilde{q}_0)L^n + \cdots + \pi(\tilde{q}_n) + \pi\left(\tilde{q}_{n+1} L_y^{-1} + \cdots\right).$$

Comparing the strictly pseudo-differential parts of the operators at the two sides of the above equality we see that:

$$\pi\left(\tilde{q}_{n+1} L_y^{-1} + \cdots\right) = 0.$$

The inequality $q_n \neq 0$ holds because n was chosen to be the minimal number with the property that $K x \partial^{nN+1} K^{-1}$ is a differential operator. □

3.2. $x = \infty$ is a regular singular point

Here we give the proof of Theorem 0.3, i.e. that the infinite point is regular for L. Take the smallest n for which $Q = K x \partial_x^{Nn+1} K^{-1}$ is a differential operator. The idea is to assume that $x = \infty$ is irregular for L and then to assign weights to x and ∂_x in such a way that the most irregular terms of L at ∞ have the highest weight. This weights will enable us to associate with each differential operator from $\mathcal{O}[\partial]$ a (ρ, σ)-homogeneous polynomial in Y with coefficients Laurent polynomials in X. Following [16] and using (0.6) we will get contradiction.

We denote the ring of Laurent polynomials by $\mathcal{L}$. The definition of ρ and σ is prompted by the theory of irregular points (see, e.g., [3]). Introduce the rational number:

$$r = \max\left(1, 2 + \frac{\operatorname{ord} V_2}{2}, \ldots, 2 + \frac{\operatorname{ord} V_n}{N}\right)$$

(called principal level). It can be expressed as r_1/r_2, where r_1 and r_2 are relatively prime. Well known fact is that $x = \infty$ is regular if and only if $r = 1$. Our assumption that ∞ is irregular point yields $r > 1$. The integers $\rho = r_2$ and $\sigma = r_1 - 2r_2$ represent the weights of x and ∂_x respectively. They satisfy the inequality:

$$\rho + \sigma > 0.$$

The next definitions are modifications of corresponding ones given by J. Dixmier [16]. In the first definition we endow the ring $\mathcal{O}[\partial]$ (of differential operators with homomorphic at ∞ coefficients) with $\mathbb{Z}$-graded structure.

Definition 3.6. Assume that $L = V_0\partial^n + V_1\partial^{n-1} + \cdots + V_n$ is an arbitrary element of $\mathcal{D}$. For each term $V(x)\partial_x^i$ define its weight

$$v_{\rho,\sigma}\left(V(x)\partial_x^i\right) = \rho(\operatorname{ord} V) + \sigma i.$$

Then the number

$$v_{\rho,\sigma}(L) := \max_{0\leqslant i\leqslant n} v_{\rho,\sigma}\big(V_i\partial^{n-i}\big)$$

will be called (ρ,σ)-order of L.

The second definition associates to each differential operator from $\mathcal{O}[\partial]$ a (ρ,σ)-homogeneous polynomial from $\mathcal{L}[Y]$.

Definition 3.7. Assume the notation of the previous definition and denote by $I(L)$ the set $\{i \in \{0,1,\ldots,n\} \mid v_{\rho,\sigma}(V_i\partial^{n-i}) = v_{\rho,\sigma}(L)\}$. The polynomial $p \in \mathcal{L}[Y]$ defined as:

$$p = \sum_{i\in I} a_i X^{\operatorname{ord} V_i} Y^{n-i}, \tag{3.3}$$

where $a_i \in \mathbb{C}$ are uniquely determined from the expansion

$$V_i = a_i x^{\operatorname{ord} V_i} + (\text{lower order terms}),$$

will be called polynomial associated with L.

The following two lemmas are also taken from [16]. Although the situation there is slightly different the proofs are essentially the same. We are going to prove only the first one. The second can be proven in a similar way.

Lemma 3.8. *Assume $L_1, L_2 \in \mathcal{O}[\partial]$ and $\rho+\sigma > 0$. The polynomial associated to the product L_1L_2 is the product of the polynomials associated with L_1 and L_2 respectively. The (ρ,σ)-order of this operator is*: $v_{\rho,\sigma}(L_1, L_2) = v_{\rho,\sigma}(L_1) + v_{\rho,\sigma}(L_2)$.

Proof. Set $\xi = \partial_x$. Then for the product of two differential operators we have:

$$L_1L_2 = \sum_{k=0}^{\infty} : \frac{\partial^k L_1}{\partial \xi^k}\frac{\partial^k L_2}{\partial_x^k} : \tag{3.4}$$

where : : is the normal ordering which always puts the differentiation on the right. Write $L_1 = a_0\xi^{N_1} + \cdots + a_{N_1}$, $L_2 = b_0\xi^{N_2} + \cdots + b_{N_2}$. From the definition of : : we have that

$$: L_1L_2 : = \sum_{0\leqslant i\leqslant N_1,\ 0\leqslant j\leqslant N_2} a_i b_j \xi^{N_1+N_2-i-j}.$$

Each term in this sum satisfies the inequality $v_{\rho,\sigma}(a_i b_j \xi^{N_1+N_2-i-j}) \leqslant v_{\rho,\sigma}(L_1) + v_{\rho,\sigma}(L_2)$. The equality is possible only when $i \in I(L_1)$ and $j \in I(L_2)$. On the other hand the coefficient in front of the highest degree of ξ in:

$$\sum_{i\in I(L_1),\ j\in I(L_2)} a_i b_j \xi^{N_1+N_2-i-j}$$

is $a_{i_1}b_{i_2} \neq 0$, where i_1 and i_2 are the minimal numbers from $I(L_1)$ and $I(L_2)$ respectively. Thus this sum (which in fact is equal to the product of the (ρ,σ)-polynomials associated with L_1 and L_2) is not zero. The conclusion of this observations is that the (ρ,σ)-polynomial associated with $: L_1L_2 :$ is the product of the polynomials associated with L_1 and L_2 and also $v_{\rho,\sigma}(: L_1L_2 :) = v_{\rho,\sigma}(L_1) + v_{\rho,\sigma}(L_2)$. To finish the proof it is enough to use formula (3.4) and the obvious fact that $v_{\rho,\sigma}(\partial_\xi^k L_1) \leqslant v_{\rho,\sigma}(L_1) - k\sigma$ and $v_{\rho,\sigma}(\partial_x^k L_2) \leqslant v_{\rho,\sigma}(L_2) - k\rho$. $\square$

Lemma 3.9. *Consider again two operators $L_1, L_2 \in \mathcal{O}[\partial]$ and denote by f_1, f_2 the polynomials associated with them and by n_1 and n_2 their (ρ,σ)-orders. If the fraction $f_1^{n_2}/f_2^{n_1}$ is not a constant and $\rho+\sigma > 0$, then the polynomial associated with $[L_1, L_2]$ is*:

$$\frac{\partial f_1}{\partial Y}\frac{\partial f_2}{\partial X} - \frac{\partial f_1}{\partial X}\frac{\partial f_2}{\partial Y}. \tag{3.5}$$

For the (ρ,σ)-order we have a formula: $v_{\rho,\sigma}([L_1, L_2]) = n_1 + n_2 - \rho - \sigma$.

In order to apply these lemmas to the string equation (0.6) we have to find the polynomials f and g associated with L and Q and their (ρ,σ)-orders v and w. This requires few auxiliary results, stated in the following two lemmas.

Lemma 3.10. (i) *The (ρ,σ)-order of L is $v = N\sigma$ and the polynomial associated with L is*:

$$f = Y^N + (\textit{at least one term}).$$

(ii) *The (ρ,σ)-order of Q is $w = (nN+1)\sigma + \rho$ and the polynomial associated with Q has the form*:

$$q = XYf^n + a_1(X,Y)f^{n-1} + \cdots + a_n(X,Y).$$

Proof. (i) Since $v_{\rho,\sigma}(\partial^N) = N\sigma$ the only thing we have to check is that $\operatorname{ord}(V_i)\rho + (N-i)\sigma \leqslant N\sigma$ for $i = 2, 3, \ldots, N$ and that equality is reached for at least one i. But this is obvious from the definition of ρ and σ.

(ii) The polynomial g has the form:

$$g(X,Y) = a_0(X,Y)f^n + a_1(X,Y)f^{n-1} + \cdots + a_n(X,Y)$$

for some $a_i \in \mathcal{L}[Y]$. Lemma 3.5 gives that a_i, $i = 1, 2, \ldots, n$, can have only negative degrees of X. But then the coefficient at the highest degree of Y in the polynomial g^v is not a constant, while the corresponding one in f^w is 1. Thus the fraction f^w/g^v is not a constant. Now from Lemma 3.9 the polynomial h associated with $[L, Q]$ is:

$$h = \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X} - \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y}$$

and $v_{\rho,\sigma}(h) = v_{\rho,\sigma}(f) + v_{\rho,\sigma}(g) - \rho - \sigma = v + w - \rho - \sigma$. On the other hand the string equation (0.6) yields: $v_{\rho,\sigma}(h) = (n+1)v$. From the last two relation we derive the formula for w. To finish the proof it is enough to notice that $v_{\rho,\sigma}(q_0L^n) = (Nn+1)\sigma + \rho$. $\square$

Lemma 3.11. *Under the above notations $g = XYf^n$.*

Proof. If h is the polynomial associated with $[L,Q]$ then using Lemma 3.9 we have the following series of equalities:

$$\begin{aligned}\rho Xh &= \rho X\left(\frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}-\frac{\partial f}{\partial X}\frac{\partial g}{\partial Y}\right)\\ &= \frac{\partial f}{\partial Y}(wg-\sigma Y\partial_Y g)-(vf-\sigma Y\partial_Y f)\frac{\partial g}{\partial Y}=wg\frac{\partial f}{\partial Y}-vf\frac{\partial g}{\partial Y}\\ &= f^{-w+1}g^{v+1}\partial_Y\left(\frac{f^w}{g^v}\right),\end{aligned}$$

where we have used that for a (ρ,σ)-homogeneous polynomial f of (ρ,σ)-degree v the following identity holds:

$$\rho X\partial_X f+\sigma Y\partial_Y f=vf.$$

Now the relation (0.6) leads to:

$$N\rho X.f^{n+1}=f^{-w+1}g^{v+1}\partial_Y\left(\frac{f^w}{g^v}\right). \tag{3.6}$$

View f and g as elements in $\mathcal{K}[Y]$, where $\mathcal{K}$ is an algebraic extension of the field of fractions of the ring $\mathcal{L}$ containing all the roots of the polynomials $f(Y)$ and $g(Y)$. Take $\alpha(X)$ to be a zero of f of order $\nu\geqslant 1$. Denote also by μ the order of $\alpha(X)$ as a zero of g. Then comparing the orders of the terms at the both sides in formula (3.6) we obtain

$$\nu(n+1)+(w-1)\nu-(v+1)\mu=\operatorname{ord}_{Y-\alpha(X)}\partial_Y\left(\frac{f^w}{g^v}\right). \tag{3.7}$$

We will treat the following 2 cases separately:

Case 1. If $w\nu\neq v\mu$, then the right side of (3.7) is $w\nu-v\mu-1$, hence $\mu=n\nu+1$.

Case 2. If $w\nu=v\mu$, then using the formulas for v and w we find

$$\mu=\frac{w}{v}\nu=\frac{(nN+1)\sigma+\rho}{N\sigma}\nu=\left(n+\frac{\rho+\sigma}{N\sigma}\right)\nu>n\nu.$$

In both cases $\mu>n\nu$, which means that $\frac{g}{f^n}$ is a polynomial in Y. Now from Lemma 3.10 f^n divides the polynomial $a_1f^{n-1}+\cdots+a_n$ whose degree in Y does not exceed $N-1+N(n-1)<nN$. But the degree of f^n is exactly $nN\Rightarrow a_1f^{n-1}+\cdots+a_n=0$. □

Now we are ready to give the proof of Theorem 0.3.

Proof of Theorem 0.3. Put $w=nv+\rho+\sigma$ and $g=XYf^n$ in (3.6) and after simplifications we get a differential equation for f:

$$Y\partial_Y\left(f^{\rho+\sigma}\right)=(\rho+\sigma)Nf^{\rho+\sigma}.$$

An immediate consequence of this equation is that $f=c(X)Y^N$. But the choice of ρ and σ was done in such a way that $f=Y^N+$(at least one term). This contradiction proves the theorem. □

The regularity of L imposes the following restrictions on the coefficients of the wave operator K.

Corollary 3.12. *Let L be an operator solving the string equation and $K = 1 + \alpha_1 \partial^{-1} + \cdots$ is the wave operator defining the corresponding string pair. Then the order of the coefficient α_i, $i = 1, 2, \ldots$, does not exceed the number $-i$, i.e.*

$$\alpha_i(x) \in \frac{1}{x^i}\mathcal{O}.$$

3.3. An estimate for n

Here we want to estimate the number n from the string equation in terms of the roots of the indicial equation for L at ∞. For us it would be convenient to write the indicial equation, using again the idea of the weights. But in order to have an analogue of Lemma 3.8 we have slightly to change the procedure of association polynomials to the elements of $\mathcal{O}[\partial]$. The next definition describes this process.

Definition 3.13. Write every $L \in \mathcal{O}[\partial]$ as

$$L = V_0 D^N + \cdots + V_{N-1} D + V_N,$$

where $D = x\partial_x$ and assume also that:

$$V_i = a_i x^{\nu_i} + (\text{lower order terms}).$$

The number

$$\mathrm{wt}(L) := \max_{0 \leqslant i \leqslant N} \mathrm{ord}(V_i)$$

will be called weight of L. The polynomial associated with L is from $\mathbb{C}[D]$ and is defined as follows:

$$p(L) := \sum_{i:\ \mathrm{ord}(V_i) = \mathrm{wt}(L)} a_i D^{N-i}.$$

In particular if the point $x = \infty$ is regular then $p(\lambda) = 0$ is explicitly the indicial equation (see [21]).

In terms of the above definition we can give the following corollary from Lemma 3.5 and Corollary 3.12.

Corollary 3.14. *Assume that (L, Q) is a string pair and n is the corresponding string number. Divide Q by L to derive*

$$Q = Q_1 L + q,$$

where q is a differential operator of order not exceeding $N - 1$. The weight of q satisfies the inequality: $\mathrm{wt}(q) \leqslant -nN$.

Obviously $p(D)x^i = x^i p(D+i)$ for every polynomial $p \in \mathbb{C}[D]$. This observation is enough to make the following conclusion:

Lemma 3.15. *Assume that $L_1, L_2 \in \mathcal{D}$ are two arbitrary differential operators and denote by f_1 and f_2 the polynomials associated with them. Then the polynomial associated with the product L_1L_2 is*:

$$f_1\big(D + \mathrm{wt}(L_2)\big) f_2(D).$$

The weight of the product is a sum of the weights of the two operators.

Now we will assume that L is an operator solving the string equation. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_N$ the roots of the indicial equation of L at ∞. The following very important fact, used in performing Darboux transformations, is the content of the next proposition.

Proposition 3.16. *Assume that L is a differential operator that solves the string equation. Then we can find numbers i and j such that*:

$$n \leqslant \frac{1}{N}|\lambda_i - \lambda_j|.$$

Proof. Let (L, Q) be a string pair. As in Corollary 3.14 divide Q by L

$$Q = Q_1 L + q.$$

Using that Q satisfies the string equation (0.6) we get

$$Lq = L_1 L \tag{3.8}$$

for some $L_1 \in \mathcal{O}[\partial]$.

Denote by f, g and h the polynomials associated with L, q and L_1 respectively. For us the weight of q will be very important and will be denoted by w.

Lemma 3.15 combined with (3.8) gives:

$$f(D+w)g(D) = h(D-N)f(D).$$

As a result we found that: $f(\lambda_i + w)g(\lambda_i) = 0$ for $i = 1, 2, \ldots, N$. Using the inequality: $\deg g \leqslant N-1$ one can find λ_i for which $g(\lambda_i) \neq 0$, hence $f(\lambda_i + w) = 0$, i.e. $\lambda_j = \lambda_i + w$ for some λ_j. Applying Corollary 3.14 we get that to $w \leqslant -nN$. This gives that

$$nN \leqslant |w| = |\lambda_i - \lambda_j|. \qquad \square$$

4. $\mathbb{Z}_r$-invariantness of bispectral operators

Let $\mathcal{A}_L$ be the ring of all differential operators commuting with L. We want to prove that if the rank of L is r then L is a $\mathbb{Z}_r$-invariant operator. The next lemma shows that it is enough to prove that $\Lambda(z, \partial_z)$ is $\mathbb{Z}_r$-invariant.

Lemma 4.1. *If Λ is $\mathbb{Z}_r$-invariant then L is also $\mathbb{Z}_r$-invariant.*

Proof. It is enough to prove that the wave operator K is $\mathbb{Z}_r$-invariant. Assume that Λ is $\mathbb{Z}_r$-invariant. Then obviously $\Theta = b(\Lambda)$ is also $\mathbb{Z}_r$-invariant.

Now by induction on i we will see that the term $\alpha_i \partial^{-i}$ is $\mathbb{Z}_r$-invariant. Compare the coefficients in front of ∂^{-j} in the relation:

$$\theta\bigl(1 + \alpha_1\partial^{-1} + \cdots\bigr) = \bigl(1 + \alpha_1\partial^{-1} + \cdots\bigr)\bigl(\Theta_0 + \Theta_1\partial^{-1} + \cdots\bigr).$$

Comparing the coefficients in front of ∂^0 and ∂^{-1} one deduces that $\Theta = \Theta_0$ is $\mathbb{Z}_r$-invariant and that $\Theta_1 = 0$. Next assume that $\alpha_1\partial^{-1}, \alpha_2\partial^{-2}, \ldots, \alpha_i\partial^{-i}$ are $\mathbb{Z}_r$-invariant and compare the coefficients in front of ∂^{-i-2}:

$$\theta\alpha_{i+2} = \sum_{s=2}^{i+2} \alpha_{i+2-s}\left(\Theta_s + \binom{s-i-2}{1}\Theta'_{s-1} + \cdots + \binom{s-i-2}{s}\Theta_0^{(s)}\right) + \alpha_{i+2}\theta + \alpha_{i+1}(\Theta_1 - \Theta'_0).$$

The last formula together with the fact that Θ is a $\mathbb{Z}_r$-invariant pseudo-differential operator and the inductive assumption give that $\alpha_{i+1}\partial^{-i-1}$ is $\mathbb{Z}_r$-invariant. □

The next lemma shows that the algebra A_L consists of $\mathbb{Z}_r$-invariant polynomials.

Lemma 4.2. *Let L be an operator of rank r. Then A_L is a subalgebra of* $\mathbb{C}[z^r]$.

Proof. Let $P \in \mathcal{A}_L$. Put $b_1(P) = f(z) \in A_L$. From Lemma 1.9 we know that $f(z)$ is a polynomial. Also the degree of $f(z)$ is a number divisible by r. Assume that $f \notin \mathbb{C}[z^r]$ and also that the coefficient in front of the highest degree is 1. We can represent f as:

$$f = f_0 + f_1,$$

where $f_0 \in \mathbb{C}[z^r]$ is formed from all terms of f whose degrees are divisible by r and $f_1 = f - f_0$. The polynomial f_0 will be called the invariant part of f and f_1 the non-invariant part of f. Denote by n_0 and n_1 the degrees of f_0 and f_1 respectively. Obviously $n_0 > n_1$ and n_1 is not divisible by r. The idea is to construct new polynomial $\tilde{f}$ from A in such a way that the difference $\tilde{n}_0 - \tilde{n}_1$ between the degrees of the invariant and the non-invariant part of $\tilde{f}$ is smaller. After finitely many steps we will end up with a polynomial for which this difference is negative, which will be a contradiction.

The polynomial $\tilde{f}$ can be constructed as follows: let $n_0 = kr$ and $N = pr$ set $\tilde{f} := f^p - z^{kN}$. Denote by $\tilde{f}_0$ and $\tilde{f}_1$ the invariant and the non-invariant parts of $\tilde{f}$ and let $\tilde{n}_0$ and $\tilde{n}_1$ be their degrees. Write the following chain of equalities:

$$\tilde{f} = f^p - z^{Nk} = (f_0 + f_1)^p - z^{Nk} = f_0^p - z^{kN} + \binom{p}{1} f_0^{p-1} f_1 + \cdots.$$

Since $pn_0 = kN$ and f_0 is a polynomial in z^r we can conclude that $\tilde{n}_0 \leqslant pn_0 - r$. The above expansion together with $n_0 > n_1$ gives that $\tilde{n}_1 = n_0(p-1) + n_1$. Now we can prove that the new difference is smaller:

$$\tilde{n}_0 - \tilde{n}_1 \leqslant pn_0 - r - \tilde{n}_1 = pn_0 - r - (p-1)n_0 - n_1 = n_0 - n_1 - r. \qquad \square$$

Proof of Theorem 0.4. It remains to prove the $\mathbb{Z}_r$-invaxiantness of Λ. Write Λ in the form:

$$\Lambda(z,\partial_z)=\sum_{i=0}^{r-1} z^i\Lambda_i\big(z^r,z\partial_z\big), \tag{4.1}$$

where

$$\Lambda_i\big(z^r,z\partial_z\big)=\sum_{j=0}^{n_i}\Lambda_{i,j}\big(z^r\big)\big(z^{nN+1}\partial_z\big)^{n_i-j}.$$

All $\Lambda_{i,j}$ are Laurent polynomials and n_i is chosen in such a way that $\Lambda_{i,0\neq 0}$, when $\Lambda_i\neq 0$. We have to prove that all Λ_i, $i=1,2,\ldots,r-1$, are 0. Thus assume that at least one $\Lambda_i\neq 0$. After applying the bispectral involution b_1 on (4.1) we will get the following relation:

$$\theta=\sum_{j=0}^{n_0} Q^{n_0-j}\Lambda_{0,j}\big(L^{r/N}\big)+\cdots+\sum_{j=0}^{n_r-1} Q^{n_{r-1}-j}\Lambda_{r-1,j}\big(L^{r/N}\big)L^{(r-1)/N}. \tag{4.2}$$

The idea is to construct an operator from $\mathcal{A}_L$ whose image under the bispectral involution is not from $\mathbb{C}[z^r]$. This will be contradiction with Lemma 4.2. We split the construction of such an operator into two cases:

Case 1. $n_0\leqslant \max\{n_1,n_2,\ldots,n_{r-1}\}$.

Denote by ρ the maximal value of the numbers $n_0,n_1,\ldots,n_{r-1}$ and by I the set of all indeces i for which $n_i=\rho$. Due to Lemma 2.1

$$(\operatorname{ad}L)^\rho\big(Q^\rho\big)=(\rho)!N^\rho L^{\rho(n+1)},$$

hence one obtains the following relation:

$$(\operatorname{ad}L)^\rho(\theta)=(\rho)!N^\rho L^{\rho(n+1)}\sum_{i\in I}\Lambda_{i,0}\big(L^{r/N}\big)L^{i/N}. \tag{4.3}$$

Since the operator at the right-hand side commutes with L, it follows that the differential operator at the left-hand side is from $\mathcal{A}_L$. After applying the bispectral involution to (4.3) we get that:

$$z^{\rho(n+1)N}\sum_{i\in I}\Lambda_{i,0}\big(z^r\big)z^i$$

is an element from A_L. This element is not polynomial in z^r because the set I includes at least one index $i\in\{1,2,\ldots,r-1\}$.

Case 2. $n_0>\max\{n_1,n_2,\ldots,n_{r-1}\}$.

Now (4.2) can be written in the form:

$$\begin{aligned}\theta-Q^{n_0}\Lambda_{0,0}\big(L^{r/N}\big) = {}&\sum_{j=0}^{n_0-1} Q^{n_0-j}\Lambda_{0,j}\big(L^{r/N}\big)+\sum_{j=0}^{n_1} Q^{n_1-j}\Lambda_{1,j}\big(L^{r/N}\big)L^{1/N}\\ &+\cdots+\sum_{j=0}^{n_r-1} Q^{n_{r-1}-j}\Lambda_{r-1,j}\big(L^{r/N}\big)L^{(r-1)/N}.\end{aligned} \tag{4.4}$$

Applying $(\operatorname{ad} L)^{n_0}$ to the above equality we see (using Lemma 2.1) that it annihilates the operator at the right-hand side. Hence the operator

$$(\operatorname{ad} L)^{n_0}\bigl(Q^{n_0}\Lambda_{0,0}\bigl(L^{r/N}\bigr)\bigr) = (\operatorname{ad} L)^{n_0}(\theta)$$

must be differential. Denote by N_1 the number $n_0(n+1)$. Using again Lemma 2.1, i.e. that

$$(\operatorname{ad} L)^{n_0}\bigl(Q^{n_0}\bigr) = n_0! N^{n_0} L^{n_0(n+1)}$$

we see that after multiplying from the right both sides of (4.4) by L^{N_1} the operator on the left-hand side will become differential. Denote this new operator by P. We re-denote $\Lambda_{i,j} L^{N_1}$ by $\Lambda_{i,j}$ to avoid complicated notation. Thus the new relation has the form:

$$\begin{aligned} P = {} & \sum_{j=0}^{n_0-1} Q^{n_0-j}\beta_{0,j}\bigl(L^{r/N}\bigr) + \sum_{j=0}^{n_1} Q^{n_1-j}\beta_{1,j}\bigl(L^{r/N}\bigr)L^{1/N} \\ & + \cdots + \sum_{j=0}^{n_{r-1}} Q^{n_{r-1}-j}\beta_{r-1,j}\bigl(L^{r/N}\bigr)L^{(r-1)/N}. \end{aligned} \tag{4.5}$$

We can repeat this procedure until no is reduced to a number smaller or equal to $\max\{n_1, n_2, \ldots, n_{r-1}\}$. Then one proceeds as in case 1. □

5. Darboux transformations

In this section we will gradually simplify the operator L by successive applications of Darboux transformations. Our goal is to obtain after a finite number of steps a Bessel operator.

According to Theorem 0.3 the point $x = \infty$ is a regular singular point for the operator L. Assume also that L is a rank r differential operator. From Theorem 0.4 we know that in this case L is $\mathbb{Z}_r$-invariant operator. Thus if we represent L as

$$L = \partial^N + V_1\partial^{n-1} + \cdots + V_{N-1}\partial + V_N$$

coefficients V_i can be expanded as

$$V_i = \frac{1}{x^i}\sum_{k=0}^{\infty} V_{i,k} x^{-rk}. \tag{5.1}$$

In what follows we need to split the set $M = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of roots of the indicial equation at ∞ for L into subsets of equivalent modulo $\mathbb{Z}$ numbers.

For an arbitrary set M_i denote by λ the number in M_i with minimal real part. The next lemma is a version of a classical result (see, e.g., [21]) and shows how one can pick an $\mathbb{Z}_r$-invariant function from $\operatorname{Ker} L$.

Lemma 5.1. *If λ is the minimal number of a set M_i, then there is a function ϕ_λ from* Ker L *which can be expanded around ∞ as*:

$$\phi_\lambda(x) = x^\lambda \sum_{k=0}^{\infty} c_k x^{-kr}, \quad c_0 = 1. \tag{5.2}$$

We omit the proof as it repeats the classical one.

Given a function ϕ_λ we construct a first order operator by setting

$$P_\lambda = \partial_x - \frac{\phi'_\lambda}{\phi_\lambda}.$$

Then the operator L can be factorised as $L = Q_\lambda P_\lambda$ and after we perform the Darboux transformation

$$L = Q_\lambda P_\lambda \to \tilde{L} = P_\lambda Q_\lambda \tag{5.3}$$

the new operator $\tilde{L}$ will have the following properties:

Proposition 5.2. *Assume that the operator L solves the string equation with a $\mathbb{Z}_r$-invariant wave operator $K = 1 + \alpha_1 \partial^{-1} + \cdots$, $\operatorname{ord}(\alpha_i) \leqslant -i$. Then*

(i) *every operator which is obtained by a Darboux transformation described above also solves the string equation*;

(ii) *if $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ are the roots of the indicial equation at ∞ of L and $\lambda = \lambda_{i_0}$ is the number with minimal real part from some M_i then the roots of the indicial equation at ∞ of $\tilde{L}$ are $\tilde{\lambda}_k = \lambda_k - 1$ for $k \neq i_0$ and $\tilde{\lambda}_{i_0} = \lambda_{i_0} + (N-1)$.*

Proof. Put

$$\widetilde{K} = P_\lambda K \partial^{-1}. \tag{5.4}$$

Now we will check that $\widetilde{L}$, $\widetilde{K}$ also satisfy the conditions of the lemma. Let's check the first condition of Definition 3.2.

$$\widetilde{L}\widetilde{K} = P_\lambda Q_\lambda P_\lambda K \partial^{-1} = P_\lambda L K \partial^{-1} = P_\lambda K \partial^N \partial^{-1} = \widetilde{K}\partial^N.$$

Further denote by $\widetilde{Q} = P_\lambda Q Q_\lambda$ and note that the following sequence of equalities holds:

$$\widetilde{Q}\widetilde{K} = P_\lambda Q Q_\lambda P_\lambda K \partial^{-1} = P_\lambda Q L K \partial^{-1}.$$

Using the equalities $LK = K\partial^N$ and $QK = Kx\partial^{nN+1}$ the last relations give

$$\widetilde{Q}\widetilde{K} = P_\lambda K x \partial^{nN+1}\partial^{N-1} = \widetilde{K}\partial x \partial^{N(n+1)} = \widetilde{K} x \partial^{(n+1)N+1} + \widetilde{K}\partial^{(n+1)N}.$$

Now it is clear that $\widetilde{K} x \partial^{(n+1)N+1}\widetilde{K}^{-1} = \widetilde{Q} - \widetilde{L}^{n+1}$ is a differential operator.

Denote by g the polynomial associated with P_λ and by h the one associated with Q_λ. Obviously $g(D) = D - \lambda$ and it has weight -1. Then using Lemma 3.15 we get:

$$h(D-1)g(D) = (D-\lambda_1)(D-\lambda_2)\cdots(D-\lambda_N),$$
$$g\big(D-(N-1)\big)h(D) = (D-\tilde{\lambda}_1)(D-\tilde{\lambda}_2)\cdots(D-\tilde{\lambda}_N).$$

From these equalities we get the second assertion in the lemma. □

After this proposition we are close to our final goal.

Proposition 5.3. *Let L be a bispectral operator with coefficients satisfying* (0.4). *Then by finitely many $\mathbb{Z}_r$-invariant Darboux transformations we can transform it into a Bessel operator.*

Proof. We will perform Darboux transformations in the following way: start with $L_0 = L$. Choose an index i, if there is any, for which the difference between numbers in M_i with maximal real part and with minimal real part exceeds N. Denote by λ the number in M_i with the minimal real part, set $P_1 = P_\lambda$ and factorise L as $L = R_1 P_1$ then the Darboux transformation will be

$$L_0 = L = R_1 P_1 \to L_1 := P_1 R_1.$$

According to Proposition 5.2 the sets $M_j^0 := M_j$ will be transformed into sets M_j^1 for which the difference between the numbers with maximal and minimal real parts are the same for $i \neq j$. When $i = j$ there are two cases:

Case 1. There is exactly one number in M_i with minimal real part. The differences between the numbers in M_i are integer. Thus there is a well defined ordering: $\lambda \geqslant \mu$, iff $\lambda - \mu \geqslant 0$, in fact $\lambda - \mu = \operatorname{Re}\lambda - \operatorname{Re}\mu$. Having in mind this remark the elements of M_i can be ordered as

$$\lambda < \mu_1 \leqslant \cdots \leqslant \mu_s.$$

Now the assumption about M_i means

$$\lambda + (N-1) \leqslant \mu_s - 1.$$

Due to Proposition 5.2 in the new set M_i^1 the following inequalities must hold: $\min M_i^1 \geqslant \lambda$, $\max M_i^1 = \mu_s - 1$. Hence, the difference between the maximal and the minimal number is reduced at least by 1.

Case 2. In M_i there is at least two numbers with minimal real part. Then the above Darboux transformation decreases the number of the roots with minimal real part at least by 1.

After finitely many Darboux transformations we obtain an operator L_m such that if M^m is the set of roots of the indicial equation at ∞ and M_j^m are the corresponding subsets modulo $\mathbb{Z}$ for M^m, then

$$\max M_j^m - \min M_j^m < N. \tag{5.5}$$

But again from Proposition 5.2 it follows that there is an operator

$$K_m = 1 + a_1\partial^{-1} + \cdots,$$

such that $L_m K_m = K_m \partial^N$ and there is an integer $n \geqslant 0$ for which

$$Q_m = K_m x \partial^{nN+1} K_m^{-1} \tag{5.6}$$

is differential. The minimal n with this property, according to Proposition 3.16 satisfies the inequality:

$$n \leqslant \frac{1}{N}\left(\max M_j^m - \min M_j^m\right).$$

Using (5.5) we see that n must be zero. Put in (5.6) $n = 0$ and compare the differential parts of the operators at both sides to conclude that

$$Q_m = x\partial_x.$$

Now comparing the coefficients at ∂^j at both sides of the string equation (0.6) with $n=0$ we obtain the equation

$$-xV_j' + jV_j = NV_j.$$

Integrating it, we obtain that

$$V_j = v_j x^{-N+j}, \quad v_j \in \mathbb{C}.$$

This shows that L_m is a Bessel operator. □

Proof of Theorem 0.2. It remains to show that the chaine of the above Darboux transformations can be replaced by one monomial. First we represent the chain by following graph:

$$L_0 = R_1P_1 \to L_1 = P_1R_1 = R_2P_2 \to L_2 = P_2R_2 = R_3P_3 \to \cdots \to L_m = P_mR_m.$$

If we set $A = R_1R_2\cdots R_m$ and $B = P_mP_{m-1}\cdots P_1$ then obviously:

$$L^m = L_0^m = R_1P_1R_1P_1\cdots R_1P_1 = R_1L_1^{m-1}P_1 = AB$$

and for the Bessel operator $L_{\beta'} := L_m$

$$L_{\beta'}^m = P_mR_mP_mR_m\cdots P_mR_m = P_mL_{m-1}^{m-1}R_m = BA. \tag{5.7}$$

The Darboux transformations do not change the rank of the operator. Thus the rank of $L_{\beta'}$ is r. If $r < N$ then according to Lemma 1.6 there is a monomial Darboux transformation which transforms $L_{\beta'}$ into L_β, where L_β is some Bessel operator of order r and rank r. But the monomial Darboux transformations connecting Bessel operators are transitive. Thus there is a monomial Darboux transformation connecting L and L_β. The only thing that we have to prove is that the operators A and B from (5.7) have rational coefficients.To prove this we need the following lemma. □

Lemma 5.4. *Assume that $P \in \mathcal{O}[\partial]$ is an operator with holomorphic at ∞ coefficients. If P divides from the right some power L_β^d of a Bessel operator*

$$L_\beta = x^{-N}(D-\beta_1)\cdots(D-\beta_N), \quad D = x\partial_x,$$

then the coefficients of P are rational.

Proof. Let n be the order of P and

$$\gamma = \beta^d = \bigl(\beta_1, \beta_1 + N, \ldots, \beta_1 + (d-1)N, \ldots, \beta_N, \beta_N + N, \ldots, \beta_N + (d-1)N\bigr).$$

First we prove that Ker P has a basis of elements f_i, $i = 1, 2, \ldots, n$, of the form:

$$f_i = x^{\gamma_i}\sum_{j=0}^{r_i} p_{ij}(x)(\ln x)^j, \quad p_{ir_i} \neq 0, \tag{5.8}$$

where p_{ij} are polynomials. In general every $f \in \operatorname{Ker} P$ can be written as:

$$f = \sum_{i=1}^{s} f_i \tag{5.9}$$

with f_i having of the form given by (5.8) and $\gamma_i - \gamma_j \notin \mathbb{Z}$ for $i \neq j$. The analytical continuation around the infinite point defines the monodromy map:

$$M_\infty : \operatorname{Ker} P \to \operatorname{Ker} P.$$

If an element $f = \sum_{i=1}^{s} f_i$ as in (5.9) and (5.8) is in $\operatorname{Ker} P$, then

$$M_\infty(f) = \sum_{i=1}^{s} \exp\bigl(2\pi\sqrt{-1}\gamma_i\bigr) x^{\gamma_i} \sum_{j=0}^{r_i} p_{ij}(x)\bigl(\ln x + 2\pi\sqrt{-1}\bigr)^j$$

is also in $\operatorname{Ker} P$.

Let s be the minimal number for which there is an element f as in (5.10), where none of the terms f_i is in $\operatorname{Ker} P$. From all such operators from $\operatorname{Ker} P$ with minimal s take one for which the number:

$$\min\{r_i \mid i = 1, 2, \ldots, s\}$$

is minimal. We can assume that $r_s = \min\{r_i \mid i = 1, 2, \ldots, s\}$. Then in the following element from $\operatorname{Ker} P$:

$$f - \exp\bigl(-2\pi\gamma_s\sqrt{-1}\bigr) M_\infty(f) = \sum_{i=1}^{s} \tilde{f}_i = x^{\tilde{\gamma}_i} \sum_{j=0}^{\tilde{r}_i} \tilde{p}_{ij}(x)(\ln x)^j$$

either the term $\tilde{f}_s$ vanishes (when $r_s = 0$) or the number $\tilde{r}_s = r_s - 1$ is less than r_s. In both cases this is a contradiction with the choice of f.

Having in mind the basis from (5.8) the action of the operator P can be written as (see [21]):

$$P\phi = \frac{\operatorname{Wr}(f_1, f_2, \ldots, f_n, \phi)}{\operatorname{Wr}(f_1, f_2, \ldots, f_n)}. \tag{5.10}$$

Note that each derivative $f_i^{(k)}$ has the form $f_i^{(k)} = x^{\gamma_i} F_{ik}(x, \ln x)$, where $F_{ik}(X, Y) \in \mathcal{L}[Y]$ is a polynomial in Y with coefficients – Laurent polynomials in X. Hence formula (5.10) gives:

$$P\phi = \frac{x^{\gamma_1+\gamma_2+\cdots+\gamma_s} \sum_{i=0}^{n} F_i(x, \ln x)\partial^i \phi}{x^{\gamma_1+\gamma_2+\cdots+\gamma_s} F_n(x, \ln x)},$$

where $F_i \in \mathcal{L}[Y]$. Thus the coefficient c_i in front of ∂^i is

$$c_i = \frac{F_i(x, \ln x)}{F_n(x, \ln x)}.$$

Since $c_i \in \mathcal{O}$ the monodromy map M_∞ preserves c_i. Hence

$$\frac{F_i(x, \ln x + 2\pi\sqrt{-1}l)}{F_n(x, \ln x + 2\pi\sqrt{-1}l)} = \frac{F_i(x, \ln x)}{F_n(x, \ln x)}$$

for every integer l and also for every $l \in \mathbb{C}$ since the above equality is equivalent to an equality between polynomials. Using that x and $\ln x$ are algebraically independent we get:

$$\frac{F_i(X, Y+l)}{F_n(X, Y+l)} = \frac{F_i(X, Y)}{F_n(X, Y)}$$

which on the other hand is equivalent to

$$\frac{F_i(X, Y+l)}{F_i(X, Y)} = \frac{F_n(X, Y+l)}{F_n(X, Y)}.$$

Take the derivative with respect to l and set $l = 0$. Then one sees that $F_i(X, Y) = c(X)F_n(X, Y)$. After putting first $Y = \ln x$, $X = x$ and then $Y = 0$, $X = x$ it follows that

$$c_i(x) = \frac{F_i(x, \ln x)}{F_n(x, \ln x)} = c(x) = \frac{F_i(x, 0)}{F_n(x, 0)}$$

is a rational function. □

6. Proof of the characterisation theorem

Essentially the proof of Theorem 0.5 has already been performed in the previous sections, as well as in [4,8]. Below we sketch a plan how to pick the pieces of the proof from these sources.

Proof of Theorem 0.5. The implication (1) → (3) is the content of Theorem 0.2. Next we consider (3) → (2). Here we use the Definition 1.4 for monomial Darboux transformations. If L_β is a Bessel operator then one factorises L_β^m as

$$L_\beta^m = QP, \tag{6.1}$$

where the operator P acts on ψ in the following way:

$$P = \frac{\mathrm{Wr}(f_1, f_2, \ldots, f_n, \psi)}{\mathrm{Wr}(f_1, f_2, \ldots, f_n)} \tag{6.2}$$

and the functions $f_1, \ldots, f_n$ have the structure prescribed in Definition 1.4. Having in mind the type of the kernel it is obvious that the operator P has only regular singularities. But then the same is true for the operator Q whose coefficients axe computed by induction from the (6.1). Then the same is true for the product PQ. At the end by the main result in [4] the latter operator is bispectral.

The implication (2) → (1) is trivial. The equivalence of (3) and (4) is the content of [8]. We briefly describe it.

First, we recall the definition of $W_{1+\infty}$, its subalgebras $W_{1+\infty}(N)$ and their bosonic representations introduced in [4]. The algebra w_∞ of the additional symmetries of the KP-hierarchy is isomorphic to the Lie algebra of regular polynomial differential operators on the circle

$$\mathcal{D} = \mathrm{span}\left\{z^\alpha \partial_z^\beta \mid \alpha, \beta \in \mathbb{Z},\ \beta \geqslant 0\right\}.$$

Its unique central extension [22] will be denoted by $W_{1+\infty}$. This algebra gives the action of the additional symmetries on tau-functions (see [2,27]). Denote by c the central element of $W_{1+\infty}$ and by $W(A)$ the image of $A \in \mathcal{D}$ under the natural embedding $\mathcal{D} \hookrightarrow W_{1+\infty}$ (as vector spaces). The algebra $W_{1+\infty}$ has a basis

$$c,\ J_k^l = W\bigl(-z^{l+k}\partial_z^l\bigr), \quad l, k \in \mathbb{Z},\ l \geqslant 0.$$

The commutation relations of $W_{1+\infty}$ can be written most conveniently in terms of generating series [22]

$$\begin{aligned}&\bigl[W\bigl(z^k \mathrm{e}^{xD_z}\bigr), W\bigl(z^m \mathrm{e}^{yD_z}\bigr)\bigr]\\ &\quad = \bigl(\mathrm{e}^{xm} - \mathrm{e}^{yk}\bigr) W\bigl(z^{k+m}\mathrm{e}^{(x+y)D_z}\bigr) + \delta_{k,-m}\frac{\mathrm{e}^{xm} - \mathrm{e}^{yk}}{1 - \mathrm{e}^{x+y}} c,\end{aligned} \tag{6.3}$$

where $D_z = z\partial_z$.

From the theory of KP-hierarchy it is well known that each operator L or its wave function (1.2) defines or can be defined by the so-called *tau-function*, which is a function $\tau(t_1, \ldots, t_n, \ldots)$ in infinite number of variables t_n, $n = 1, \ldots$. We denote the tau-functions of the Bessel operators L_β by τ_β. In [5] a family of highest weight modules $\mathcal{M}_\beta$ over $W_{1+\infty}$ has been constructed, using as a highest weight vector τ_β. We briefly describe them.

Introduce the subalgebra $W_{1+\infty}(N)$ of $W_{1+\infty}$ spanned by c and J_{kN}^l, $l, k \in \mathbb{Z}$, $l \geqslant 0$. It is a simple fact that $W_{1+\infty}(N)$ is isomorphic to $W_{1+\infty}$ (see [22]). Now put

$$\mathcal{M}_\beta = \operatorname{span}\bigl\{J_{k_1N}^{l_1} \cdots J_{k_pN}^{l_p}\tau_\beta \mid k_1 \leqslant \cdots \leqslant k_p < 0\bigr\}. \tag{6.4}$$

The main result of [8] can be summed up as:

Theorem 6.1. *If an element in a module $\mathcal{M}_\beta$ is a tau-function then the corresponding operator L is a monomial Darboux transformation of some Bessel operator $L_{\beta'}$ (with eventually different β'). If an operator L is a monomial Darboux transformation of a Bessel operator L_β then the corresponding tau-function belongs to the module $\mathcal{M}_\beta$.*

Obviously the above cited theorem gives the equivalence between (3) and (4). □

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