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Families of Biorthogonal Wavelets

A. KAROUI AND R. VAILLANCOURT
Department of Mathematics, University of Ottawa
Ottawa, ON, Canada K1N 6N5
rxvsg@acadvm1.uottawa.ca

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Abstract—Several families of biorthogonal wavelet bases are constructed with various properties. In particular, for a given filter, $\mathcal{F}_n(\xi)$, of finite length $2n+1$, a parametric family of dual filters, $\mathcal{F}_N^{\nu}(\xi)$, of length $2N+1$ is constructed. The parametric nature of the dual filters makes it possible to design the optimum dual filter $\mathcal{F}_N^{\nu_0}(\xi)$ corresponding to a fixed filter $\mathcal{F}_n(\xi)$.

1. INTRODUCTION

Recently, the growing interest in orthogonal wavelets is due, in great part, to their ability to represent wide classes of functions and operators without redundancy, and to the fast wavelet transform which makes possible the computer implementation of new, very efficient algorithms.

In pure mathematics, wavelets are used to characterize some functional spaces, such as $L^p(\mathbb{R}^n)$ for $0 < p < \infty$, Hölder and Hardy spaces, etc., [1]. A proof by means of wavelets of the famous $T(1)$ theorem of David and Journé [2] on the L^2-continuity of a class of linear singular integral operators is found in [3, pp. 267–278].

In numerical analysis, wavelets are used as an efficient tool for the rapid numerical application of certain types of linear operators to arbitrary vector-valued functions [4]. They are also used in the numerical solution of partial differential equations by mean of finite element methods [1, pp. 57–60]. They are widely and efficiently applied in engineering, for example, in sound analysis [5], image processing [6–8] to cite but a few.

An orthonormal wavelet basis for $L^2(\mathbb{R})$ is a family of functions

$$\psi_{j,k}(x) = 2^{-j/2}\psi\left(2^{-j}x - k\right), \qquad x \in \mathbb{R}, \qquad j,\ k \in \mathbb{Z}, \tag{1.1}$$

obtained by dilations and translations of a single (mother) wavelet $\psi \in L^2(\mathbb{R})$. Thus, any function f in $L^2(\mathbb{R})$ can be expressed in terms of the wavelets $\psi_{j,k}$:

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(f, \psi_{jk})\psi_{jk}(x), \tag{1.2}$$

where the equality holds in the strong L^2-topology and the wavelet coefficients are given by the scalar products

$$(f, \psi_{jk}) = \int_{-\infty}^{\infty} f(x)\,\overline{\psi_{jk}(x)}\,dx. \tag{1.3}$$

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The (nonsmooth) Haar basis [9] constitutes the first known wavelets. J. O. Strömberg [10] constructed the first orthonormal basis of the form (1.1) with a function ψ of class C^m for an arbitray integer m. In 1985, Y. Meyer [11] constructed an orthonormal wavelet basis which is an unconditional basis for various functional spaces. In [12], P. G. Lemarié constructed a wavelet basis for $L^2(\mathbb{R}^n)$ with bounded regularity, but with exponential decay. In 1988, Ingrid Daubechies [13] constructed orthonormal wavelet bases with compact support and arbitrarily high regularity.

We remark that the construction of the majority of useful wavelet bases is a consequence of the design of some 2π-periodic functions called wavelet filters by the signal processing community (see [14] in an early stage, and [15,16]. Moreover, any wavelet filter with finite or infinite length is a finite impulse response (FIR) or infinite impulse response (IIR) filter, respectively.

In many applications [17, p. 113; 18], it is necessary to use linear phase FIR filters. Unfortunately, such filters are impossible to design. On the other hand, biorthogonal wavelet bases provide us with compactly supported symmetric wavelets [19]. Biorthogonal wavelets are formed by a pair of families of dual (see Definition 2 below) wavelets, $\psi_{jk}(x)$ and $\widetilde{\psi}_{jk}(x)$, derived from two mother wavelets, $\psi(x)$ and $\widetilde{\psi}(x)$, respectively, and such that any function f in $L^2(\mathbb{R})$ can be written in either forms:

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} (f, \psi_{jk}) \widetilde{\psi}_{jk}(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left(f, \widetilde{\psi}_{jk}\right) \psi_{jk}(x). \tag{1.4}$$

Hence, f is decomposed by one family and reconstructed by the other. Ph. Tchamitchian [20] constructed the first family of biorthogonal wavelets. In [20], it is shown that it is possible to construct symmetric biorthogonal wavelet bases with arbitrary high preassigned regularity.

In this work, biorthogonal wavelet bases are constructed by an approach which differs from the one used in [19]. By this new approach, it is possible to construct a new class of biorthogonal wavelet bases with the following remarkable properties:

- symmetry,
- compact support,
- regularity,
- the dual filter, corresponding to a fixed wavelet filter $\mathcal{F}_0(\xi)$, is given in parametric form.

This paper is divided as follows. In Section 2, some necessary or sufficient conditions are stated for the construction of regular wavelets. In Section 3, we provide the numerical techniques for the construction of the wavelets and estimating their regularities and to extend a fixed family of biorthogonal wavelet filters to an infinite family. Numerical results are quoted in Section 4.

2. BIORTHOGONAL WAVELET BASES

In this section, we describe the fundamental steps of the construction of wavelet bases. The first step consists in designing a wavelet filter and constructing a scaling function. In the second step, the wavelets are constructed.

2.1. Preliminaries

An orthonormal wavelet basis $\{\psi_{j,k};\ j,k \in \mathbb{Z}\}$ is directly related to a multiresolution analysis (MRA) [21,22]. Let $V_0 \in L^2(\mathbb{R})$ be the subspace spanned by the orthonormal functions $\phi(x-k)$, $k \in \mathbb{Z}$. Define the space V_j obtained by dilating V_0 by 2^j,

$$f \in V_j \Leftrightarrow f\left(2^j \cdot\right) \in V_0. \tag{2.1}$$

An orthonormal basis of V_j is given by $\{\phi_{j,k};\ k \in \mathbb{Z}\}$, with $\phi_{j,k}(x) = 2^{-j/2}\phi\left(2^{-j}x - k\right)$. Then,

$$\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots, \tag{2.2}$$

and

$$\bigcap_{-\infty}^{\infty} V_j = \{0\}, \qquad \overline{\bigcup_{-\infty}^{\infty} V_j} = L^2(\mathbb{R}).$$

Since $V_1 \subset V_0$, there exists a sequence, $(\alpha_n)_{n\in\mathbf{Z}}$, of complex numbers [23] such that the function $\phi\left(2^{-1}x\right) \in V_1$, by (2.1), satisfies the two-scale difference equation

$$\frac{1}{2}\phi\left(\frac{x}{2}\right) = \sum_{n\in\mathbf{Z}} \alpha_n \phi(x-n), \qquad \alpha_n = \frac{1}{2}\int_{-\infty}^{\infty} \phi\left(\frac{x}{2}\right)\overline{\phi(x+n)}\,dx, \tag{2.3}$$

where $\alpha_n = O(|n|^{-m})$ for any integer $m \geq 1$. The Fourier transform of the first expression in (2.3) is

$$\widehat{\phi}(2\xi) = \left[\sum_{n\in\mathbf{Z}} \alpha_n e^{in\xi}\right]\widehat{\phi}(\xi). \tag{2.4}$$

The wavelet ψ is defined by

$$\psi(x) = 2\sum_{n\in\mathbf{Z}} (-1)^n \overline{\alpha_{1-n}}\phi(2x-n). \tag{2.5}$$

Let W_j be the orthogonal complement of V_j in V_{j+1}. It is shown in [1, pp. 71–73], that $\psi_{0n}(x) = \psi(x-n)$, $n \in \mathbb{Z}$, is an orthonormal basis of W_0. It then follows that, for fixed $j \in \mathbb{Z}$, the sequence $(\psi_{jk})_{k\in\mathbf{Z}}$ defined as in (1.1) is an orthonormal basis of W_j. Moreover, $\bigcup_{j\in Z} W_j$ is dense in $L^2(\mathbb{R})$.

DEFINITION 1. *Scaling functions, orthornormal wavelets and wavelet filters are defined as follows:*

(a) *the functions $(\phi_{jk})_{j,k\in\mathbf{Z}}$ are called the scaling functions generated from the (father) scaling function $\phi(x)$,*
(b) *the functions $(\psi_{jk})_{j,k\in\mathbf{Z}}$ are the orthonormal wavelets generated from the mother wavelet $\psi(x)$,*
(c) *$\mathcal{F}_0(\xi) = \sqrt{2}\left(\sum_{n\in\mathbf{Z}} \alpha_n e^{in\xi}\right)$ is a wavelet filter where the α_n are given by (2.3).*

We shall use the following definition [24, p. 151].

DEFINITION 2. *Two scaling functions ϕ and $\widetilde{\phi}$, generating possibly different multiresolution analyses of $L^2(\mathbb{R})$, are said to be dual scaling functions if their scalar product satisfy the following condition:*

$$\left(\phi(\cdot - j), \widetilde{\phi}(\cdot - k)\right) := \int_{-\infty}^{\infty} \phi(x-j)\overline{\widetilde{\phi}(x-k)}\,dx = \delta_{j,k}, \qquad j,\, k \in \mathbb{Z}. \tag{2.6}$$

In the biorthogonal case, we shall need two dual scaling functions $\phi(x)$, $\widetilde{\phi}(x)$ satisfying

$$\phi(x) = 2\sum_{n\in\mathbf{Z}} \alpha_n \phi(2x-n), \qquad \widetilde{\phi}(x) = 2\sum_{n\in\mathbf{Z}} \widetilde{\alpha}_n \widetilde{\phi}(2x-n). \tag{2.7}$$

By using condition (2.7) and the techniques employed for proving Theorem 1 in [1, pp. 72–73], the reader can easily verify that the pair of functions

$$\psi(x) = 2\sum_{n\in\mathbf{Z}} (-1)^n \overline{\widetilde{\alpha}_{1-n}}\phi(2x-n), \qquad \widetilde{\psi}(x) = 2\sum_{n\in\mathbf{Z}} (-1)^n \overline{\alpha_{1-n}}\,\widetilde{\phi}(2x-n), \tag{2.8}$$

are (in general, nonorthonormal) dual wavelets associated with the scaling functions $\phi(x)$, $\widetilde{\phi}(x)$. We remark that if the scaling functions, ϕ_{jk} and $\widetilde{\phi}_{jk}$, and their corresponding wavelets, ψ_{jk} and $\widetilde{\psi}_{jk}$, are defined as in (1.1), then any $f \in L^2(\mathbb{R})$, can be written in the forms

$$\begin{aligned} f(x) &= \sum_{k\in\mathbf{Z}} (f, \phi_{jk})\widetilde{\phi}_{jk}(x) + \sum_{\substack{n\in\mathbf{Z}\\ n<j}}\sum_{k\in\mathbf{Z}} (f, \psi_{nk})\widetilde{\psi}_{nk}(x) \\ &= \sum_{k\in\mathbf{Z}} \left(f, \widetilde{\phi}_{jk}\right)\phi_{jk}(x) + \sum_{\substack{n\in\mathbf{Z}\\ n<j}}\sum_{k\in\mathbf{Z}} \left(f, \widetilde{\psi}_{nk}\right)\psi_{nk}(x). \end{aligned} \tag{2.9}$$

By taking the Fourier transform of $\phi(x/2)$ and $\widetilde{\phi}(x/2)$, we see by (2.7) that there exist a pair of 2π-periodic functions,

$$m_0(\xi) = \sum_{n\in\mathbf{Z}} \alpha_n e^{in\xi}, \qquad \widetilde{m}_0(\xi) = \sum_{n\in\mathbf{Z}} \widetilde{\alpha}_n e^{in\xi}, \tag{2.10}$$

such that

$$\widehat{\phi}(2\xi) = m_0(\xi)\widehat{\phi}(\xi), \qquad \widehat{\widetilde{\phi}}(2\xi) = m_0(\xi)\widehat{\widetilde{\phi}}(\xi). \tag{2.11}$$

Then the following important question arises: under what conditions on m_0 and $\widetilde{m}_0$ can one have a pair of dual scaling functions and consequently a biorthogonal wavelet basis. An answer, given in [19], is briefly summarized in the following subsection.

2.2. Necessary and Sufficient Conditions for the Existence of Biorthogonal Wavelet Bases

From now on, we assume that the coefficients, α_n and $\widetilde{\alpha}_n$, of m_0 and $\widetilde{m}_0$, respectively, as defined in (2.7), are real, satisfy the symmetry relations $\alpha_{-n} = \alpha_n$ and $\widetilde{\alpha}_{-n} = \widetilde{\alpha}_n$, and are finite in number. The last assumption is equivalent to the compact support property of the constructed wavelets.

To construct a pair of dual scaling functions leading to a biorthogonal family of wavelets, we use an efficient method (see [19]), which results from the biorthogonal version of the multiresolution analysis. This method is essentially based on (2.4) which implies that

$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0\left(2^{-j}\xi\right), \qquad \widehat{\widetilde{\phi}}(\xi) = \prod_{j=1}^{\infty} \widetilde{m}_0\left(2^{-j}\xi\right). \tag{2.12}$$

To obtain scaling functions that lead to regular biorthogonal wavelets, the functions $m_0(\xi)$ and $\widetilde{m}_0(\xi)$ have to satisfy certain conditions.

In [19], a set of conditions is provided on the dual 2π-periodic functions $m_0(\xi)$ and $\widetilde{m}_0(\xi)$ and consequently on the corresponding filters which are given, respectively, by

$$\mathcal{F}_0(\xi) = \sqrt{2}\, m_0(\xi), \qquad \widetilde{\mathcal{F}}_0(\xi) = \sqrt{2}\, \widetilde{m}_0(\xi).$$

These conditions insure that our biorthogonal wavelet bases have preassigned regularities.

For completeness, we briefly summarize these conditions. It follows from the biorthogonality conditon (2.6) on ϕ and $\widetilde{\phi}$, that m_0 and $\widetilde{m}_0$ satisfy the identity:

$$m_0(\xi)\overline{\widetilde{m}_0(\xi)} + m_0(\xi+\pi)\overline{\widetilde{m}_0(\xi+\pi)} = 1, \qquad \forall\, \xi \in [0,\pi]. \tag{2.13}$$

Hence, the coefficients α_n and $\widetilde{\alpha}_n$ satisfy the relation

$$\sum \alpha_n\, \widetilde{\alpha}_n = 1. \tag{2.14}$$

Moreover, (2.12) implies that ϕ and $\widetilde{\phi}$ are in $L^2(\mathbb{R})$ only if

$$m_0(0) = \widetilde{m}_0(0) = 1. \tag{2.15}$$

On the other hand, if the scaling functions are to be continuous, it is necessary that $m_0(\xi)$ and $\widetilde{m}_0(\xi)$ vanish at $\xi = \pi$:

$$m_0(\pi) = \widetilde{m}_0(\pi) = 0. \tag{2.16}$$

Note that conditions (2.13) and (2.14) imply neither the biorthogonality of the scaling functions nor that these are in $L^2(\mathbb{R})$. However, a positive answer is provided by the following proposition.

PROPOSITION. *Assume that both* $m_0(\xi)$ *and* $\widetilde{m}_0(\xi)$ *can be factored in the form:*

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^L f(\xi), \qquad \widetilde{m}_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{\widetilde{L}} \tilde{f}(\xi), \tag{2.17}$$

where L, $\widetilde{L} \geq 1$, *and suppose, that for some* k, $\tilde{k} > 0$,

$$B_k = \sup_{\xi} \left| f(\xi) f(2\xi) \cdots f\left(2^{k-1}\xi\right)\right|^{1/k} < 2^{L-1/2}, \tag{2.18}$$

$$\widetilde{B}_{\tilde{k}} = \sup_{\xi} \left| \tilde{f}(\xi)\, \tilde{f}(2\xi) \cdots \tilde{f}\left(2^{\tilde{k}-1}\xi\right)\right|^{1/\tilde{k}} < 2^{\widetilde{L}-1/2}. \tag{2.19}$$

Then, ϕ, $\widetilde{\phi} \in L^2(\mathbb{R})$ *and*

$$\int_{-\infty}^{\infty} \phi(x)\overline{\widetilde{\phi}(x-n)}\, dx = \delta_{0n}.$$

PROOF. See [19]. ∎

It has been shown (see [19]) that if (2.18) and (2.19) are satisfied, then there exist two positive numbers, ϵ, $\widetilde{\epsilon} > 0$, and a positive constant c such that

$$\left|\widehat{\phi}(\xi)\right| < c\,(1+|\xi|)^{-L+\log(B_k)/\log(2)-\epsilon}, \tag{2.20}$$

$$\left|\widehat{\widetilde{\phi}}(\xi)\right| < c\,(1+|\xi|)^{-\widetilde{L}+\log(B_{\tilde{k}})/\log(2)-\tilde{\epsilon}}. \tag{2.21}$$

Now, by Theorem 3.8 in [19], if $m_0(\xi)$ and $\widetilde{m}_0(\xi)$ satisfy (2.13) and if (2.20) and (2.21) are satisfied, then the dual wavelets constructed from the scaling functions $\phi(x)$ and $\widetilde{\phi}(x)$ generate two biorthogonal wavelet bases, in the sense that any $f \in L^2(\mathbb{R})$ can be written in both forms:

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} (f, \psi_{jk})\widetilde{\psi}_{jk}(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \left(f, \widetilde{\psi}_{jk}\right) \psi_{jk}(x). \tag{2.22}$$

Thus, the problem of constructing a biorthogonal wavelet basis has been reduced to the much easier problem of constructing 2π-periodic functions that satisfy conditions (2.13), (2.15), (2.18), and (2.19).

REMARK 1. If there exists a constant $c > 0$ such that (2.20) and (2.21) are satisfied, then ϕ and $\widetilde{\phi}$ belong to the Hölder spaces $C^{\epsilon}(\mathbb{R})$ and $C^{\tilde{\epsilon}}(\mathbb{R})$, respectively, for all $\epsilon < L - 1 - \log(B_k)/\log 2$ and $\widetilde{\epsilon} < \widetilde{L} - 1 - \log\left(B_{\tilde{k}}\right)/\log 2$.

In the previous sections, we have seen the basic theoretical steps for the construction of biorthogonal wavelet bases; however, our actual construction relies on the special approximation techniques described in the next section.

3. NUMERICAL TECHNIQUES FOR THE CONSTRUCTION OF BIORTHOGONAL WAVELETS

The aim of this section is to provide and justify the numerical techniques used in the construction of biorthogonal wavelet bases. In particular, we shall prove that it is possible to construct a pair of dual filters, such that the coefficients of one of them are given in parametric form. The idea of our numerical method for the construction of dual trigonometric polynomials is based on two major steps. The first step consists in the construction of a function, $m_0(\xi)$, which satisfies the conditions of Section 2. In the second step, a dual function, $\widetilde{m}_0(\xi)$, is determined in a straightforward way. Both steps are analyzed in detail in the following subsections.

3.1. The Construction of $m_0(\xi)$

By using a result from [13], one can easily see that if a 2π-periodic function, $m_0(\xi)$, satisfies the first parts of (2.17), (2.15), and (2.16), respectively, that is

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^L f(\xi), \qquad m_0(0) = 1, \qquad m_0(\pi) = 0, \tag{3.1}$$

and (2.18) for some $k > 0$, then $m_0(\xi)$ is a candidate for generating a biorthogonal wavelet basis. Hence, let $n_0 \geq 1$ be a positive integer and consider the function

$$m_0(\xi) = \sum_{n=-n_0}^{n_0} \alpha_n e^{in\xi}. \tag{3.2}$$

We note that the symmetry of $m_0(\xi)$, that is $m_0(-\xi) = m_0(\xi)$, implies the symmetry of the associated wavelet. To have symmetric wavelets, we require that the coefficients α_n satisfy the following relations:

$$\alpha_n = \alpha_{-n}, \qquad 1 \leq n \leq n_0. \tag{3.3}$$

From condition (3.1) at $\xi = 0$ and π, we derive the pair linear equations:

$$\alpha_0 + 2\sum_{n=1}^{n_0} \alpha_n = 1, \qquad \alpha_0 + 2\sum_{n=1}^{n_0} (-1)^n \alpha_n = 0, \tag{3.4}$$

in the $n_0 + 1$ unknowns α_j, $j = 0, 1, \dots, n_0$. By fixing $n_0 - 1$ of these, one obtains a unique solution to the system.

It remains to verify condition (2.18). This verification, in general, is not easy and cannot be done explicitly. Hence, one resorts to numerical methods to find a good approximation to the upper bound of B_k.

We remark that the problem simplifies considerably if, instead of estimating B_k, we estimate the maximum of the absolute value of a piecewise polynomial which approximates the function $\left|f(\xi)f(2\xi)\cdots f\left(2^{k-1}\xi\right)\right|$. This method, which turns out to be very efficient in our case, is given by the following theorem.

THEOREM 1. *Consider a 2π-periodic function,*

$$m_0(\xi) = \sum_{j=-n}^{N} \alpha_j \, e^{ij\xi}, \qquad \alpha_j \in \mathbb{R},\ -n \leq j \leq N. \tag{3.5}$$

Suppose that $m_0(0) = 1$ and $m_0(\xi)$ can be factored in the form

$$m_0(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^L f(\xi), \qquad L \geq 1. \tag{3.6}$$

If we write

$$F_k(\xi) = f(\xi)f(2\xi)\cdots f\left(2^{k-1}\xi\right), \tag{3.7}$$

then, for all $\epsilon > 0$ and $k > 0$, there exist a positive integer r and a finite partition of $[0, 2\pi]$, say $(I_i)_{i\in I}$, such that

$$\left|\sup_{\xi} |F_k(\xi)|^{1/k} - \sup_{\xi} |P_{F_k}(\xi)|^{1/2k}\right| < \epsilon,$$

where, for each i, the function $P_{F_k}(\xi)$ is equal to a polynomial of degree r if $\xi \in I_i$, and 0 otherwise.

Proof. The proof consists in three parts.

(a) In the first part, if

$$Q_n(\cos\xi) = a_0 + a_1\cos\xi + \cdots + a_n\cos n\xi, \qquad a_\nu \in \mathbb{R}, \qquad 0 \le \nu \le n,$$

we obtain an estimate of the upper bound of $\sup_\xi |Q_n(\cos\xi)|$. For $\nu = 1, 2, \dots, n$, we expand $\cos\nu\xi$ in a Taylor series around a point ξ_ν to be fixed later. Thus, we have

$$\cos\nu\xi = \sum_{j=0}^{r} (-1)^j \frac{[\nu(\xi-\xi_\nu)]^j}{j!}\cos^{(j)}(\xi_\nu) + R_{r+1}(\xi),$$

where

$$|R_{r+1}(\xi)| \le \nu^{r+1}\frac{|\xi-\xi_\nu|^{r+1}}{(r+1)!}.$$

Since $Q_n(\cos\xi)$ is 2π-periodic, it suffices to find the supremum over the interval $[0, 2\pi]$:

$$\sup_{\xi\in[0,2\pi]} |Q_n(\cos\xi)|.$$

Because

$$[0, 2\pi] = \bigcup_{m=1}^{([2\pi n]+1)/2} I_m, \qquad I_m = \left[\frac{2m-2}{n}, \frac{2m}{n}\right],$$

where $[r]$ denotes the integer part of the real number r, it follows that

$$\sup_{\xi\in[0,2\pi]} |Q_n(\cos\xi)| = \max_m\left\{\sup_{\xi\in I_m} |Q_n(\cos\xi)|\right\}.$$

Now, for a fixed m, $1 \le m \le ([2\pi n]+1)/2$, let $\xi_m = (2m-1)/n$. Then, for $\xi \in I_m$, a Taylor expansion of order r of $Q_n(\cos\xi)$ around ξ_m gives

$$\begin{aligned} Q_n(\cos\xi) &= a_0 + \sum_{\nu=1}^{n} a_\nu \left(\sum_{j=1}^{r} \frac{[\nu(\xi-\xi_m)]^j}{j!}\cos^{(j)}(\xi_m) + R_{r+1}(\xi)\right) \\ &= P^m_{Q_n}(\xi) + R(\xi). \end{aligned} \tag{3.8}$$

(b) In the second part, if $P_{Q_n}(\xi)$ is a function whose restriction on each I_m is equal to the polynomial $P^m_{Q_n}(\xi)$ and 0 outside, then it is clear that

$$\sup_{\xi\in[0,2\pi]} |Q_n(\cos\xi) - P_{Q_n}(\xi)| \le \sum_{\nu=1}^{n} \frac{|a_\nu|}{(r+1)!}.$$

Since, $m_0(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^L f(\xi)$, then necessarily

$$f(\xi)\overline{f}(\xi) = \sum_{-n}^{N-L} \beta_n^2 + 2\sum_{\substack{-n\le i,j\le N-L \\ i\ne j}} \beta_i\beta_j\cos((i-j)\xi).$$

Since $f(\xi)\overline{f(\xi)}$ is real and symmetric in $e^{i\xi}$, then, with $M = N - L + n$, we have

$$f(\xi)\overline{f(\xi)} = \sum_{j=-M}^{M} \gamma_j\, e^{ic_j\xi}, \qquad c_{-j} = -c_j,\ \gamma_{-j} = \gamma_j,\ 1 \le j \le M.$$

By the first part of the proof, there exists a function $P_f(x)$, associated with a finite partition, $(I_n)_n$, of $[0, 2\pi]$, such that

$$\sup_{\xi} |f(\xi)\overline{f(\xi)} - P_f(\xi)| \le \sum_{j=-M}^{M} \frac{|\gamma_j|}{(r+1)!}.$$

Since $f(0) = 1$, then $\sup_\xi f(\xi) \ge 1$. Thus,

$$\left| \sup_{\xi} |f(\xi)| - \sup_{\xi} |P_f(\xi)|^{1/2} \right| \le \left(\sum_{j=-M}^{M} \frac{|\gamma_j|}{(r+1)!} \right)^{1/2}.$$

(c) In the third and last part, we prove the general case. Take any positive integer k, $k \ge 1$, and define the function $F_k(\xi)$ by the product:

$$F_k(\xi) = f(\xi)\, f(2\xi) \cdots f(2^{k-1}\xi).$$

The reader can easily check that if $K = \left(1 + 2^{k-1}\right) 2^{k-2}$, then

$$\overline{F_k(\xi)} F(\xi) = \sum_{j=-KM}^{KM} \beta_j e^{i d_j \xi}, \qquad d_{-j} = -d_j,\ \beta_{-j} = \beta_j,\ 1 \le j \le KM.$$

By choosing the partition

$$[0, 2\pi] = \bigcup_{l=1}^{([2KM\pi]+1)/2} I_l, \qquad I_l = \left[\frac{2l-2}{KM}, \frac{2l}{KM} \right],$$

and using the techniques of the first part of the proof, we construct a function P_{F_k}, whose restriction on each of the intervals I_l is a polynomial of degree r, and which is equal to zero outside $[0, 2\pi]$. Moreover, $P_{F_k}(\xi)$ satisfies:

$$\sup_{\xi \in [0, 2\pi]} \left| \overline{F(\xi)} F_k(\xi) - P_{F_k}(\xi) \right| \le \sum_{j=-KM}^{KM} \frac{|\beta_j|}{(r+1)!}.$$

Again, since $f(0) = 1$, then $\sup_\xi |F_k(\xi)| \ge 1$, which implies the following inequalities:

$$\begin{aligned} \sup_{\xi} |P_{F_k}(\xi)| &\le \sup_{\xi} |F_k(\xi)|^2 + \sum_{j=-KM}^{KM} \frac{|\beta_j|}{(r+1)!} \\ &\le \sup_{\xi} |F_k(\xi)|^2 \left[1 + \frac{1}{\sup_\xi |F_k(\xi)|^2} \sum_{j=-KM}^{KM} \frac{|\beta_j|}{(r+1)!} \right] \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\left| \left[\sup_{\xi} |P_{F_k}(\xi)| \right]^{1/2k} - \left[\sup_{\xi} |F_k(\xi)| \right]^{1/k} \right| \\ &\le \left[\sup_{\xi} |F_k(\xi)| \right]^{1/k} \left\{ \left[1 + \frac{1}{\sup_\xi |F_k(\xi)|^2} \sum_{j=-KM}^{KM} \frac{|\beta_j|}{(r+1)!} \right]^{1/2k} - 1 \right\} \\ &\le \text{const} \left\{ \left[1 + \frac{1}{\sup_\xi |F_k(\xi)|^2} \sum_{j=-KM}^{KM} \frac{|\beta_j|}{(r+1)!} \right]^{1/2k} - 1 \right\}. \end{aligned} \tag{3.10}$$

Finally, since $m_0(0) = 1$ implies that $\sup_\xi |F_k(\xi)| \ge 1$, then the right-hand side of (3.10) can be made arbitrarily small by choosing r arbitrarily big. ∎

REMARK 2. AN APPROXIMATION TECHNIQUE. In practice, it is convenient to discretize the function $P_{F_k}(\xi)$ as follows. For a given finite partition, $(I_i)_i$, of $[0, 2\pi]$, the restriction of $P_{F_k}(\xi)$ on I_i is approximated by the constant value $\overline{F_k(\xi_i)}F_k(\xi_i)$, where ξ_i is the midpoint of the interval I_i. Note that, from the proof of the previous theorem, it follows that the error made in approximating $\sup_\xi |F_k(\xi)|^{1/k}$ is bounded by

$$\text{const}\left\{\left[1 + \frac{1}{\sup_\xi |F_k(\xi)|^2} \sum_{j=-KM}^{KM} |\beta_j|\, h\right]^{1/2k} - 1\right\}, \tag{3.11}$$

where h is the size of the partition.

We shall use this technique whenever an approximation to the upper bound of a real function is required. In particular, we shall use it to decide whether or not the trigonometric function $m_0(\xi)$ is a candidate for generating a biorthogonal wavelet basis.

3.2. The Construction of $\widetilde{m}_0(\xi)$

Once $m_0(\xi)$ is constructed, its dual, $\widetilde{m}_0(\xi)$, is constructed in a straightforward way from the identity (2.13):

$$m_0(\xi)\overline{\widetilde{m}_0(\xi)} + m_0(\xi+\pi)\overline{\widetilde{m}_0(\xi+\pi)} = 1, \qquad \forall\, \xi \in [0, \pi]. \tag{3.12}$$

Since $m_0(0) = 1$ and $m_0(\pi) = 0$, then necessarily

$$\widetilde{m}_0(0) = 1. \tag{3.13}$$

If we require some regularity (at least continuity) on the wavelet $\widetilde{\psi}(x)$, then by an argument given in [13], $\widetilde{m}_0(\xi)$ also has to satisfy the condition:

$$\widetilde{m}_0(\pi) = 0. \tag{3.14}$$

It is trivial to see that if the two functions

$$m_0(\xi) = \sum_{n=-n_0}^{n_0} \alpha_n e^{in\xi}, \tag{3.15}$$

and

$$\widetilde{m}_0(\xi) = \sum_{n=-N_0}^{N_0} \beta_n e^{in\xi}, \qquad N_0 = n_0 + (2k+1), \tag{3.16}$$

satisfy (3.12), then the number $N_0 + n_0$ has to be an odd integer.

If $m_0(\xi)$ is of the form (3.15), then by choosing a dual symmetric function $\widetilde{m}_0(\xi)$ of the form (3.16) for some integer k, and applying conditions (3.12)–(3.14), one obtains the following linear system of equations in β_j:

$$\sum_{i+j=0} \alpha_i \beta_j = 1, \qquad \sum_{i+j=2n} \alpha_i \beta_j = 0, \quad 1 \le n \le \frac{n_0 + N_0 - 1}{2}, \qquad \beta_0 + 2\sum_{j=1}^{N_0} (-1)^j \beta_j = 0, \tag{3.17}$$

the solution of which are the coefficients of $\widetilde{m}_0(\xi)$. The numerical method given in Remark 2 is then used to decide whether or not the dual trigonometric polynomial $\widetilde{m}_0(\xi)$ generates a biorthogonal wavelet basis.

Unlike other known methods for finding $\widetilde{m}_0(\xi)$, this one has the interesting feature of providing an infinite family of dual filters all of the same length. In fact, it is possible to extend the solution of (3.17) in such a way as to generate an infinite set of dual filters all of the same length. This is given by the following theorem.

THEOREM 2. *Consider a 2π-periodic function,*

$$m_{n_0}(\xi) = \sum_{j=-n_0}^{n_0} \alpha_j e^{ij\xi}, \qquad \alpha_j \in \mathbb{R}, \quad \alpha_j = \alpha_{-j}, \qquad 1 \le j \le n_0. \tag{3.18}$$

Assume that, for some $N_0 > n_0$, there exists a real dual trigonometric function

$$\widetilde{m}_{N_0}(\xi) = \sum_{j=-N_0}^{N_0} \beta_j e^{ij\xi}, \qquad \beta_j \in \mathbb{R}, \quad \beta_j = \beta_{-j}, \qquad 1 \le j \le N_0, \tag{3.19}$$

such that $m_{n_0}(\xi)$ and $\widetilde{m}_{N_0}(\xi)$ satisfy condition (3.12) and $\widetilde{m}_{N_0}(\xi)$ factors in the form

$$\widetilde{m}_{N_0}(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^2 \tilde{f}(\xi), \tag{3.20}$$

where $\tilde{f}(\xi)$ is a trigonometric function satisfying

$$\sup_{\xi} \left| \tilde{f}(\xi)\, \tilde{f}(2\xi) \cdots \tilde{f}\left(2^{k-1}\xi\right) \right|^{1/k} \le 2^{3/2-\epsilon}, \tag{3.21}$$

for some positive integer $k \ge 1$ and $\epsilon > 0$. Then, for all $N = N_0 + 2l$, l a positive integer, there exists a set S of trigonometric functions of length $2N+1$, dual to $m_{n_0}(\xi)$ and having parametric coefficients.

PROOF. To prove, for a fixed positive intger l, that there exists an infinite set of dual trigonometric functions of length $N_0 + 2l$, it is enough to prove the result for $N = N_0 + 2$. Hence, if we let

$$\widetilde{m}_N(\xi) = \sum_{j=-N_0-2}^{N_0+2} \gamma_j e^{ij\xi},$$

where

$$\gamma_j = \begin{cases} \beta_j + \delta_j, & \text{if } -N_0 \le j \le N_0, \\ \delta_j, & \text{if } N_0 + 1 \le |j| \le N_0 + 2, \end{cases}$$

then,

$$\widetilde{m}_N(\xi) = \widetilde{m}_{N_0}(\xi) + \widetilde{m}_{N_0}^1(\xi),$$

where

$$\widetilde{m}_{N_0}^1(\xi) = \sum_{j=-N_0-2}^{N_0+2} \delta_j e^{ij\xi},$$

and $\delta_j = \delta_{-j}$. Since $\widetilde{m}_N(\xi)$ has to satisfy the identity

$$m_{n_0}(\xi)\, \widetilde{m}_N(\xi) + m_{n_0}(\xi+\pi)\, \widetilde{m}_N(\xi+\pi) = 1, \tag{3.22}$$

then the coefficients of $\widetilde{m}_{N_0}^1(\xi)$ have to satisfy the following homogeneous system of $(n_0+N_0+3)/2$ linear equations in δ_j:

$$\begin{aligned} &\delta_0 + 2 \sum_{j=1}^{N_0+2} (-1)^j \delta_j = 0, \\ &\sum_{i+j=2n} \alpha_i \delta_j = 0, \qquad 0 \le n \le \frac{n_0 + N_0 + 1}{2}, \end{aligned} \tag{3.23}$$

in the N_0+3 unknowns δ_j, $j=0,1,\dots,N_0+2$. Since $N_0 > n_0$, this system has a parametric solution of the form:

$$\delta_j(\delta_{N_0+2}) = \tau_j\,\delta_{N_0+2}, \qquad 0 \le j \le N_0+1.$$

Hence,

$$\widetilde{m}_N(\xi) = \widetilde{m}_{N_0}(\xi) + \sum_{j=-N_0-2}^{N_0+2} \tau_j\,\delta_{N_0+2}\,e^{ij\xi}$$

satisfies equation (3.12).

To prove that, under some conditions on the τ_j, we obtain dual filters that lead to the construction of an infinite family of biorthogonal wavelet bases, we consider the matrix, A, associated with the linear system (3.17) in the unknowns β_j. Since the elements of A are bounded, then for all j, $0 \le j \le N_0+2$, we have $|\tau_j| \le C$ for some constant C. Moreover, because $\widetilde{m}^1_{N_0}(\xi)$ is symmetric and $\widetilde{m}^1_{N_0}(\pi)=0$, this function can be factored in the form

$$\widetilde{m}^1_{N_0}(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^2 \tilde{f}^1(\xi).$$

This implies that

$$\widetilde{m}_N(\xi) = \widetilde{m}_{N_0}(\xi) + \widetilde{m}^1_{N_0}(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^2 \left[\tilde{f}(\xi) + \tilde{f}^1(\xi)\right].$$

Since, $|\tau_j| \le C$, then there exist two real numbers, $l_{N_0+2} < L_{N_0+2}$, such that for all $\delta_{N_0+2} \in [l_{N_0+2}, L_{N_0+2}]$, we have

$$\sup_\xi \left|\tilde{f}^1(\xi)\right| < 2^{3/2}\left(2^{-\epsilon/2} - 2^{-\epsilon}\right).$$

If we write $\widetilde{F}(\xi) := \tilde{f}(\xi) + \tilde{f}^1(\xi)$, then it is clear that

$$\sup_\xi \left|\widetilde{F}(\xi)\widetilde{F}(2\xi)\cdots\widetilde{F}\left(2^{k-1}\xi\right)\right|^{1/k} < 2^{3/2}\times 2^{-\epsilon/2}.$$

Consequently, for all $\delta_{N_0+2} \in [l_{N_0+2}, L_{N_0+2}]$, there exists a dual trigonometric function of length N_0+2.

By repeating the above technique as many times as required, one easily proves that there exist two real numbers, $l_N < L_N$, such that for all $\delta_N \in [l_N, L_N]$, there exists a dual trigonometric function $\widetilde{m}_{\delta_N}(\xi)$ of length $2N+1$, the coefficients of which depend linearly on the parameter δ_N. Furthermore, $m_{n_0}(\xi)$ and $\widetilde{m}_{\delta_N}(\xi)$ generate a biorthogonal wavelet basis. ∎

4. NUMERICAL RESULTS

The techniques of the previous section have been used to construct filters of length five, seven and nine, respectively. The coefficients of the dual of each filter are given in parametric form. Let $m_N(\xi)$ and $\widetilde{m}_N(\xi)$ denote the dual trigonometric functions that generate a set of biorthogonal wavelet bases. Here, the integer N stands for the number of vanishing moments [4] of the corresponding wavelets. Since, generally, the coefficients of $\widetilde{m}_N(\xi)$ are given in parametric form, we have used the numerical techniques of the previous section to obtain an approximation to the range, $[l_N, L_N]$, of the parameter μ, for which condition (2.19) is satisfied by $\widetilde{m}_N(\xi)$.

To obtain the filters associated with $m_N(\xi)$ and $\widetilde{m}_N(\xi)$, it suffices to multiply their coefficients by $\sqrt{2}$.

The decay associated with the Fourier transform of a scaling function $\phi(x)$ is defined as the largest positive real number ϵ such that, for some constant C, the following inequality holds:

$$\int_{-\infty}^{\infty} |\widehat{\phi}(\xi)|(1+|\xi|)^\epsilon\,d\xi < C.$$

In this case the scaling function $\phi(x)$ and the corresponding wavelet $\psi(x)$ are at least of class $C^{\epsilon-1}$.

In Table 1, we list the coefficients α_n of $m_N(\xi)$ and β_n of $\widetilde{m}_N(\xi)$, for $N = 2$, 4, 6.

In Table 2, we give the range $[l_N, L_N]$ of the parameter μ appearing in $\widetilde{m}_N(\xi)$ of Table 1, and list an estimate of the decays ϵ_N, $\widetilde{\epsilon}_N$ associated with ϕ_N and $\widetilde{\phi}_N$, respectively, where the parameter μ is set, respectively, to 1.0, 1.25 and 2.5.

Table 1. The coefficients α_n of $m_N(\xi)$ and β_n of $\widetilde{m}_N(\xi)$.

N	n	α_n	β_n
2	0	0.550	$0.569\,105\,691\,0 + 0.060\,975\,609\,8\,\mu$
	±1	0.250	$0.365\,650\,406\,5 - 0.067\,073\,170\,7\,\mu$
	±2	−0.025	$-0.083\,333\,333\,3$
	±3	0.000	$-0.120\,528\,455\,3 + 0.070\,121\,951\,2\,\mu$
	±4	0.000	$0.048\,780\,487\,7 - 0.030\,487\,880\,4\,\mu$
	±5	0.000	$0.004\,878\,048\,7 - 0.003\,048\,788\,0\,\mu$
4	±0	0.593 750 0	$0.520\,023\,738\,6 + 0.011\,332\,417\,4\,\mu$
	±1	0.304 687 5	$0.295\,215\,790\,9 - 0.009\,043\,040\,2\,\mu$
	±2	−0.046 875 0	$-0.043\,576\,845\,4 - 0.003\,434\,065\,9\,\mu$
	±3	−0.054 687 5	$-0.066\,317\,288\,8 + 0.014\,079\,670\,3\,\mu$
	±4	0.000 000 0	$0.044\,939\,760\,7 - 0.005\,265\,567\,7\,\mu$
	±5	0.000 000 0	$0.022\,701\,399\,3 - 0.005\,380\,036\,6\,\mu$
	±6	0.000 000 0	$-0.013\,241\,336\,3 + 0.003\,434\,065\,9\,\mu$
	±7	0.000 000 0	$-0.001\,599\,901\,5 + 0.003\,434\,065\,9\,\mu$
	±8	0.000 000 0	$0.001\,866\,551\,8 - 0.000\,400\,641\,0\,\mu$
6	±0	0.480 468 750	$0.652\,550\,142\,8 + 0.002\,034\,171\,4\,\mu$
	±1	0.301 562 500	$0.311\,632\,251\,0 - 0.001\,218\,971\,7\,\mu$
	±2	0.026 562 500	$-0.147\,006\,292\,1 - 0.000\,888\,136\,7\,\mu$
	±3	−0.051 562 500	$-0.094\,861\,784\,2 + 0.002\,128\,062\,5\,\mu$
	±4	−0.016796875	$0.102\,878\,181\,4 - 0.000\,728\,562\,3\,\mu$
	±5	0.000 000 000	$0.037\,606\,500\,2 - 0.001\,058\,792\,7\,\mu$
	±6	0.000 000 000	$-0.043\,722\,261\,0 + 0.000\,861\,540\,9\,\mu$
	±7	0.000 000 000	$-0.003\,013\,442\,6 + 0.000\,101\,345\,9\,\mu$
	±8	0.000 000 000	$0.013\,307\,065\,2 - 0.000\,288\,523\,5\,\mu$
	±9	0.000 000 000	$-0.001\,927\,659\,8 + 0.000\,057\,019\,6\,\mu$
	±10	0.000 000 000	$-0.001\,731\,765\,3 + 0.000\,026\,595\,7\,\mu$
	±11	0.000 000 000	$0.000\,564\,135\,6 - 0.000\,008\,663\,7\,\mu$

Table 2. The range $[l_N, L_N]$, for $N = 2$, 4, 6, of the parameter μ appearing in $\widetilde{m}_N(\xi)$ of Table 1 and the optimum lower bounds, ϵ_N and $\widetilde{\epsilon}_N$, associated, respectively, with $\widehat{\phi}_N(\xi)$ and $\widehat{\widetilde{\phi}}_N(\xi)$.

N	$[l_N, L_N]$	ϵ_N	$\tilde{\epsilon}_N$
2	[−0.13, 2.21]	1.609 2	1.452 9
4	[−9.50, 8.70]	2.111 5	2.916 6
6	[−39.50, 41.5]	4.021 7	2.544 3

Eight iterations of the constructive cascade algorithm given in [25, pp. 202–205], produce a good approximation to the graphs of the scaling functions and the corresponding wavelets.

In Figures 1, 2 and 3, we present three sets of graphs of $\phi_N(x)$, $\widetilde{\phi}_N(x)$, $\psi_N(x)$, and $\widetilde{\psi}_N(x)$, corresponding to $N = 2$, 4, and 6, respectively. In these figures, the parameter μ was set to 1.0, 1.25, 2.5, respectively.

Lastly, in Figure 4, the decays $\tilde{\epsilon}(\mu)$, associated with the Fourier transforms $\widehat{\widetilde{\phi}}_N(\xi)$, of the parametric scaling functions, are graphed against the parameter μ, for N =2, 4, and 6, respectively.

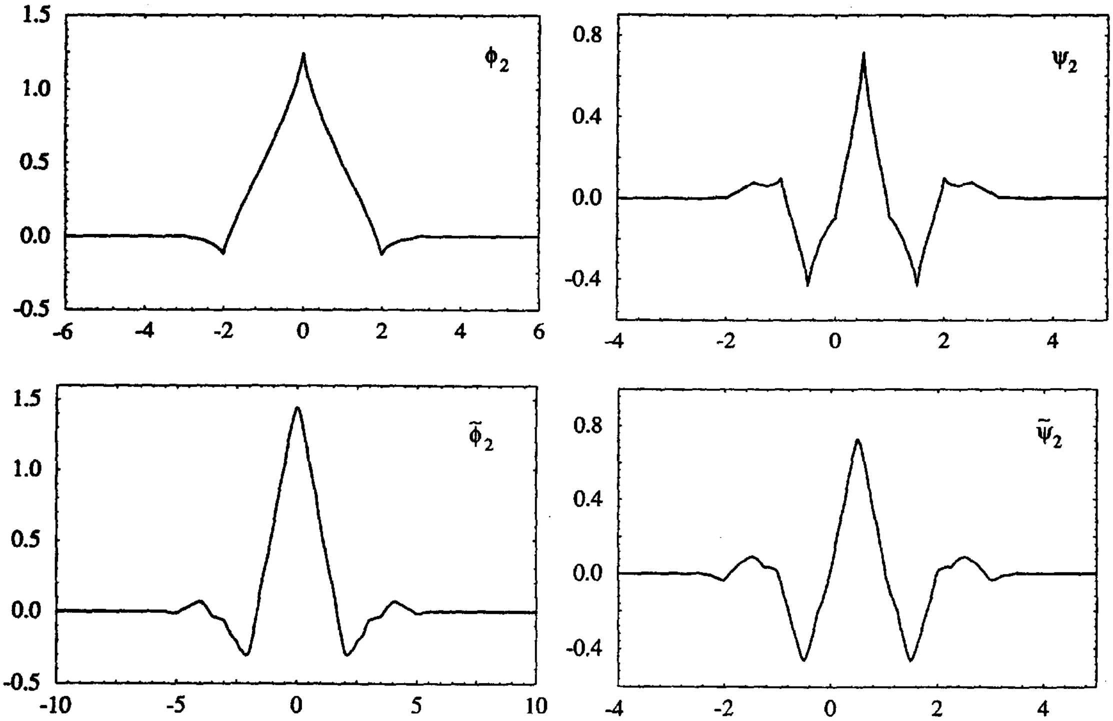

Figure 1. Dual scaling functions, $\phi_N(x)$, $\widetilde{\phi}_N(x)$, and corresponding wavelets, $\psi_N(x)$, $\widetilde{\psi}_N(x)$, for $N = 2$ and $\mu = 1.0$.

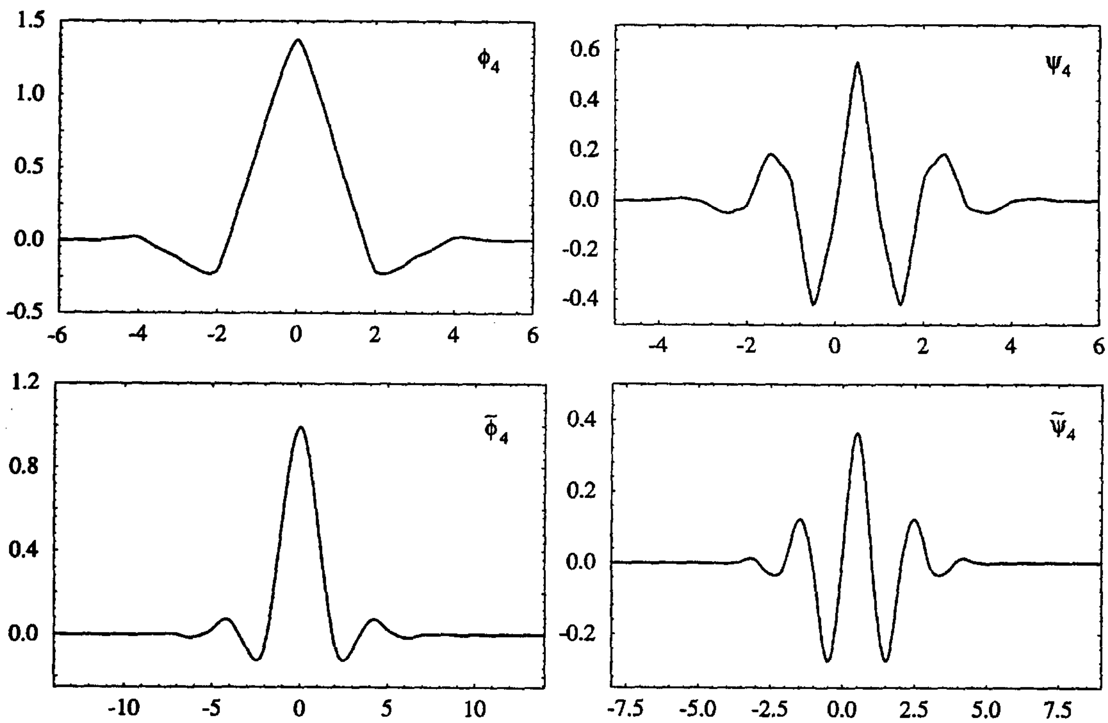

Figure 2. Dual scaling functions, $\phi_N(x)$, $\widetilde{\phi}_N(x)$, and corresponding wavelets, $\psi_N(x)$, $\widetilde{\psi}_N(x)$, for $N = 4$ and $\mu = 1.25$.

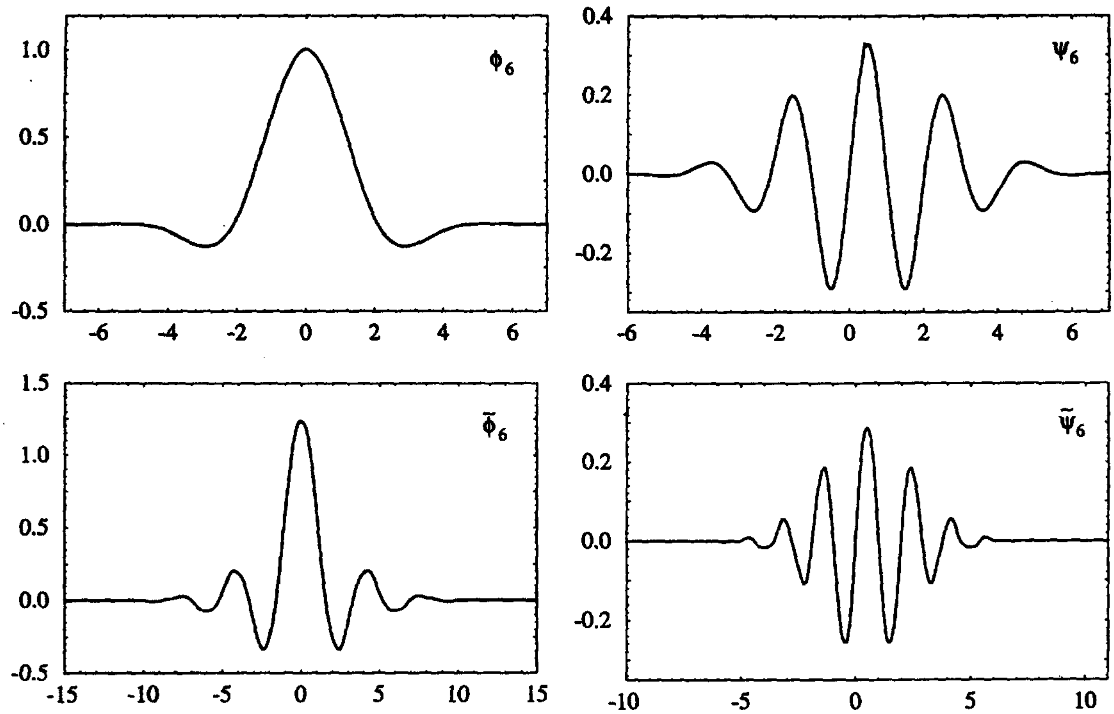

Figure 3. Dual scaling functions, $\phi_N(x)$, $\tilde{\phi}_N(x)$, and corresponding wavelets, $\psi_N(x)$, $\tilde{\psi}_N(x)$, for $N = 6$ and $\mu = 2.5$.

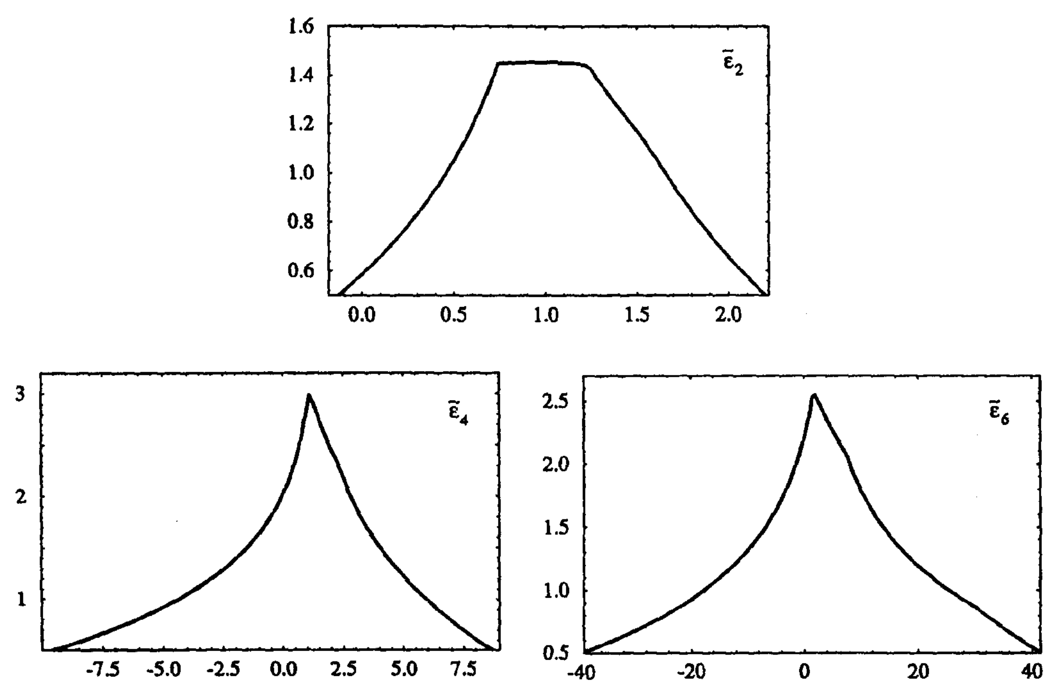

Figure 4. Graphs of decay function, $\tilde{\varepsilon}_N(\mu)$, of $\widehat{\tilde{\phi}}_N(\xi)$ for $N = 2$, 4 and 6, respectively.

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