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Numerical Solution of a Class of Random Boundary Value Problems*

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This paper deals with the nonlinear two point boundary value problem

$$y'' = f(x, y, y', R_1, ..., R_n), \qquad x_0 < x < x_f$$
$$S_1\, y(x_0) + S_2\, y'(x_0) = S_3\,, \qquad S_4\, y(x_f) + S_5\, y'(x_f) = S_6$$

where $R_1, ..., R_n$, $S_1, ..., S_6$ are bounded continuous random variables. An approximate probability distribution function for $y(x)$ is constructed by numerical integration of a set of related deterministic problems. Two distinct methods are described, and in each case convergence of the approximate distribution function to the actual distribution function is established. Primary attention is placed on problems with two random variables, but various generalizations are noted. As an example, a nonlinear one-dimensional heat conduction problem containing one or two random variables is studied in some detail.

1. INTRODUCTION

In many areas of application there has recently been increasing interest in mathematical models that include random effects, for example, initial or boundary value problems for random differential equations. While there are powerful and fairly general methods available for the treatment of certain types of random differential equations, these methods sometimes are difficult to apply to specific problems, and may also involve undesirable restrictions, such as that the random terms must be of small amplitude.

An alternative approach involves a direct numerical construction of useful information about the solution of a random differential equation. In recent years there have been a number of papers dealing with direct numerical methods: for example, see [1]–[6]. Of these papers, [1] and [2] deal with Ito equations,

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[3] with first order linear equations, [4] with initial value problems for first order nonlinear equations, and [5] and [6] with initial and boundary value problems respectively for nth order linear equations. In this paper we will be concerned primarily with boundary value problems for nonlinear second order equations.

Any direct numerical method involves the discretization of the random input of the problem, for example, by assuming that this input is described by a finite number of random variables with known properties. Some physical problems naturally occur in this form with the random variables representing such physical quantities as Young's modulus, refractive index, coefficient of diffusivity, etc. In other cases a mathematical approximation is involved such as the replacement of a stochastic process by a random polynomial or the truncation of an appropriate series expansion.

In any case, we will be primarily concerned with the nonlinear two-point boundary value problem

$$\begin{gathered} y'' = f(x, y, y', R_1, R_2, \dots, R_n) \\ S_1 y(x_0) + S_2 y'(x_0) = S_3, \qquad S_4 y(x_f) + S_5 y'(x_f) = S_6 \end{gathered} \tag{1.1}$$

on the interval $[x_0, x_f]$, where $R_1, \dots, R_n, S_1, \dots, S_6$ are bounded continuous random variables. Our main object is to provide a feasible algorithm for the computation of the marginal distribution function

$$F_x(z) = P\{y(x) \leqslant z\}. \tag{1.2}$$

By numerical integration of a set of related deterministic problems we construct an approximate distribution function $\hat{F}_x(z)$. The case in which the problem involves two random variables is discussed in detail. Two methods are given, first with S_2 and S_5 as the only random variables, and the second with R_1 and R_2 random. In each case convergence of $\hat{F}_x(z)$ to $F_x(z)$ is established in the sense that for any $\epsilon > 0$ one can insure that

$$|F_x(z) - \hat{F}_x(z)| < \epsilon \tag{1.3}$$

by a suitable choice of mesh size. In proving the convergence theorem the values that the random variables can assume are essentially dealt with as parameters in a deterministic problem. Thus the only restrictions on the function f in Eq. (1.1) are those necessary to insure that the solution $y(x)$ is a continuously differentiable function of those parameters.

In Section 2 we establish the convergence of a numerical procedure for a problem involving one random variable. In Section 3 the method is extended to problems in which there are two independent random variables. A somewhat different approach is examined in Section 4; here the assumption of independence is replaced by other conditions. Section 5 contains an example concerning heat conduction in a tapered bar in the presence of a random nonlinear heat source.

Finally, we note that while our discussion deals with two point boundary value problem, there is no difficulty in using the methods described here for random initial value problems as well. Also, they can be extended, at least formally, to problems containing any number of random variables.

2. One random variable

We first consider the boundary value problem

$$\begin{aligned} y'' &= f(x, y, y'), \qquad x_0 < x < x_f, \\ y(x_0) + Ay'(x_0) &= \alpha, \qquad y(x_f) + By'(x_f) = \beta. \end{aligned} \tag{2.1}$$

We will assume that A is a bounded random variable taking on values in the closed interval $I_A = [A_1, A_2]$ with probability distribution function $F_A(a) = P(A \leqslant a)$ for $a \in I_A$. We assume that the problem (2.1) has a unique solution for each value in I_A. For a fixed $x \in [x_0, x_f]$ we want to determine a numerical approximation, $\hat{F}_x(z)$, to the probability distribution function, $F_x(z)$, of the solution $y(x)$ of (2.1), where $F_x(z) = P(y(x) \leqslant z)$. We want to determine $\hat{F}_x(z)$ so that given $\epsilon > 0$ we can insure that $|\hat{F}_x(z) - F_x(z)| < \epsilon$.

The basic approach here is to solve (2.1) numerically for a finite number of possible values of A. If $a \in I_A$ and $\tilde{y}(x)$ is the solution of (2.1) with A replaced by a then the probability of the solution $\tilde{y}(x)$ is just the probability associated with a. Thus, knowledge of $\tilde{y}(x)$ for each $a \in I_A$ completely determines $F_x(z)$. However, in most cases the relation between $\tilde{y}(x)$ and a cannot be found exactly. Hence we replace A by a suitable discrete random variable, construct an approximate distribution function, $\hat{F}_x(z)$, for the solution, and show that $\hat{F}_x(z)$ can be made as accurate as desired by sufficiently refining the procedure. There are two sources of error in determining $\hat{F}_x(z)$, namely, the replacement of A by a discrete random variable and the numerical solution of (2.1).

Let the set $\{a_0, ..., a_M\}$ be a partition of I_A with $a_i < a_{i+1}$, $i = 0, ..., M-1$, and $\Delta a = \sup_i(a_{i+1} - a_i)$. Using a method of order of accuracy at least p, solve numerically the $M + 1$ boundary value problems:

$$\begin{aligned} y_i'' &= f(x, y_i, y_i'), \qquad i = 0, ..., M \\ y_i(x_0) + a_i y_i'(x_0) &= \alpha, \qquad y_i(x_f) + By_i'(x_f) = \beta \end{aligned} \tag{2.2}$$

on a net $\{x_j\}$ where $x_j = x_0 + jh$, $j = 1, ..., J$, with $h = (x_f - x_0)/J$. If y_{ij} is the value calculated for $y_i(x_j)$, there exist constants C_i such that

$$|y_{ij} - y_i(x_j)| \leqslant C_i h^p, \qquad i = 0, 1, ..., M, \quad j = 0, 1, ..., J. \tag{2.3}$$

Using linear interpolation, we can approximate $y_i(x)$ between mesh points by

$$\begin{aligned} \hat{y}_i(x) &= y_{ij} + \frac{(x - x_j)}{(x_{j+1} - x_j)} (y_{i,j+1} - y_{ij}), \\ x_j &\leqslant x \leqslant x_{j+1}, \qquad j = 0, 1, \ldots, J - 1. \end{aligned} \tag{2.4}$$

By a standard argument it follows that

$$| \hat{y}_i(x) - y_i(x) | \leqslant Ch^p + \tfrac{1}{2}h^2 Y, \tag{2.5}$$

where

$$C = \max_i C_i, \qquad Y = \max_i Y_i, \qquad Y_i = \sup_{x \in [a,b]} | y_i''(x) | . \tag{2.6}$$

Since A is the only random variable in (2.1), the probability attached to a given solution, $y_i(x)$, is the same as the probability corresponding to a_i. If, for a fixed $x \in [x_0, x_f]$, $y(x, a)$ is a monotonically increasing function of a, then, if $z_i = y_i(x)$, we have that $F_x(z_i) = P(y(x) \leqslant z_i) = F_A(a_i) = P(A \leqslant a_i)$. We will assume for the remainder of this section that $y(x, a)$ is indeed a monotonically increasing function of a. In fact, we will assume that $\partial y/\partial a > 0$ and that both $\partial y/\partial a$ and $\partial^2 y/\partial a^2$ are continuous for $x \in [x_0, x_f]$ and $a \in I_A$. As a consequence there exist m, M_1, and M_2 such that

$$\begin{gathered} 0 < m \leqslant \left| \frac{\partial y}{\partial a}(x, a) \right| \leqslant M_1, \qquad \left| \frac{\partial^2 y}{\partial a^2}(x, a) \right| \leqslant M_2, \\ x \in [x_0, x_f], \qquad a \in I_A . \end{gathered} \tag{2.7}$$

For a given value of z let $i \in [0, \ldots, M - 1]$ be such that $\hat{y}_i(x) < z \leqslant \hat{y}_{i+1}(x)$. Let $z_i = y_i(x)$, the actual value of y_i at x, and let $\hat{z}_i = \hat{y}_i(x)$, the value calculated from (2.4). Then we define $\hat{F}_x(z)$ to be

$$\begin{aligned} \hat{F}_x(z) &= F_x(z_i) + \frac{(z - \hat{z}_i)}{(\hat{z}_{i+1} - \hat{z}_i)} [F_x(z_{i+1}) - F_x(z_i)] \\ &= F_A(a_i) + \frac{(z - \hat{z}_i)}{(\hat{z}_{i+1} - \hat{z}_i)} [F_A(a_{i+1}) - F_A(a_i)]. \end{aligned} \tag{2.8}$$

Since $\hat{z}_i$ is a numerical approximation to $y_i(x)$ rather than the exact value, it actually corresponds not to a_i but to some nearby value $\hat{a}_i$. Using this value, we obtain the approximate distribution function $\tilde{F}_x(z)$ given by

$$\begin{aligned} \tilde{F}_x(z) &= F_x(\hat{z}_i) + \frac{(z - \hat{z}_i)}{(\hat{z}_{i+1} - \hat{z}_i)} [F_x(\hat{z}_{i+1}) - F_x(\hat{z}_i)] \\ &= F_A(\hat{a}_i) + \frac{(z - \hat{z}_i)}{(\hat{z}_{i+1} - \hat{z}_i)} [F_A(\hat{a}_{i+1}) - F_A(\hat{a}_i)]. \end{aligned} \tag{2.9}$$

Then by the triangle inequality the error can be expressed as

$$| F_x(z) - \hat{F}_x(z)| \leqslant | F_x(z) - \tilde{F}_x(z)| + | \tilde{F}_x(z) - \hat{F}_x(z)|. \tag{2.10}$$

We will now construct an upper bound for each of the terms on the right side of (2.10). The first term is the error due to numerical integration and the second term is the error due to replacing A by a discrete random variable.

For the first term we have from Newton's Interpolation Formula that

$$F_x(z) = \tilde{F}_x(z) + \tfrac{1}{2}(z - \hat{z}_i)(z - \hat{z}_{i+1}) F_x''(\xi) \tag{2.11}$$

for some $\xi \in [\hat{z}_i , \hat{z}_{i+1}]$. Thus

$$| F_x(z) - \tilde{F}_x(z)| \leqslant \tfrac{1}{8}(\hat{z}_{i+1} - \hat{z}_i)^2 \sup_z | F_x''(z)| . \tag{2.12}$$

Using (2.5) we have that

$$\begin{aligned} | \hat{z}_{i+1} - \hat{z}_i | &\leqslant | \hat{z}_{i+1} - z_{i+1} | + | z_{i+1} - z_i | + | z_i - \hat{z}_i | \\ &\leqslant | z_{i+1} - z_i | + 2(Ch^p + \tfrac{1}{2}h^2 Y). \end{aligned} \tag{2.13}$$

Since $y \in C^1(I_A)$ then

$$z_{i+1} - z_i = y(x, a_{i+1}) - y(x, a_i) = \frac{\partial y(x, a')}{\partial a}(a_{i+1} - a_i), \qquad a' \in [a_i , a_{i+1}] \tag{2.14}$$

and it follows that

$$| F_x(z) - \tilde{F}_x(z)| \leqslant \tfrac{1}{8}[M_1 \Delta a + 2(Ch^p + \tfrac{1}{2}h^2 Y)]^2 \sup_z | F_x''(x)| . \tag{2.15}$$

Hence, if $\sup_z | F_x''(z)|$ is finite, and if Δa and h are sufficiently small, then

$$| F_x(z) - \tilde{F}_x(z)| < \tfrac{1}{2}\epsilon. \tag{2.16}$$

For the second term on the right side of (2.10) we have that

$$\begin{aligned} | F_x(\hat{z}_i) - F_x(z_i)| &= | F_x'(z')(\hat{z}_i - z_i)| \\ &\leqslant \sup_z | F_x'(z)| (Ch^p + \tfrac{1}{2}h^2 Y), \qquad z' \in [z_i , \hat{z}_i] \end{aligned} \tag{2.17}$$

and thus

$$\begin{aligned} &| \tilde{F}_x(z) - \hat{F}_x(z)| \\ &= [F_x(\hat{z}_i) - F_x(z_i)] \frac{\hat{z}_{i+1} - z}{\hat{z}_{i+1} - \hat{z}_i} + [F_x(\hat{z}_{i+1}) - F_x(z_{i+1})] \frac{z - \hat{z}_i}{\hat{z}_{i+1} - \hat{z}_i} \\ &\leqslant \sup_z | F_x'(z)| (Ch^p + \tfrac{1}{2}h^2 Y). \end{aligned} \tag{2.18}$$

It follows that if $\sup_z |F_x'(z)|$ is finite, and if h is sufficiently small, then

$$|\tilde{F}_x(z) - \hat{F}_x(z)| < \tfrac{1}{2}\epsilon. \tag{2.19}$$

Combining (2.16) and (2.19) with (2.10) gives the desired result.

It remains to show that $F_x'(z)$ and $F_x''(z)$ are bounded. Since $\partial y(x, a)/\partial a > 0$ there exists, by the Inverse Function Theorem, a well-defined differentiable inverse function y^{-1}. Further, if $\zeta = y(a)$, then $a = y^{-1}(\zeta)$ and

$$\frac{da}{d\zeta} = \frac{dy^{-1}(\zeta)}{d\zeta} = \frac{1}{\partial y(x, a)/\partial a}. \tag{2.20}$$

Now, if $\tilde{a} = y^{-1}(z)$, then

$$\begin{aligned} F_x(z) &= P(a \mid y(x, a) \leqslant z) = P(a \mid a \leqslant y^{-1}(z)) \\ &= F_A(y^{-1}(z)) = F_A(\tilde{a}). \end{aligned}$$

Thus by the chain rule

$$F_x'(z) = F_A'(\tilde{a}) \frac{da}{d\zeta}(z) = \frac{F_A'(\tilde{a})}{\partial y(x, \tilde{a})/\partial a}.$$

Consequently,

$$|F_x'(z)| \leqslant \sup_{I_A} F_A'(a)/m.$$

Similarly,

$$F_x''(z) = \frac{F_A''(\tilde{a})\, \partial y(x, \tilde{a})/\partial a - F_A'(\tilde{a})\, \partial^2 y(x, \tilde{a})/\partial a^2}{(\partial y(x, \tilde{a})/\partial a)^3},$$

and hence

$$|F_x''(z)| \leqslant [\sup_{I_A} |F_A''(a)|\, M_1 + \sup_{I_A} |F_A'(a)|\, M_2]/m^3.$$

This completes the proof of the following

THEOREM 1. *Let $y(x)$ be the solution of*

$$\begin{aligned} y'' &= f(x, y, y'), \qquad x_0 < x < x_f \\ y(x_0) + Ay'(x_0) &= \alpha, \qquad y(x_f) + By'(x_f) = \beta, \end{aligned} \tag{2.21}$$

where** α, β, **and** B **are constants, and** A **is a bounded random variable, taking on values in** $I_A = [A_1, A_2]$ **with distribution function** $F_A(a) \in C^2(I_A)$. **Assume that for each possible value of** A, **the deterministic problem corresponding to** (2.21) **has a unique solution. Assume that** $\partial y/\partial a$ **and** $\partial^2 y/\partial a^2$ **are continuous for** $x \in [x_0, x_f]$ **and** $a \in I_A$ **and that** $\partial y/\partial a > 0$ **there. Let** $F_x(z)$, $\hat{F}_x(z)$, Δa, **and** h **be as defined above. Then for any** $\epsilon > 0$ **it is possible to choose** Δa **and** h **so small that

$$|F_x(z) - \hat{F}_x(z)| < \epsilon.$$

3. Differential Equations Containing Two Independent Random Variables

In this section we will extend the method of Section 2 to problems containing two independent random variables. First let us look at the problem from a geometrical point of view.

In Section 2 we assumed that $\partial y/\partial a > 0$ and we looked for a means of approximating the set $A_x = (a \in I_A \mid y(x, a) \leqslant z)$; see Figure 1. If $\partial y/\partial a = 0$ for some values of a, then y may no longer be a strictly monotonic function of a. However, as long as $\partial y/\partial a = 0$ only a finite number of times, say $(\partial y/\partial a)(x, a_1) = (\partial y/\partial a)(x, a_2) = \cdots = (\partial y/\partial a)(x, a_k) = 0$, the results in Section 2 can be applied to each line segment between these points. In this case A_x is possibly the union of several line segments, and hence $F_x(z)$ is the sum of several terms. For instance, $F_x(z) = P(A_x) = P([A_1, \tilde{a}_1] \cup [\tilde{a}_2, \tilde{a}_3])$ in Figure 2.

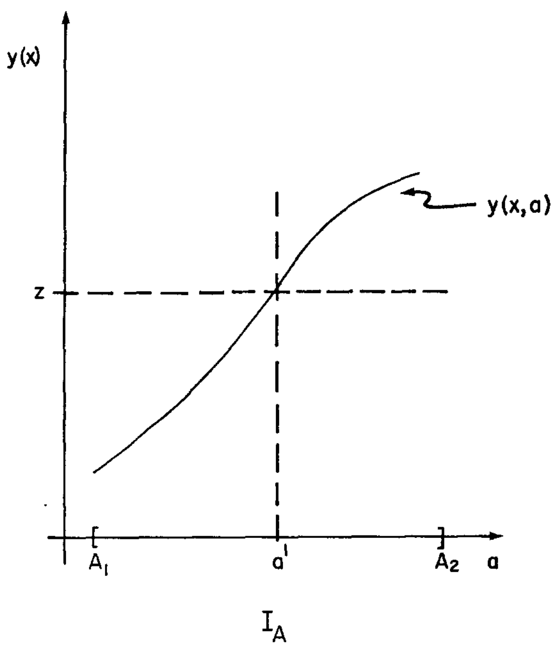

Figure 1

When two of the R_i or S_i in (1.1) are random variables, we have to approximate a region in R^2 in order to construct the distribution function of $y(x)$. Suppose that $R_1 = A$ and $R_2 = B$ are random variables taking on values in the real intervals $[A_1, A_2]$ and $[B_1, B_2]$, respectively. Let $F_{AB}(a, b)$ be their joint distribution function and suppose that the joint density function $f_{AB}(a, b) = \partial^2 F_{AB}(a, b)/\partial a\, \partial b$ exists for all $(a, b) \in S = [A_1, A_2] \times [B_1, B_2]$. Then for a

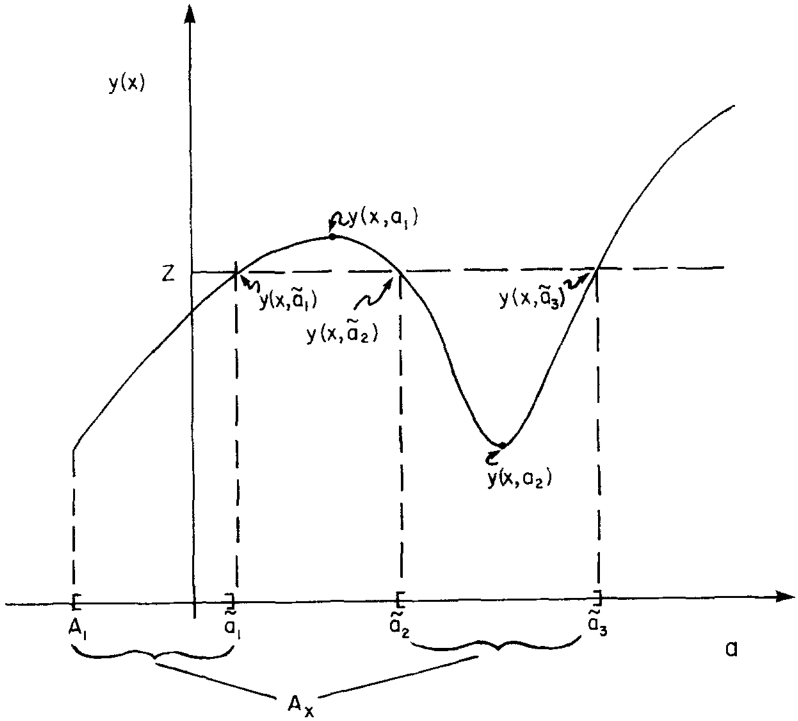

FIGURE 2

fixed x, the solution of (1.1) defines a mapping from S into the R^2 plane with coordinates $(y'(x), y(x))$.

If

$$\underline{S} = \{(a, b) \mid y(x, a, b) \leqslant z\}, \tag{3.1}$$

then

$$F_x(z) = \iint_{\underline{S}} f_{AB}(a, b)\, da\, db. \tag{3.2}$$

If

$$\bar{S} = \{(a, b) \mid y(x, a, b) > z\}, \tag{3.3}$$

then we could alternately write

$$F_x(z) = 1 - \iint_{\bar{S}} f_{AB}(a, b)\, da\, db. \tag{3.4}$$

Thus $\underline{S}$ and $\bar{S}$ are the regions we want to approximate. Extending the procedure in Section 2, we do so by partitioning S and solving (1.1) as a deterministic boundary value problem a finite number of times. For each partition of S we can define a set $\hat{\underline{S}} \subset \underline{S}$ and an approximate distribution function $\hat{F}_x(z) \leqslant F_x(z)$. Alternately we can define a set $\hat{\bar{S}} \subset \bar{S}$ and use it to approximate $F_x(z)$.

To insure that $\hat{\underline{S}} \to \underline{S}$ or $\hat{\bar{S}} \to \bar{S}$ in area as the mesh size approaches zero, we need to know that $y \in C(S)$. This is assured by standard theorems; see [7], for example.

To be able to use either $\hat{\underline{S}}$ or $\hat{\bar{S}}$ interchangeably to define $\hat{F}_x(z)$ we need to know that the curve

$$\Gamma_x = \{(a, b) \mid y(x, a, b) = z\} \tag{3.5}$$

has zero area. Sufficient conditions for this are that $y \in C^1(S)$ and that $\partial y/\partial a$ and $\partial y/\partial b$ be equal to zero only a finite number of times and never at the same point. Figure 3 shows a typical surface $y(x, a, b)$ and curve Γ_x.

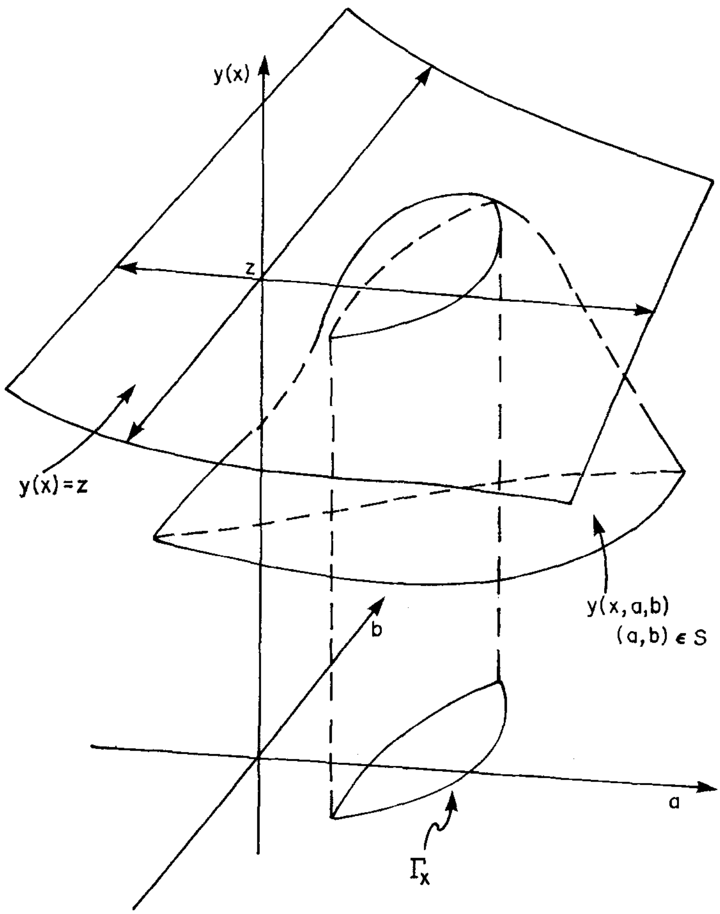

FIGURE 3

We will now extend in a more formal way the discussion in Section 2 to the case where there are two random boundary conditions. The case in which the random variables are in the differential equation is similar. As in Section 2 we consider the following boundary value problem:

$$y'' = f(x, y, y'), \quad y(x_0) + Ay'(x_0) = \alpha, \quad y(x_f) + By'(x_f) = \beta, \tag{3.6}$$

where A and B are independent random variables taking on values in the real intervals $I_A = [A_1, A_2]$ and $I_B = [B_1, B_2]$, respectively. Suppose that A and B are defined by the respective distribution functions, $F_A(a)$ and $F_B(b)$ for $a \in I_A$ and $b \in I_B$. We assume that the density functions $f_A(a)$ and $f_B(b)$ exist. Suppose that (3.6) is such that it has a unique solution for each of the possible values of A and B. Using the conditional probability distribution of $y(x) = z$ given $B = b$, we have that

$$F_x(z) = \int_{I_B} F_x(z \mid B = b) f_B(b)\, db. \tag{3.7}$$

Let $\{a_0, a_1, ..., a_M\}$ and $\{b_0, b_1, ..., b_J\}$ be partitions of I_A and I_B. Using a numerical method of order of accuracy at least p, solve the $(M+1)(J+1)$ boundary value problems:

$$\begin{gathered} y''_{ij} = f(x, y_{ij}, y'_{ij}); \qquad i = 0, 1, ..., M; \quad j = 0, 1, ..., J, \\ y_{ij}(x_0) + a_i y'_{ij}(x_0) = \alpha, \qquad y_{ij}(x_f) + b_j y'_{ij}(x_f) = \beta, \end{gathered} \tag{3.8}$$

on some net $x_k = x_0 + kh$, $k = 0, 1, ..., K$, where $h = (x_f - x_0)/K$. We then have a triply indexed array of numbers, y_{ijk} and y'_{ijk} such that

$$| y_{ijk} - y_{ij}(x_k)| \leqslant Ch^p,$$
$$| y'_{ijk} - y'_{ij}(x_k)| \leqslant Ch^p,$$

where C is a suitable positive constant. We define $\hat{F}_x(z)$ by

$$\begin{aligned} \hat{F}_x(z) = \sum_{j=1}^{J-1} & \hat{F}_x(z \mid B = b_j)\, P(b_j - \tfrac{1}{2}\Delta b < B \leqslant b_j + \tfrac{1}{2}\Delta b) \\ & + \hat{F}_x(z \mid B = B_1)\, P(B \leqslant B_1 + \tfrac{1}{2}\Delta b) \\ & + \hat{F}_x(z \mid B = B_2)\, P(B > B_2 - \tfrac{1}{2}\Delta b), \end{aligned} \tag{3.9}$$

where $\hat{F}_x(z \mid B = b_j)$ is the numerical approximation to the distribution function of the solution of (2.1) with $B = b_j$. The approximations made in replacing (3.7) by (3.9) are to

(1) substitute the single point b_j for the interval $(b_j - \frac{1}{2}\Delta b, b_j + \frac{1}{2}\Delta b]$, and

(2) replace $F_x(z \mid B = b_j)$ by the numerical approximation $\hat{F}_x(z \mid B = b_j)$ from Section 2.

Let

$$\tilde{F}_x(z) = \sum_{j=1}^{J-1} F_x(z \mid B = b_j)\, P(b_j - \tfrac{1}{2}\Delta b < B \leqslant b_j + \tfrac{1}{2}\Delta b)$$
$$+ F_x(z \mid B = B_1)\, P(B \leqslant B_1 + \tfrac{1}{2}\Delta b)$$
$$+ F_x(z \mid B = B_2)\, P(B > B_2 - \tfrac{1}{2}\Delta b). \tag{3.10}$$

Then

$$| F_x(z) - \hat{F}_x(z)| \leqslant | F_x(z) - \tilde{F}_x(z)| + | \tilde{F}_x(z) - \hat{F}_x(z)|. \tag{3.11}$$

The second term on the right side of (3.11) reflects the error due to the approximation in Section 2.

Suppose that Δa and h are chosen so that

$$| F_x(z \mid B = b_j) - \hat{F}_x(z \mid B = b_j)| < \frac{\epsilon}{2}; \qquad j = 0, 1,..., J.$$

Then

$$| \tilde{F}_x(z) - \hat{F}_x(z)| \leqslant \frac{\epsilon}{2} \int_{I_B} f_B(b)\, db = \frac{\epsilon}{2}. \tag{3.12}$$

To see that the first term on the right hand side of (3.11) can be made small, we need to show that $F_x(z \mid B = b)$ is uniformly continuous in b. For any $b \in I_B$ we have that

$$F_x(z \mid B = b) = P(A_b \mid B = b) \tag{3.13}$$

for some set $A_b \subset I_A$. Since A and B are independent random variables

$$P(A_b \mid B = b) = \int_{A_b} f_A(a)\, da.$$

Hence for any $b', b'' \in I_B$ we have that

$$| F_x(z \mid B = b'') - F_x(z \mid B = b')| = \int_{A_{b''} \Delta A_{b'}} f_A(a)\, da$$
$$= \int_{a'}^{a''} f_A(a)\, da \leqslant K \mid a'' - a' \mid , \tag{3.14}$$

where $\sup_{a \in I_A} f_A(a) \leqslant K < \infty$. We assume here that $A_{b''} \Delta A_{b'}$ is an interval, but it may in fact be a finite collection of intervals.

If $f(x, y, y')$ is sufficiently smooth then $\partial y/\partial a$ and $\partial y/\partial b$ exist and by the Mean Value Theorem

$$\Delta y(x) = y(x, a'', b'') - y(x, a', b')$$
$$= \frac{\partial y}{\partial a}(x, a' + \theta_1 \Delta a, b')\, \Delta a + \frac{\partial y}{\partial b}(x, a'', b' + \theta_2 \Delta b)\, \Delta b,$$

where $\Delta a = (a'' - a')$, $\Delta b = (b'' - b')$ and $0 < \theta_1, \theta_2 < 1$. Since we are looking at points $(a, b) \in S$ such that $y(x, a, b) = z$ we have that $\Delta y(x) = 0$ and

$$\Delta a \leqslant K'\Delta b \qquad \text{where} \qquad K' = \sup \left| \frac{\partial y}{\partial a} \right| \Big/ \inf \left| \frac{\partial y}{\partial b} \right| . \tag{3.15}$$

We now can prove the following:

THEOREM 2. *Let A, f, y satisfy the hypotheses of Theorem 1. Let B be a bounded random variable with distribution function $F_B(b) \in C^1(I_B)$. Assume that A and B are independent and that for each $a \in I_A$ and $b \in I_B$ the deterministic problem corresponding to* (3.6) *has a unique solution. Suppose that*

$$f_A(a) \leqslant K < \infty$$

and that

$$K' = \sup_{x,a,b} \left| \frac{\partial y}{\partial b} \right| \Big/ \inf_{x,a,b} \left| \frac{\partial y}{\partial a} \right| .$$

Let $\epsilon > 0$. Let $\hat{F}_x(z)$ be given by (3.9). *Then Δa, Δb, and h can be chosen so that*

$$| F_x(z) - \hat{F}_x(z) | < \epsilon.$$

Proof. Use Theorem 1 to choose Δa and h so that

$$| F_x(z \mid B = b_j) - \hat{F}_x(z \mid B = b_j) | < \frac{\epsilon}{2}, \qquad j = 0, 1, \ldots, J.$$

Let $\Delta b \leqslant \epsilon/2KK'$. Then from (3.14) and (3.15) we have that

$$| F_x(z \mid B = b'') - F_x(z \mid B = b') | \leqslant \frac{\epsilon}{2}$$

for any $b', b'' \in I_B$. Thus

$$\begin{aligned} &| F_x(z) - \tilde{F}_x(z) | \\ &\quad \leqslant \sum_{j=1}^{J-1} \sup_{b_j - (\Delta b/2) \leqslant b \leqslant b_j + (\Delta b/2)} | F_x(z \mid B = b) - F_x(z \mid B = b_j) | \int_{b_j - (\Delta b/2)}^{b_j + (\Delta b/2)} f_B(b)\, db \\ &\quad + \sup_{B_1 \leqslant b \leqslant B_1 + (\Delta b/2)} | F_x(z \mid B = b) - F_x(z \mid B = B_1) | \int_{B_1}^{B_1 + (\Delta b/2)} f_B(b)\, db \\ &\quad + \sup_{B_2 - (\Delta b/2) \leqslant b \leqslant B_2} | F_x(z \mid B = b) - F_x(z \mid B = B_2) | \int_{B_2 - (\Delta b/2)}^{B_2} f_B(b)\, db \leqslant \frac{\epsilon}{2}. \end{aligned} \tag{3.16}$$

Combining (3.16), (3.12), and (3.11) it follows that

$$| F_x(z) - \hat{F}_x(z) | \leqslant \epsilon.$$

4. Two Random Variables—Second Method

We now consider the boundary value problem

$$\begin{aligned} y'' &= f(x, y, y', A, B), \qquad x_0 < x < x_f \\ S_1 y(x_0) + S_2 y'(x_0) &= S_3, \qquad S_4 y(x_f) + S_5 y'(x_f) = S_6, \end{aligned} \tag{4.1}$$

where A and B are random variables assuming values on the real intervals $I_A = [A_1, A_2]$ and $I_B = [B_1, B_2]$, respectively, with joint distribution function $F_{AB}(a, b) = P(A \leqslant a, B \leqslant b)$. We assume that (4.1) has a unique solution for all values of A and B. We also assume as before that f is sufficiently smooth to insure that $\partial y/\partial b$ and $\partial y/\partial a$ exist and are continuous for all values of A and B.

The method used here in constructing $\hat{F}_x(z)$ is based on the direct use of the joint probability distribution of A and B rather than on conditional probabilities. We do not necessarily assume that A and B are independent as the simplification this allowed in Section 3 is not needed here. However, giving up this restriction means that we do need to assume a better knowledge of the derivative of y with respect to b. This will be made more specific in what follows.

Let $S = [A_1, A_2] \times [B_1, B_2]$. Define a partition of S by the sets

$$S_{ij} = \{(a, b) \mid a_i \leqslant a < a_{i+1}, b_j \leqslant b < b_{j+1}\}$$

$i = 0, 1,..., M - 1$ and $j = 0, 1,..., J - 1$ for some integers M and J. Let $A_1 = a_0$, $A_2 = a_M$, $B_1 = b_0$, and $B_2 = b_J$. Let

$$\Delta a = \max_{0 \leqslant i \leqslant M-1} | a_{i+1} - a_i |, \qquad \Delta b = \max_{0 \leqslant j \leqslant J-1} | b_{j+1} - b_j |. \tag{4.2}$$

Let $y_{ij}(x)$ denote the solution of

$$\begin{aligned} y'' &= f_{ij}(x, y, y'), \qquad x_0 < x < x_f \\ S_1 y(x_0) + S_2 y'(x_0) &= S_3, \qquad S_4 y(x_f) + S_5 y'(x_f) = S_6, \end{aligned} \tag{4.3}$$

where $f_{ij}(x, y, y') = f(x, y, y', a_i, b_j)$ for each i and j. We will use these exact solutions and $F_{AB}(a, b)$ to construct upper and lower bounds, $\bar{F}_x(z)$ and $\underline{F}_x(z)$, for $F_x(z)$. We will show that for any $\eta > 0$, we can find Δa and Δb sufficiently small to insure that $|\bar{F}_x(z) - \underline{F}_x(z)| < \eta$.

Suppose that $y(x, a, b)$ is strictly increasing as a function of a. We also assume that the partition points of I_B have been chosen so that for any given interval, $[b_j, b_{j+1})$ we know that either $\partial y/\partial b \geqslant 0$ or $\partial y/\partial b \leqslant 0$ throughout the interval.

Suppose that $y_{ij}(x) > z$ and $y_{i,j+1}(x) > z$. Then it follows that $y(x, a, b) > z$ for all $(a, b) \in S_{ij}$. Similarly suppose that $y_{i+1,j}(x) \leqslant z$ and $y_{i+1,j+1}(x) \leqslant z$. Then $y(x, a, b) \leqslant z$ for all $(a, b) \in S_{ij}$. Thus we can define upper and lower bounds for $F_x(z)$ as follows:

$$\bar{F}_x(z) = 1 - \sum_{i,j} P(S_{ij}), \tag{4.4}$$

where the sum is over all those i, j such that both $y_{ij}(x) > z$ and $y_{i,j+1}(x) > z$, and

$$\underline{F}_x(z) = \sum_{i,j} P(S_{ij}), \tag{4.5}$$

where the sum is over all those i, j such that both $y_{i+1,j}(x) \leqslant z$ and $y_{i+1,j+1}(x) \leqslant z$. Clearly

$$\underline{F}_x(z) \leqslant F_x(z) \leqslant \bar{F}_x(z). \tag{4.6}$$

For each $b \in [B_1, B_2]$ let $\bar{a}(b)$ be defined as that number $a \in [A_1, A_2]$ such that

$$\bar{F}_x(z) = \int_{B_1}^{B_2} \int_{A_1}^{\bar{a}(b)} \frac{\partial^2 F_{AB}(a, b)}{\partial a\, \partial b}\, da\, db. \tag{4.7}$$

Such an a exists for each b since for any $j \in (0, 1,..., J)$ there exists an $i_j \in (0, 1,..., M)$ such that $P(S_{ij})$ is included in $\bar{F}_x(z)$ for all $i \geqslant i_j$ and excluded for all $i < i_j$. For $b \in [b_j, b_{j+1})$ define $\bar{a}(b) = a_{i_j}$. Thus $\bar{a}(b)$ is defined for all $b \in [B_1, B_2]$ and is piecewise continuous since it is constant in each interval, $[b_j, b_{j+1})$. Hence writing $\bar{F}_x(z)$ as the double integral (4.7) makes sense.

Similarly there exists for each j an i_j such that $P(S_{ij})$ is included in $\underline{F}_x(z)$ for all $i \leqslant i_j - 1$ and excluded for all $i \geqslant i_j$. If we define $\underline{a}(b) = a_{i_j}$ for $b \in [b_j, b_{j+1})$, then $\underline{a}(b)$ is piecewise continuous and we can write

$$\underline{F}_x(z) = \int_{B_1}^{B_2} \int_{A_1}^{\underline{a}(b)} \frac{\partial^2 F_{AB}(a, b)}{\partial a\, \partial b}\, da\, db. \tag{4.8}$$

If we assume that $|\, \partial^2 F_{AB}(a, b)/\partial a\, \partial b \,| \leqslant C < \infty$, then we have that

$$|\bar{F}_x(z) - \underline{F}_x(z)| \leqslant (B_2 - B_1)\, CK\Delta a, \tag{4.9}$$

where K is an integer. It is possible to show that for a given Δa, the partition of I_B can be defined so that $K \leqslant 2$. This result is needed to insure that as $\Delta a \to 0$, K does not become large in such a way that $K\Delta a$ remains finite.

There are several possible cases; a typical one is shown in Figure 4. In the case shown there, $\underline{a}(b) = a_{i-3}$, $\bar{a}(b) = a_{i+1}$, and $\bar{a}(b) - \underline{a}(b) = 4\Delta a$ for $b_j \leqslant b < b_{j+1}$ and for a uniform mesh size Δa on the a-axis. The difficulty is that three curves on which a is constant (namely, the ones for which a has the values a_i, a_{i-1}, a_{i-2}) cross the line $y(x) = z$ in the interval $[b_j, b_{j+1})$. The solution is to introduce additional mesh points β_1 and β_2, as shown in Figure 4, chosen so that only one of these a-curves crosses the line $y(x) = z$ in each of the intervals $[b_j, \beta_1)$, $[\beta_1, \beta_2)$, $[\beta_2, b_{j+1})$ respectively. This geometrical argument can be made rigorous in a straightforward way, and other cases can be handled similarly.

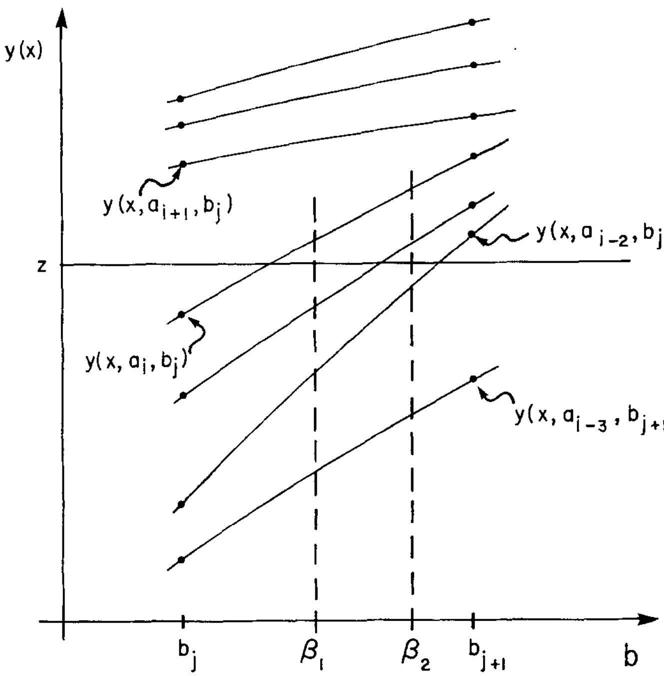

FIGURE 4

Thus suppose that the partition on I_B is refined as just described and that (4.3) is solved for the additional mesh points indicated. Then, redefining $\bar{F}_x(z)$ and $\underline{F}_x(z)$ (as well as $\bar{a}(b)$ and $\underline{a}(b)$) and using the adjusted set of subsets S_{ij} of S, we have $K = 2$; it follows that

$$|\bar{F}_x(z) - \underline{F}_x(z)| \leqslant 2(B_2 - B_1)\, C\Delta a. \tag{4.10}$$

Thus we have established the following theorem:

THEOREM 3. *Let $y(x)$ be the solution of* (4.1). *Suppose that $\partial y/\partial a > 0$ and $\partial^2 F_{AB}(a, b)/\partial a\, \partial b \leqslant C$ for $a \in [A_1, A_2]$, $b \in [B_1, B_2]$. Let $\Delta a, \Delta b, \bar{F}_x(z)$ and $\underline{F}_x(z)$ be as defined above. Assume that the partition of $[B_1, B_2]$ is chosen so that for each $[b_j, b_{j+1})$, $\partial y/\partial b \geqslant 0$ or $\partial y/\partial b \leqslant 0$. Let $\eta > 0$. Then $\underline{F}_x(z) \leqslant F_x(z) \leqslant \bar{F}_x(z)$. If $\Delta a < \eta/2(B_2 - B_1)C$ then there exists a partition $[b_0, b_1, ..., b_J]$ of $[B_1, B_2]$ which insures that $|\bar{F}_x(z) - \underline{F}_x(z)| < \eta$.*

This theorem shows that we can approximate the distribution function by using the exact solutions of the boundary value problems obtained by replacing A and B in (4.1) by a finite number of their respective values. Since such exact solutions are rarely available we want to use numerical solutions of (4.1), again replacing A and B by an appropriate discrete set of values, to calculate an approximation, $\hat{F}_x(z)$, to $F_x(z)$.

Thus suppose that we have numerically solved the boundary value problem (4.3) and that we have values y_{ijk} such that

$$| y_{ijk} - y_{ij}(x_k)| < \tilde{C}h^p; \qquad i = 0, 1,..., M, \quad j = 0, 1,..., J, \quad k = 0, 1,..., K \tag{4.11}$$

for some $\tilde{C} > 0$. We define $\hat{\bar{F}}_{x_k}(z) = 1 - \sum P(S_{ij})$, where the summation is over all those ij for which $y_{ijk} > z$ and $y_{i,j+1,k} > z$. We define $\hat{\underline{F}}_{x_k}(z) = \sum_{ij} P(S_{ij})$ where the summation is over all those ij for which $y_{i+1,j,k} \leqslant z$ and $y_{i+1,j+1,k} \leqslant z$. We want to show that for h sufficiently small we can approximate $\bar{F}_{x_k}(z)$ and $\underline{F}_{x_k}(z)$ by $\hat{\bar{F}}_{x_k}(z)$ and $\hat{\underline{F}}_{x_k}(z)$, respectively. The following lemma indicates how small h must be.

LEMMA 1. *If h is chosen so that*

$$\tilde{C}h^p < \left[\inf_{a,b} \frac{\partial y(x_k)}{\partial a}\right] \frac{\Delta a}{4}, \tag{4.12}$$

then for any fixed b_j there exists at most one a_i such that $| y_{ijk} - z | < \tilde{C}h^p$.

The proof of the lemma is not difficult and is omitted.

For a fixed $j \in [0, 1,..., J]$ and $k \in [0, 1,..., K]$ let $\hat{\imath}_j$ be the largest i such that $y_{ijk} \leqslant z$. For $b \in [b_j, b_{j+1})$, let

$$\hat{a}(b) = \max(a_{\hat{\imath}_j+1}, a_{\hat{\imath}_{j+1}+1}) \tag{4.13}$$

and

$$\hat{\underline{a}}(b) = \min(a_{\hat{\imath}_j}, a_{\hat{\imath}_{j+1}}). \tag{4.14}$$

We then have that

$$\hat{\bar{F}}_x(z) = \int_{B_1}^{B_2} \int_{A_1}^{\hat{a}(b)} \frac{\partial^2 F_{AB}(a, b)}{\partial a\, \partial b}\, da\, db \tag{4.15}$$

and

$$\hat{\underline{F}}_x(z) = \int_{B_1}^{B_2} \int_{A_1}^{\hat{\underline{a}}(b)} \frac{\partial^2 F_{AB}(a, b)}{\partial a\, \partial b}\, da\, db. \tag{4.16}$$

Thus it follows that

$$| \hat{\bar{F}}_x(z) - \hat{\underline{F}}_x(z)| \leqslant \int_{B_1}^{B_2} C\, | \hat{a}(b) - \hat{\underline{a}}(b)|\, db \leqslant (B_2 - B_1)\, CK'\Delta a \tag{4.17}$$

where K' is an integer. We wish to show that we can refine the partition of $[B_1, B_2]$ to insure that $K' \leqslant 4$.

If i_j is the largest i such that $y_{ij}(x_k) \leqslant z$, then it follows from Lemma 1 that $\hat{i}_j$ can be equal to i_j, $i_j + 1$, or $i_j - 1$. Suppose, for example, that $\hat{i}_j = i_j + 1$ and $\hat{i}_{j+1} = i_{j+1} - 1$. We have defined

$$\bar{a}(b) = \max(a_{i_j+1}, a_{i_{j+1}+1}),$$

$$\underline{a}(b) = \min(a_{i_j}, a_{i_{j+1}}),$$

$$\hat{a}(b) = \max(a_{\hat{i}_j+1}, a_{\hat{i}_{j+1}+1}),$$

and

$$\underline{\hat{a}}(b) = \min(a_{\hat{i}_j}, a_{\hat{i}_{j+1}})$$

for $b \in [b_j, b_{j+1})$.

We then have that

$$\hat{a}(b) - \Delta a \leqslant \bar{a}(b) \leqslant \hat{a}(b) + \Delta a,$$

$$\underline{\hat{a}}(b) - \Delta a \leqslant \underline{a}(b) \leqslant \underline{\hat{a}}(b) + \Delta a.$$

If, for instance, $i_j = i + 1$ and $i_{j+1} = i$, then $\hat{i}_j = i + 2$, $\hat{i}_{j+1} = i - 1$, $\bar{a}(b) = \hat{a}(b) - \Delta a = a_{i+2}$, and $\underline{a}(b) = \underline{\hat{a}}(b) + \Delta a = a_i$. The other possible cases are similar and the combined results are given by the following:

LEMMA 2.

(i) $|\bar{a}(b) - \hat{a}(b)| \leqslant 2\Delta a$

(ii) $|\underline{a}(b) - \underline{\hat{a}}(b)| \leqslant 2\Delta a$

(iii) $|\bar{a}(b) - \underline{a}(b)| \leqslant |\hat{a}(b) - \underline{\hat{a}}(b)| + 2\Delta a$

(iv) $|\hat{a}(b) - \underline{\hat{a}}(b)| \leqslant |\bar{a}(b) - \underline{a}(b)| + 2\Delta a.$

We showed in proving Theorem 3 that there exists a partition of $[B_1, B_2]$ which insures that $|\bar{a}(b) - \underline{a}(b)| \leqslant 2\Delta a$. Using this partition it then follows from (iv) of Lemma 2 that $|\hat{a}(b) - \underline{\hat{a}}(b)| \leqslant 4\Delta a$. Thus there exists at least one partition which insures that $K' \leqslant 4$ in (4.17).

Now let

$$\hat{F}_x(z) = \frac{\hat{F}_x(z) + \underline{\hat{F}}_x(z)}{2}, \qquad \check{F}_x(z) = \frac{\bar{F}_x(z) + \underline{F}_x(z)}{2}. \tag{4.18}$$

Then we have the following theorem:

THEOREM 4. *Let $y(x)$, $\partial y/\partial a$, C, Δa, Δb, $\bar{F}_x(z)$, $\underline{F}_x(z)$ and $\partial y/\partial b$ be as in Theorem 3. Let $\tilde{C}$, $\hat{\bar{F}}_x(z)$, $\hat{\underline{F}}_x(z)$, $\tilde{F}_x(z)$ and $\hat{F}_x(z)$ be as defined above. Let $\epsilon > 0$. Suppose that Δa and h are chosen so that*

$$\Delta a < \frac{\epsilon}{6(B_2 - B_1)\, C}, \qquad \tilde{C}h^p < \left[\inf \frac{\partial y}{\partial a}\right] \frac{\Delta a}{4}. \tag{4.19}$$

Then the partition of $[B_1, B_2]$ can be refined to insure that $|F_x(z) - \hat{F}_x(z)| < \epsilon$.

Proof. Let $\epsilon > 0$ be given. Choose Δa and h so that (4.19) is satisfied. It follows from Theorem 3, Lemma 2, and the preceding discussion that there exists at least one partition of $[B_1, B_2]$ for which $|\hat{a}(b) - \hat{\underline{a}}(b)| \leqslant 4\Delta a$ for all $b \in [B_1, B_2]$. Choose such a partition.

Then from Lemma 2(iii) we have that for this partition, $|\bar{a}(b) - \underline{a}(b)| \leqslant 6\Delta a$ for all $b \in [B_1, B_2]$. Thus

$$|\bar{F}_x(z) - \underline{F}_x(z)| \leqslant 6(B_2 - B_1)\, C\Delta a < \epsilon.$$

Further, since $\underline{F}_x(z) \leqslant F_x(z) \leqslant \bar{F}_x(z)$, then $|F_x(z) - \tilde{F}_x(z)| \leqslant \epsilon/2$.

From (i) and (ii) of Lemma 2 we have that

$$|\bar{F}_x(z) - \hat{\bar{F}}_x(z)| = \left| \int_{B_1}^{B_2} \int_{\hat{a}(b)}^{\bar{a}(b)} \frac{\partial^2 F_{AB}(a, b)}{\partial a\, \partial b}\, da\, db \right|$$
$$\leqslant 2(B_2 - B_1)\, C\Delta a < \tfrac{1}{3}\epsilon$$

and

$$|\underline{F}_x(z) - \hat{\underline{F}}_x(z)| = \left| \int_{B_1}^{B_2} \int_{\hat{\underline{a}}(b)}^{\underline{a}(b)} \frac{\partial^2 F_{AB}(a, b)}{\partial a\, \partial b}\, da\, db \right|$$
$$\leqslant 2(B_2 - B_1)\, C\Delta a < \tfrac{1}{3}\epsilon.$$

Consequently

$$\begin{aligned}
&|F_x(z) - \hat{F}_x(z)| \\
&= \left| F_x(z) - \frac{\hat{\bar{F}}_x(z) + \hat{\underline{F}}_x(z)}{2} - \frac{\bar{F}_x(z)}{2} + \frac{\bar{F}_x(z)}{2} - \frac{\underline{F}_x(z)}{2} + \frac{\underline{F}_x(z)}{2} \right| \\
&\leqslant |F_x(z) - \tilde{F}_x(z)| + \frac{|\bar{F}_x(z) - \hat{\bar{F}}_x(z)|}{2} + \frac{|\underline{F}_x(z) - \hat{\underline{F}}_x(z)|}{2} \\
&\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{6} + \frac{\epsilon}{6} < \epsilon,
\end{aligned}$$

and the theorem is proved.

Note that in the theorems in this section and the one preceding it, assumptions are made about the derivatives of $y(x)$ with respect to a and b. We restricted the discussion to problems where $\partial y/\partial a > 0$ and $\partial y/\partial b \geqslant 0$ or $\partial y/\partial b \leqslant 0$ but not both. We would obviously like to be able to consider problems where these derivatives are allowed to change sign. In Section 3 we discussed what to do in the case of one random variable, that is, divide the interval I_A into segments, each of which satisfied the requirements of Theorem 1. In the case of two random variables, we would likewise divide $I_A \times I_B$ into regions which satisfy the hypotheses of Theorems 2 or 4. The calculated distribution function, $\hat{F}_x(z)$, would then be the sum of several calculations, one for each of the different regions of the ab-plane.

If the numerical method used to calculate $y(x)$ also calculates $y'(x)$ then $F_{y'(x)}(z)$ can be estimated in exactly the same manner as $F_{y(x)}(z)$.

To estimate the joint distribution of $y(x)$ and $y'(x)$, that is, $F_{y(x)y'(x)}(z_1, z_2) = P(y(x) \leqslant z_1 ; y'(x) \leqslant z_2)$, when there is only one degree of randomness we need to approximate the probability of the intersection of the sets $S_y = \{a \in I_A \mid y(x, a) \leqslant z_1\}$ and $S_{y'} = \{a \in I_A \mid y'(x, a) \leqslant z_2\}$. If y and y' are both monotonically increasing functions of a then $\hat{F}_{y(x)y'(x)}(z_1, z_2)$ is simply the smaller of the two quantities $\hat{F}_{y(x)}(z_1)$ and $\hat{F}_{y'(x)}(z_2)$.

If there are two random variables in the problem, either of the methods of Sections 3 or 4 could be adapted to approximate $F_{y(x)y'(x)}(z_1, z_2)$, as well as the joint distribution

$$F_{y(x)y(x')}(z_1, z_2) = P(y(x) \leqslant z_1 ; y(x') \leqslant z_2), \qquad x \neq x'.$$

When there are more than two random variables, say $A_1, \ldots, A_n$, occurring in the problem, either in the differential equation or in the boundary conditions, or both, the method of Section 3 can easily be extended provided that A_1 is independent of $A_2, \ldots, A_n$.

5. Numerical Example

To illustrate the methods discussed in the preceding sections, we consider the following version of the one-dimensional heat equation:

$$\frac{\partial}{\partial \xi}\left[a(\xi)\frac{\partial u}{\partial \xi}\right] + F(\xi, u) = 0, \tag{5.1}$$

$$u(0) = u(1) = 1,$$

where

$$a(\xi) = \frac{(b_0 - b_1 \xi l)(h_0 - h_1 \xi l)}{l^2},$$

and

$$F(\xi, u) = \frac{a_1 Tu}{KT}(a_2 - Tu)\left(d - \xi l + \frac{l}{2}\right)\left(d + \xi l - \frac{l}{2}\right), \qquad \left|\xi l - \frac{l}{2}\right| \leqslant d$$
$$= 0, \qquad \text{otherwise.}$$

Here $F(\xi, u)$ represents heat added or taken away over the interval $[\frac{1}{2} - d/l, \frac{1}{2} + d/l]$. Heat is added if $u(\xi) < a_2/T$ and taken away if $u(\xi) > a_2/T$. The constants l, b_0, b_1, h_0, h_1 represent dimensions of a tapered bar and the quantity $(d - \xi l + l/2)(d + \xi l - l/2)$ is used to make $F(\xi, u)$ continuous for $\xi \in [0, 1]$.

The solution $u(\xi)$ was computed numerically using PROGRAM PEARSON by J. Flaherty which uses Pearson's method for solving second order quasilinear boundary value problems. The example shown is for $l = 2, d = .1, b_0 = h_0 = .4, b_1 = h_1 = .1, K = 117$, and $T = 50$. The parameter a_1 was varied from .5 to 1.5 and a_2 ranged from 50 to 100. Table 1 shows the maximum value of the numerical solution $\hat{u}(\xi)$ for various values of a_1 and a_2.

TABLE I

Values of $\hat{u}(\xi, a_1, a_2)$ for $\xi = 0.51$
$[l = 2, d = 0.1, b_0 = h_0 = 0.4, b_1 = h_1 = 0.1, K = 117.0, T = 50.0]$

	a_2										
a_1	50	55	60	65	70	75	80	85	90	95	100
0.5	1.0	1.0390	1.0799	1.1226	1.1672	1.2138	1.2623	1.3128	1.3652	1.4196	1.4760
0.6	1.0	1.0436	1.0893	1.1371	1.1871	1.2392	1.2935	1.3499	1.4084	1.4690	1.5316
0.7	1.0	1.0476	1.0974	1.1496	1.2042	1.2610	1.3201	1.3814	1.4448	1.5103	1.5778
0.8	1.0	1.0510	1.1046	1.1605	1.2189	1.2797	1.3428	1.4082	1.4756	1.5452	1.6166
0.9	1.0	1.0541	1.1108	1.1701	1.2318	1.2960	1.3625	1.4312	1.5020	1.5748	1.6495
1.0	1.0	1.0568	1.1163	1.1784	1.2431	1.3102	1.3796	1.4512	1.5248	1.6003	1.6777
1.1	1.0	1.0592	1.1212	1.1859	1.2531	1.3227	1.3946	1.4685	1.5446	1.6224	1.7019
1.2	1.0	1.0614	1.1256	1.1925	1.2620	1.3338	1.4078	1.4839	1.5619	1.6417	1.7231
1.3	1.0	1.0634	1.1296	1.1985	1.2699	1.3436	1.4195	1.4974	1.5772	1.6586	1.7416
1.4	1.0	1.0652	1.1332	1.2039	1.2770	1.3525	1.4300	1.5095	1.5908	1.6736	1.7580
1.5	1.0	1.0668	1.1365	1.2088	1.2835	1.3604	1.4394	1.5203	1.6029	1.6870	1.7726

Figure 5 shows the computed probability distribution functions of the solution $u(\xi)$ at various points along the bar, when $a_1 = 1$ is held constant and a_2 is assumed to have a triangular distribution with density function

$$f(a_2) = \frac{25 - |a_2 - 75|}{(25)^2}, \qquad 50 \leqslant a_2 \leqslant 100$$
$$= 0, \qquad \text{elsewhere.}$$

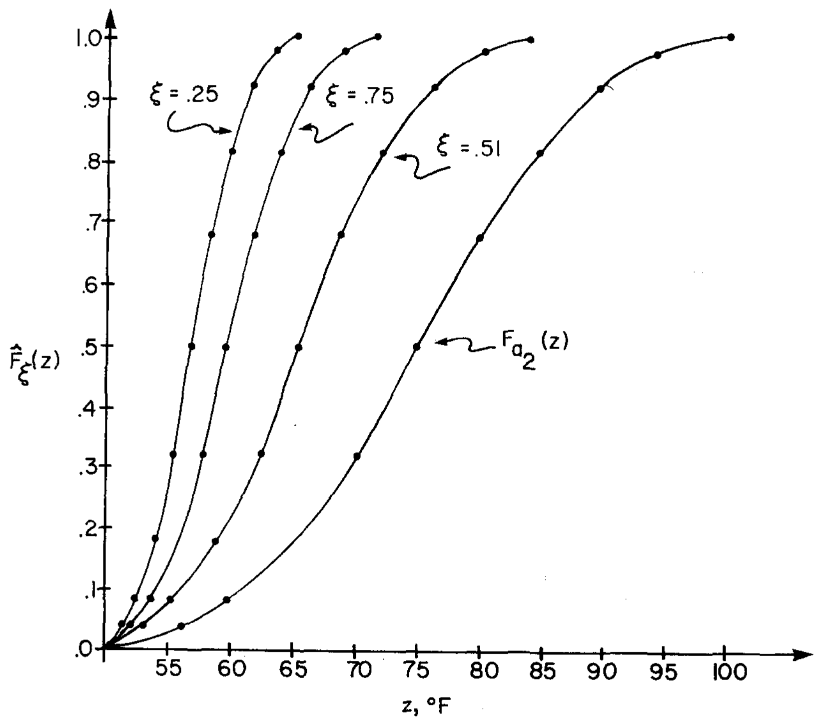

FIG. 5. Probality Distribution Function, $\hat{F}_\xi(z)$, of $u(\xi)$ for $\xi = .25, .75$ and $.51$ when $a_2 \in [50, 100]$ has a triangular distribution.

That is, $\hat{F}_\xi(z) \cong P(u(\xi) \leqslant z)$. The values shown are for $\xi = .25, .51$ and $.75$. The solution, $u(\xi)$, reaches its maximum value at approximately $\xi = .51$.

To calculate $\hat{F}_\xi(z)$ for a given value of z, as described in Section 2, we interpolate between the appropriate points. Suppose, for example, that we want to estimate

$$F_{.51}(1.4) = P(u_{\max} \leqslant 1.4).$$

For $a_2 = 80$, $\hat{u}(.51) = 1.3796$. For $a_2 = 85$, $\hat{u}(.51) = 1.4512$. See Table 1. Thus

$$\hat{F}_{.51}(1.4) = F_{a_2}(80) + \frac{1.4 - 1.3796}{1.4512 - 1.3796}(F_{a_2}(85) - F_{a_2}(80)) = .7199.$$

We can determine an upper bound for the error using the results of Section 2. We have

$$\frac{dF_{a_2}}{da_2} \leqslant .04, \qquad \frac{d^2F_{a_2}}{da_2^{\,2}} \leqslant (.04)^2, \qquad \Delta a_2 = 5, \qquad Ch^p \cong .0005.$$

We estimate $\partial u/\partial a_2$ and $\partial^2 u/\partial a_2{}^2$ using Table 1, that is,

$$\frac{\partial u(.51, a_2)}{\partial a_2} \leqslant \frac{\hat{u}(.51, 90) - \hat{u}(.51, 85)}{5} = .01472 \leqslant .02 = M_1,$$

$$\frac{\partial u(.51, a_2)}{\partial a_2} \geqslant \frac{\hat{u}(.51, 80) - \hat{u}(.51, 75)}{5} = .01388 \geqslant .01 = m,$$

and

$$\frac{\partial^2 u(.51, a_2)}{\partial a_2{}^2} \leqslant \frac{\hat{u}(.51, 80) - 2\hat{u}(.51, 75) + \hat{u}(.51, 70)}{25} = 9.6 \times 10^{-5}$$

$$\leqslant 10^{-4} = M_2 .$$

Then from (2.15), (2.18), and (2.10) where we neglect the $h^2 Y/2$ terms since we are not using an interpolated value of $\hat{u}(\xi)$, we have that

$$|F_{.51}(1.4) - \hat{F}_{.51}(1.4)| \leqslant .05.$$

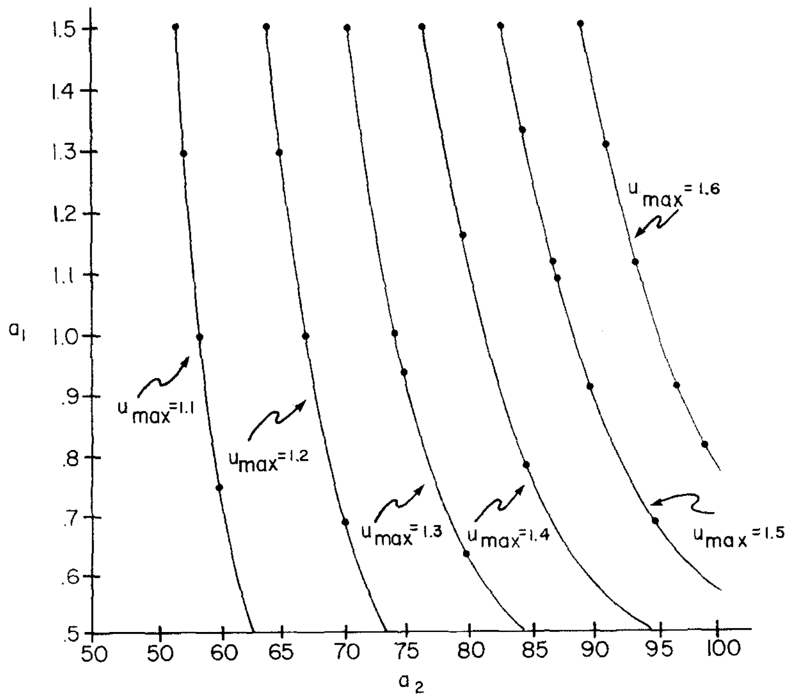

FIG. 6. Values of a_1 and a_2 for which the maximum value of the solution $u_{\max} = u(.51)$ is 1.1, 1.2, 1.3, 1.4, 1.5, and 1.6.

When both a_1 and a_2 are assumed to be random, the distribution function of $u(\xi)$ for a given ξ is found by integrating the joint density function of a_1 and a_2 over the appropriate region of the a_1a_2-plane as described in Sections 3 and 4. Such regions are illustrated in Figure 6 for various values of the maximum value of $u(\xi)$. So, for instance, the probability that $u_{\max}$ is less than 1.4 would be found by integrating $\partial^2 F(a_1, a_2)/\partial a_1 \partial a_2$ over the region to the left of the curve labelled $u_{\max} = 1.4$.

Suppose, for instance, that a_1 and a_2 are independent random variables. Let a_2 have the triangular distribution described above. Suppose a_1 also has a triangular distribution with density function

$$f(a_1) = \frac{\frac{1}{2} - |a_1 - 1|}{(\frac{1}{2})^2}, \qquad .5 \leqslant a_1 \leqslant 1.5,$$
$$= 0, \qquad \text{elsewhere.}$$

Using the method in Section 4 with $\Delta a_1 = .1$ and $\Delta a_2 = 5$, we have that

$$\hat{F}_{.51}(1.4) = .814, \qquad \hat{\underline{F}}_{.51}(1.4) = .6336.$$

Thus

$$\hat{F}_{.51}(1.4) = .7238.$$

If the error estimate as described in Theorem 4 is used here, the results do not appear very encouraging. We have

$$\frac{\partial F_{a_2}}{\partial a_2} \leqslant .04, \qquad \frac{\partial F_{a_1}}{\partial a_1} \leqslant 2 \qquad \text{and} \qquad C = .08.$$

According to the theorem, if we want to insure that we can make the error less than ϵ, then we need to choose

$$\Delta a_2 < \frac{\epsilon}{6(1.5 - .5)\,C} = \frac{25}{12}\epsilon$$

and

$$Ch^p < \inf \left| \frac{\partial u}{\partial a_2} \right| \frac{\Delta a_2}{4} = .0025 \Delta a_2 .$$

Thus if we wanted to have $\epsilon = .01$ we would choose $\Delta a_2 < .02$ and $Ch^p <$ .00005. In the numerical example described above, we used $\Delta a_2 = 5$ and had $Ch^p \cong .0005$. For these values the theorem would say that $\epsilon > 2.4$ or that we have an error of about 250%.

Obviously the error is not this large and we can estimate it better by noting that the functions $\hat{a}(b)$ and $\hat{\underline{a}}(b)$ used to calculate the values $\hat{F}_{.51}(1.4)$ and $\hat{\underline{F}}_{.51}(1.4)$

yield actual upper and lower bounds, respectively, for $F_{.51}(1.4)$. Thus we have that

$$|F_{.51}(1.4) - \hat{F}_{.51}(1.4)| \leqslant \frac{\hat{\bar{F}}_{.51}(1.4) - \hat{\underline{F}}_{.51}(1.4)}{2} = .0902 < .1$$

and, for this problem, reducing Δa_2 to 1 would probably give an acceptable error.

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