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Stability of asymmetric tetraquarks in the minimal-path linear potential

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ABSTRACT

The linear potential binding a quark and an antiquark in mesons is generalized to baryons and multiquark configurations as the minimal length of flux tubes neutralizing the color, in units of the string tension. For tetraquark systems, i.e., two quarks and two antiquarks, this involves the two possible quark–antiquark pairings, and the Steiner tree linking the quarks to the antiquarks. A novel inequality for this potential demonstrates rigorously that within this model the tetraquark is stable in the limit of large quark-to-antiquark mass ratio.

The quark–antiquark confinement in ordinary mesons is often described by a linear potential $V_2 = r$, in units where the string tension is set to unity. For a given interquark separation r, it can be interpreted as the minimal gluon energy if the field is localized in a flux tube of constant section linking the quark to the antiquark.

The natural extension to describe the confinement of three quarks in a baryon is the so-called Y-shape potential

$$V_3(v_1, v_2, v_3) = \min_s (d_1 + d_2 + d_3), \tag{1}$$

where d_i is the distance of the ith quark located at v_i ($i = 1, 2, 3$) to a junction s whose location is adjusted to minimize V_3. This potential has been proposed in Refs. [1–7], among others. It has been used, e.g., in Refs. [8,9] for studying the spectroscopy of baryons. See, also [10]. The optimization in (1) corresponds to the well-known problem of Fermat and Torricelli to link three points with the minimal network. See Fig. 1.

We now turn to the tetraquark systems $(Q, Q, \bar{q}, \bar{q})$, with the notation (v_1, v_2, v_3, v_4) for the locations, and (M, M, m, m) for the masses which will be used shortly. The potential is assumed to be (with $d_{ij} = \|v_i v_j\|$)

$$U = \min\{d_{13} + d_{24}, d_{14} + d_{23}, V_4\},$$
$$V_4 = \min_{s_1, s_2}\big(\|v_1 s_1\| + \|v_2 s_1\| + \|s_1 s_2\| + \|s_2 v_3\| + \|s_2 v_4\|\big). \tag{2}$$

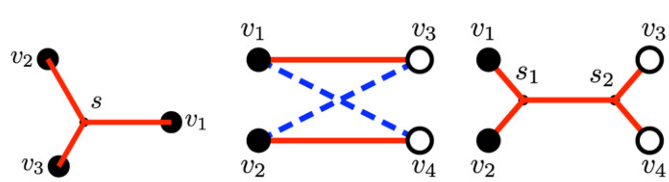

Fig. 1. Generalization of the linear quark–antiquark potential of mesons to baryons (left) and to tetraquarks, where the minimum is taken of the flip–flop (center) and Steiner tree (right) configurations.

The first two terms of U describe the two possible quark–antiquark links, and their minimum is sometimes referred to as the "flip–flop" model, schematically pictured in Fig. 1. It was introduced by Lenz et al. [11], who used, however, a quadratic instead of linear rise of the potential as a function of the distance. The last term, V_4, is represented in Fig. 1 and corresponds to a connected flux tube. It is given by a Steiner tree, i.e., it is minimized by varying the location of the Steiner points s_1 and s_2. The choice of this potential is inspired by Refs. [3,12–14], and has been discussed in the context of lattice QCD [15,16].

The four-body problem in quantum mechanics is notoriously difficult. For instance, Wheeler proposed in 1945 the existence of a positronium molecule (e^+, e^+, e^-, e^-) which is stable in the limit where internal annihilation is neglected, i.e., lies below its threshold for dissociation into two positronium atoms. In 1946, Ore published a four-body calculation of this system [17] and con-

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cluded that his investigation "counsels against the assumption that clusters of this (or even of higher) complexity can be formed". However, in 1947, Hylleraas and the same Ore published an elegant analytic proof that this molecule is stable [18]. It has been discovered recently [19].

Similarly, the above model (2), in its linear version, was considered by Carlson and Pandharipande, who entitled their paper [20] "Absence of exotics in the flux tube model", i.e., did not find stable tetraquarks.[1] However, Vijande et al. [21] used a more systematic variational expansion of the wave function and in their numerical solution of the four-body problem found a stable tetraquark ground state. Moreover, unlike [20], they considered the possibility of unequal masses, and found that stability improves if the quarks are heavier (or lighter) than the antiquarks, in agreement with previous investigations (see, e.g., [21] for references).

It is thus desirable to check whether this minimal-path model supports or not bound states. The present attempt is based on an upper bound on the potential, which leads to an *exactly solvable* four-body Hamiltonian.

With the Jacobi vector coordinates

$$x = v_2 - v_1, \qquad y = v_4 - v_3, \qquad z = \frac{v_3 + v_4 - v_1 - v_2}{2}, \tag{3}$$

and their conjugate momenta, the relative motion is described by the Hamiltonian

$$H = \frac{p_x^2}{M} + \frac{p_y^2}{m} + \frac{p_z^2}{4\mu} + U(x, y, z), \tag{4}$$

where μ, given by $\mu^{-1} = m^{-1} + M^{-1}$, is the quark–antiquark reduced mass. Using the scaling properties of H, one can set $m = 1$ without loss of generality.

The simplest bound on the potential U is

$$U \leqslant V_4 \leqslant \|x\| + \|y\| + \|z\|, \tag{5}$$

as the tree with optimized Steiner points s_1 and s_2 is shorter than if the junctions are set at the middles of the quark separation $v_1 v_2$ and antiquark separation $v_3 v_4$. This leads to a separable upper bound for the Hamiltonian

$$H \leqslant H' = \frac{p_x^2}{M} + \|x\| + p_y^2 + \|y\| + \frac{p_z^2}{4\mu} + \|z\|. \tag{6}$$

Now, the ground state e_0 of $p_x^2 + \|x\|$ corresponds to the radial equation $-u''(r) + ru(r) = e_0 u(r)$ with $u(0) = u(\infty) = 0$ and is the negative of the first zero of the Airy function, $e_0 = 2.3381\ldots$. By scaling, the ground state of $\alpha p_x^2 + \beta\|x\|$, with $\alpha > 0$ and $\beta > 0$ is $\alpha^{1/3}\beta^{2/3} e_0$. Thus the lowest eigenvalue of H' is

$$E' = e_0\left[M^{-1/3} + 1 + (4\mu)^{-1/3}\right], \tag{7}$$

with $\mu = M/(1+M)$. By comparison, the threshold of $(QQ\bar{q}\bar{q})$ is made of two identical $(Q\bar{q})$ mesons, each governed by the Hamiltonian $h = p^2/(2\mu) + \|r\|$, where p is conjugate to the quark–antiquark separation r. Thus the threshold energy is

$$E_{\rm th} = 2e_0(2\mu)^{-1/3}, \tag{8}$$

and it is easily seen that $E' > E_{\rm th}$ for any value of the quark-to-antiquark mass ratio M, i.e., the bound (5) cannot demonstrate binding.

A better bound will be proved below. If there is a genuine Steiner tree[2] linking the quarks to the antiquarks, then

$$V_4 \leqslant \frac{\sqrt{3}}{2}\left(\|x\| + \|y\|\right) + \|z\|. \tag{9}$$

[1] The authors used a relativistic form of kinetic energy and considered also the possibility of short-range corrections, but this seemingly does not affect their conclusion.

[2] This will be made more precise in the proof given in Appendix A.

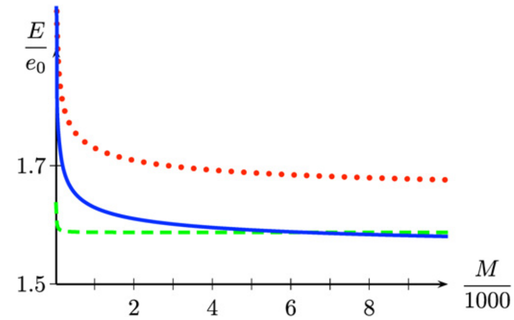

Fig. 2. Simple bound E' (Eq. (7), dotted line) and improved upper bound E'' (Eq. (11), solid line) on the tetraquark ground-state energy as a function of quark-to-antiquark mass ratio M. Also shown is the threshold energy $E_{\rm th}$ (Eq. (8), dashed line). The energies are in units of e_0, the ground state of $-\Delta + \|r\|$.

But if V_4 is not associated to a genuine Steiner tree, this inequality is often violated. Consider for instance a rectangular configuration with $\|v_1 v_2\| = \|v_3 v_4\| \gg \|v_1 v_3\| = \|v_2 v_4\|$ (in this case the mathematical Steiner tree problem would require a Steiner point linking v_1 and v_3, another Steiner point linking v_2 and v_4, but the corresponding fluxes are not permitted by the color coupling in QCD), then $\|z\| \sim 0$ and $V_4 \sim \|x\| + \|y\|$, so (9) does not hold.

However, it will be shown that

$$U \leqslant \frac{\sqrt{3}}{2}\left(\|x\| + \|y\|\right) + \|z\|, \tag{10}$$

for any configuration of the quarks and antiquarks, i.e., for any x, y and z. Then the ground state of H is bounded as

$$E < E'' = e_0\left[\left(\frac{3}{4}\right)^{1/3}\left(M^{-1/3} + 1\right) + (4\mu)^{-1/3}\right]. \tag{11}$$

As shown in Fig. 2, this bound E'' significantly improves the previous one, E'. It is easily seen than E'' becomes smaller than $E_{\rm th}$ for very large values of the mass ratio, more precisely for $M > 6402$, and thus that the tetraquark is bound at least in this range of M. The numerical estimate of [21] actually indicates stability for all values of M, even $M = 1$.

To summarize, we obtained an analytic upper bound on the ground state energy of tetraquarks systems with two units of open flavor, $(QQ\bar{q}\bar{q})$, using a model of linear confinement inspired by the strong-coupling regime of QCD. The key is an inequality on the length of a Steiner tree with four terminals. The bound confirms a recent numerical investigation, in which this potential was shown to bind these tetraquarks below the threshold for dissociation into two mesons. It remains to investigate whether this stability survives refinements in the dynamics, such as short range corrections, spin-dependent forces, etc.

It is our intention to extend this investigation to the case of the pentaquark (one antiquark and four quarks) and hexaquark configurations (six quarks), which have been much debated in recent years.

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Appendix A. Results on the Steiner problem

Three terminals. The three-point problem is very much documented in textbooks [22–26]. Let $v_1 v_2 v_3$ be the triangle, with side

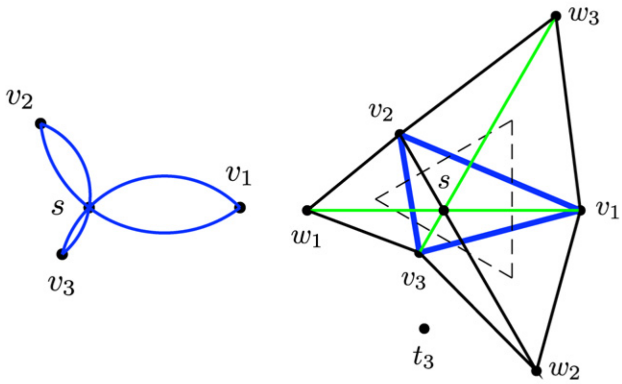

Fig. 3. Left: the junction lies on each arc from which a side is seen under 120°. Right: the three-terminal Steiner problem as a side product of Napoleon's theorem.

lengths $a_1 = \|v_2v_3\|, \ldots$ and angles $\alpha_1 = \angle v_1v_2v_3$, etc. The problem of finding a path of minimal length $\|sv_1\| + \|sv_2\| + \|sv_3\|$ linking the three vertices has been solved by Fermat and Torricelli. See, e.g., [22]. The result is the following: if one of the angles, say α_1, is larger than 120°, s coincides with v_1, otherwise each side of the triangle is seen from s with an angle of 120°. The Steiner point s is thus at the intersection of three arcs of circles, see Fig. 3.

The three-terminal problem is also linked to Napoleon's theorem, which states that if one draws external equilateral triangles on each side, $v_1v_2w_3$, $v_1v_3w_1$ and $v_1v_1w_2$, the centers of these triangles form an *equilateral* triangle (dashed lines in Fig. 3), a nice example of symmetry restoration. The junction s is just the intersection of v_1w_1, v_1w_2 and v_3. Note that $\|sv_1\| + \|sv_2\| = \|sw_3\|$, and similar relations, and thus the potential is simply $V_3 = \|v_1w_1\| = \|v_2w_2\| = \|v_3w_3\|$.

The point w_3 and its reflection with respect to v_1v_2, t_3 form the toroidal domain associated to the subset $\{v_1, v_2\}$. The length of the minimal Steiner tree is the maximal distance between v_3 and the domain $\{w_3, t_3\}$. From the above properties, one can estimate the string potential in a closed form [7].

$$V_3 = \ell_1 + \ell_2 + \ell_3 = \sqrt{a_1^2 + a_2^2 + a_3^2 + \sqrt{3\lambda(a_1, a_2, a_3)}},$$
$$\lambda(x, y, z) = (x + y + z)(x + y - z)(x - y + z)(-x + y + z). \qquad (12)$$

The planar tetraquark problem. For the four-point problem, there are many special cases, which can be treated by inspection. If, for instance the quark v_2 is on the back of v_1, as in Fig. 4, the problem reduces to the Steiner problem for $\{v_1, v_3, v_4\}$. Another special case is shown in Fig. 4, where the quarks are close to the antiquarks. For the standard Steiner problem of geometry, the solution would correspond to the Steiner tree shown as a dotted line, with a Steiner point s_3 linked to v_1 and v_3 and another one, s_4, linked to v_2 and v_4. This is not allowed by the different color properties of quarks and antiquarks, hence our best tree, shown as a solid line, has only one junction. But in estimating the potential U of Eq. (2) for this configuration, the minimum is the flip–flop term $d_{13} + d_{24}$.

Let us turn to the case of a genuine Steiner tree $(v_1v_2)s_1s_2(v_3v_4)$ as in Fig. 5. The string of Fig. 1 is minimized with respect to s_1 and s_2. Hence for fixed s_2, it assumes the Fermat–Torricelli minimization of $v_1v_2s_1$, a well-known iteration property of Steiner trees. Hence $\angle v_1s_1v_2 = 120°$ and v_1v_2 is the bissector of $\angle v_1s_1v_2$ and passes through the point w_{12} which completes an equilateral triangle $v_1v_2w_{12}$ in the quark sector. Similarly, it also passes through w_{34} which makes $v_3v_4w_{34}$ equilateral in the antiquark sector.

The junction points s_1 and s_2 are just the other intersections of the straight line $w_{12}w_{34}$ with the circumcircles of $v_1v_2w_{12}$ and

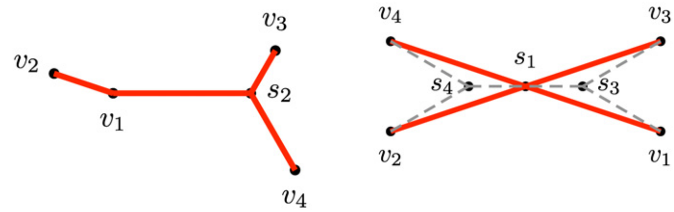

Fig. 4. Examples of special configurations. Left: one junction coincides with v_1. Right: the two junctions merge (the dotted gray line corresponds to the Steiner tree if the four points v_i play the same role, unlike the tetraquark problem with quarks and antiquarks having conjugate colors).

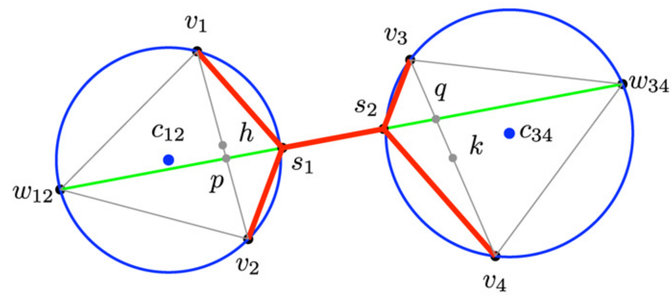

Fig. 5. Construction of the minimal string in the planar case.

$v_3v_4w_{34}$, as shown in Fig. 5. There is a possible ambiguity about on which side s_1 or s_2 should be, but this is easily solved by the requirement that the total length of the string is minimum. Crucial is the observation that $V = \|w_{12}w_{34}\|$, so that the determination of the Steiner points s_1 and s_2 is not required to compute V_4.

A variant is that is t_{12} is the reflection of w_{12} with respect to v_1v_2, the set $\{w_{12}t_{12}\}$ is the toroidal domain associated to the quarks, and similarly $\{w_{34}t_{34}\}$ for the antiquarks, the length of the Steiner trees is the maximal distance between these two sets.

This construction, which is a special case of the Melzak's algorithm [27], leads to a very easy computation. If each vector v_i is identified with its affix (complex number) v_i, etc., then those of w_{12} and w_{34} are easily deduced, for instance $w_{12} = -j^2v_1 - jv_2$ or $-jv_1 - j^2v_2$ (depending on which side is w_{12}), if one uses the familiar root of unity $j = \exp(2i\pi/3)$. Once w_{12} and w_{34} are determined, $V = \|w_{12}w_{34}\|$. If one wishes to locate the Steiner points, it is sufficient to remark that $w_{12}s_2.w_{12}w_{34} = \|w_{12}c_{34}\|^2 - r_{34}^2$ and $w_{34}s_1.w_{34}w_{12} = \|w_{34}c_{12}\|^2 - r_{12}^2$, where c_{12} is the center of the circle $v_1v_2w_{12}$ and $r_{12} = d_{12}\sqrt{3}/2$ its radius and c_{34} and r_{34} are defined similarly in the antiquark sector.

The spatial tetraquark problem. In general, the four constituents do not belong to the same plane. The minimum is achieved for $v_1v_2s_1s_2$ coplanar, and $v_3v_4s_1s_2$ also coplanar, but in a different plane. The toroidal domain to which the point w_{12} belongs is the Melzak circle, of axis v_1v_2 and radius $r_{12} = \|v_1v_2\|\sqrt{3}/2$, and similarly for w_{34} in the antiquark sector. The straight line $w_{12}w_{34}$ has to intersect these two circles as well as the lines v_1v_2 and v_3v_4. The problem consists of constructing such a straight line.

The reasoning can be made on Fig. 5, if one imagines that $v_3v_4s_2$ is not coplanar to $v_1v_2s_1$. As stressed in [28], the key is to determine p and q, the intersections of s_1s_2 with v_1v_2 and v_3v_4, respectively. In this Letter, the following coupled equations are derived

$$x_{\{p,q\}} = \frac{\{m,n\}\sqrt{h^2 + x_{\{q,p\}}^2 \sin^2\phi} + r_{\{12,34\}}\, v\cos\phi}{r_{\{34,12\}} + \sqrt{h^2 + x_{\{q,p\}}^2 \sin^2\phi}}, \qquad (13)$$

for the abscissa x_p of p along v_1v_2 and x_q of q along v_3v_4. These abscissas are from the common perpendicular uv to v_1v_2

and v_3v_4 ($u \in v_1v_2$ and $v \in v_3v_4$), with $\|uv\| = h$, $\|uh\| = m$ and $\|vk\| = n$, where h is the middle of v_1v_2 and k that of v_3v_4. Eq. (13) can be solved by iterations, with remarkably fast convergence. Once x_p and x_q, i.e., p and q, are determined, the Steiner points are determined by imposing they are on the circles $v_1v_2w_{12}$ and $v_3v_4w_{34}$, respectively. For instance, if $s_1 = p + t(q - p)$, t obeys a second order equation.[3]

If one is interested only in the length of the Steiner tree and not in the position of the Steiner points, an alternative formalism consists of locating p through $p = h + x(v_2 - h)$ and $q = k + y(v_4 - k)$. With this notation, the length of the tree is simply

$$V_4 = \min_{x,y}\left[\|pq\| + \frac{r_{ab}}{\sqrt{3}}\sqrt{3+x^2} + \frac{r_{cd}}{\sqrt{3}}\sqrt{3+y^2}\right], \tag{14}$$

which is easily minimized by varying x and y. The minimisation is equivalent to solving the coupled equations

$$x = \sqrt{3+x^2}\,\frac{v_1v_2.pq}{\|v_1v_2\|\|pq\|}, \qquad y = \sqrt{3+y^2}\,\frac{v_3v_4.qp}{\|v_3v_4\|\|pq\|}, \tag{15}$$

which expresses that w_{12}, p, s_1, s_2, q and w_{34} are collinear. These equations are easily solved by iteration or any other means.

We believe that, besides checking the particular cases with large angles or a single Steiner point, the fastest computation of the connected four-quark potential consists of minimising (14) or solving (15). We expect a dramatic improvement in computing time from the above algorithm.

However, it is aesthetically appealing to attempt a further reduction of the number of variables to be determined numerically, and to provide an almost analytic estimate of the interaction as a function of the coordinates of the quarks and antiquarks. Finding $V_4 = \|w_{12}w_{34}\|$, the maximal distance between the Melzak circles $\mathcal{C}_{12}$ and $\mathcal{C}_{34}$, is very similar to the problem of the minimal distance between two circles in space, as addressed e.g., in [29,30]. Neff [29] has shown that with the help of Lagrange multipliers and Gröbner type of elimination performed by computer-algebra software, the squared stationary distance V_4^2 obeys an eighth-order polynomial equation whose coefficients are rational functions of the coordinates of v_1, v_2, v_3 and v_4. (See Fig. 6.)

Eberly [30] showed that if m is associated to an angle θ along $\mathcal{C}_{12}$, and n to ϕ along $\mathcal{C}_{34}$, then imposing $\|mn\|^2$ to be stationary, results in two equations of the type

$$\alpha_i \cos\theta + \beta_i \sin\theta + \gamma_i = 0, \quad i = 1, 2, \tag{16}$$

where α_i, β_i and γ_i contain constants and terms linear in $\cos\phi$ and $\sin\phi$. Solving (16) as two linear equations, as if $\cos\theta$ and $\sin\theta$ were independent, and then imposing $\cos\theta^2 + \sin\theta^2 = 1$ gives an equation for $\cos\phi$ and $\sin\phi$, which is transformed into an 8th order equation in $\cos\phi$.

It is slightly faster to rewrite (16) using $t = \tan(\theta/2)$ and $u = \tan(\phi/2)$ as

$$\delta_i t^2 + \eta_i t + \epsilon_i = 0, \quad i = 1, 2, \tag{17}$$

where the coefficients are quadratic in u. The compatibility of two such equations is simply

$$W(\delta,\eta)W(\eta,\epsilon) = W(\delta,\epsilon)^2, \quad W(x,y) = x_1y_2 - x_2y_1, \tag{18}$$

and is directly a polynomial in u, of order 8.

Proof of the inequality (10). If we have a positively oriented edge from s_1 to s_2, i.e., the Steiner tree is non-degenerate, then we have

$$V_4 \leqslant \big(\|x\| + \|y\|\big)\sqrt{3}/2 + \|z\| = B, \tag{19}$$

using Melzak circles.

[3] There is a misprint in [28] which propagated in the numerical calculation given as an example.

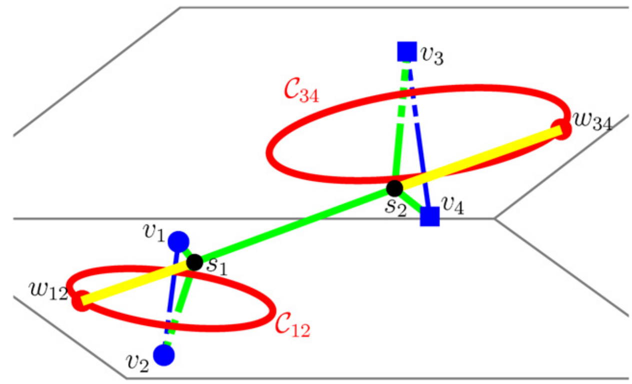

Fig. 6. The confining potential V_4 for the tetraquark system $(v_1v_2v_3v_4)$ is the minimal length of the tree $\|v_1s_1\| + \|v_2s_1\| + \|s_1s_2\| + \|s_2v_3\| + \|s_2v_4\|$ when s_1 and s_2 are varied. It is also the maximal distances between the circles $\mathcal{C}_{12}$ and $\mathcal{C}_{34}$, i.e., the distance $w_{12}w_{34}$. The Melzak circle $\mathcal{C}_{12}$ is centered at the middle of v_1 and v_2, has v_1v_2 as axis and a radius $\|v_1v_2\|\sqrt{3}/2$, and $\mathcal{C}_{34}$ has analogous properties in the antiquark sector.

However the bound required is for $U = \min\{d_{13} + d_{24}, d_{14} + d_{23}, V_4\}$. So we want to confirm that

$$U \leqslant \big(\|x\| + \|y\|\big)\sqrt{3}/2 + \|z\| = B, \tag{20}$$

is valid, regardless of whether V_4 is a degenerate or non-degenerate Steiner tree.

We follow the variational method introduced in [31]. The problem is formulated as a global optimisation problem as follows;

Define L as the length of the *formal* Steiner tree spanned by the four vertices. This length is obtained from the distance between the farthest points on the two Melzak circles. In terms of the usual Steiner tree components, $L = \|v_1s_1\| + \|v_2s_1\| \pm \|s_1s_2\| + \|v_3s_2\| + \|v_4s_2\|$). We get the positive sign for $\|s_1s_2\|$ if there is a real Steiner tree. On the other hand, if the Steiner vertices have interchanged position, so that on the line between the two farthest Melzak points, s_2 is closer to the Melzak point for v_1, v_2 than s_1, then we have the negative sign for $\|s_1s_2\|$. So we can construct a formal tree on the six vertices $v_1, v_2, v_3, v_4, s_1, s_2$ where the edge joining the two Steiner vertices is 'negatively oriented'.

Now it is easy to see that $L \leqslant (\|x\| + \|y\|)\sqrt{3}/2 + \|z\|$. So if $V = L$ then the desired inequality follows trivially. So we only need to consider the situation where $L < V$, i.e. the Steiner tree is formal rather than a real Steiner tree. Now by the inequality above, if either of $d_{13} + d_{24}, d_{14} + d_{23}$ is not larger than L, then clearly the required inequality follows. So we only need to consider the case when $d_{13} + d_{24} > L$ and $d_{14} + d_{23} > L$.

We can parametrise the points v_1, v_2, v_3, v_4 by the numbers $\|v_1s_1\|$, $\|v_2s_1\|$, $\pm\|s_1s_2\|$, $\|v_3s_2\|$, $\|v_4s_2\|$. (It is easy to see that these four points are determined up to rotation, translation by five parameters.) By rescaling, we can assume that the sum of these five numbers is 1, without loss of generality for the inequality. It is easy to see that all the numbers are then bounded so the domain becomes compact. So we seek a maximum of the ratio of $R = \min\{d_{13} + d_{24}, d_{14} + d_{23}\}$ and $(\|x\| + \|y\|)\sqrt{3}/2 + \|z\| = B$ over this domain.

Now suppose that we rotate the triangles $v_1v_2v_3$ and $v_1v_2v_4$ around an axis line through v_1v_2. Clearly we can think of one triangle as being fixed and the other as moving relative to the first one. The quantity R does not change by this rotation, but obviously B does. Hence a maximum of the ratio R/B corresponds to a minimum for B under such a rotation.

Now an elementary argument shows that such a minimum for B occurs for the configuration being planar, i.e. when the vertex v_4 moves into the plane of v_1, v_2, v_3. Now assume that some initial configuration satisfies $R/B > 1$ and the Steiner tree is formal rather than real. As the triangle $v_1v_2v_4$ rotates around an axis line through v_1v_2, it is easy to see that the two Melzak circles move apart. At some intermediate point, if they cross, then we find that the Steiner tree changes from being formal to being real. At this intermediate point, it is trivial to see that $R/B < 1$. But this is impossible, since we have initially $R/B > 1$ and R/B is increasing, since B is decreasing and R is fixed.

On the other hand, if the Melzak circles never intersect, then this must be true for the planar configuration. So we would have such a configuration for which the Steiner tree is still formal but $R/B > 1$. It is elementary to prove that this is impossible. So this completes the argument.

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