Zeros of the Wronskian of a Polynomial

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1. Introduction

An important part of the “geometry of polynomials” is concerned with locating the zeros of $Df(z) := f'(z)$, given those of a polynomial $f(z)$. For a thorough treatment of this topic, see, e.g., the book by Marden [10, Chap. 2]. If we rewrite

$$W(z) = f(z) - \log f(z),$$

it becomes clear that we can consider the operator $W$, defined by

$$Wf(z) = f^2(z) \frac{d^2}{dz^2} \log f(z),$$

as an analogue of the differential operator $D$. Obviously,

$$Wf(z) = f(z) f''(z) - (f'(z))^2 = \begin{vmatrix} f(z) & f'(z) \\ f'(z) & f''(z) \end{vmatrix}.\quad (1.2)$$

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This determinant form prompted the second author to call the polynomial $Wf(z)$ the "Wronskian of the polynomial $f(z)$." Much of [10] is concerned with the problem of relating the zero distributions of $p(z)$ and $Lp(z)$ where $p(z)$ is a polynomial and $L$ is a linear differential operator. It seems both natural and nontrivial to generalize this to algebraic differential operators. We believe the "Wronskian operators" are a rather attractive set of non-linear operators to begin with. They have recently attracted some attention from experts in the study of general algebraic differential equations; see [7, 12] and the references therein. Also, the Wronskian of an entire function plays an important role in the recent proof of the "Pólya-Wiman Conjecture" (see [3]). An unsolved problem concerning the connection between the number of real zeros of a polynomial $p(z)$ and those of $Wp(z)$ is also mentioned in [3]. Furthermore, the operator $W$ is related to the "Laguerre inequality"; see, e.g., [11].

In this paper we study the distribution of the zeros of $Wf(z)$, given some information on the location of those of $f(z)$. We begin by proving, in Section 2, some general properties of the polynomial $Wf(z)$ and its zeros. By a shifting and scaling argument we may assume without loss of generality that all the zeros of $f(z)$, if they are real, lie in the interval $[-1, 1]$. For such a polynomial, the zeros of $Df(z)$ lie also in the interval $[-1, 1]$; we show that the corresponding result for $Wf(z)$ is that the zeros lie inside or on the unit circle.

The following are our main results:

1. If $d$ is the minimum distance between two consecutive zeros of $f(z)$, then the imaginary part of the zeros of $Wf(z)$ cannot be less than $\sqrt{3d}/4$ (Theorem 2.8).

2. If the zeros of $f(z)$ are located at $z_j := -1 + (j/n)$, $j = 0, 1, \ldots, 2n$, then very exact bounds on the zeros of $Wf(z)$ can be given. The imaginary parts of the zeros are asymptotically $\pm (\log n)/2\pi n$ as $n \to \infty$ (Corollary 4.1). This is closely related to the truncated partial fraction expansion of $\pi^2/\sin^2 \pi z$ (Theorem 3.1).

3. If the zeros of $f(z)$ are not too far from the $z_j$, then the zeros of $Wf(z)$ are still close to the real axis (Corollaries 4.3–4.5).

Finally, Section 9 contains some remarks on constructing polynomials whose Wronskians have preassigned properties.

2. General Properties of $Wf(z)$

First we list some general properties of the operator $W$. In what follows, $f$, $g$, and $h$ are always polynomials.
LEMMA 2.1. (a) \( W(gh) = g^2W_1 + h^2W_2 \),
(b) \( W(g^n) = ng^{2n-2}W_n \),
(c) \( W(z - a) = -1 \).

(a) and (c) follow from (1.2) by direct computation; (b) is a direct consequence of (a). With (a) and (c) we get immediately the following.

LEMMA 2.2. If \( f(z) = (z - \alpha_1)^{m_1} \cdots (z - \alpha_k)^{m_k} \), \( m_j \in \mathbb{N} \), then

\[
Wf(z) = -(f(z))' \left( \frac{m_1}{(z - \alpha_1)^2} + \cdots + \frac{m_k}{(z - \alpha_k)^2} \right). 
\]

It follows from Lemma 2.2 that if \( \alpha \) is a zero of \( f(z) \) of multiplicity \( \geq 2 \) then it is also a (multiple) zero of \( Wf(z) \). In this case we call \( \alpha \) a "trivial zero" of \( Wf(z) \). We note that by Lemma 2.2 trivial zeros occur only with even multiplicities.

PROPOSITION 2.3. If all the zeros of \( f(z) \) lie on a straight line in \( \mathbb{C} \), then no zeros of \( Wf(z) \) other than the trivial ones lie on this line.

Proof. Let \( z \) be collinear with \( \alpha_1, \ldots, \alpha_k \). Then there is some \( \theta \in [0, 2\pi) \) such that

\[
z_0 - \alpha_j = e^{i\theta} \beta_j, \quad \beta_j \in \mathbb{R}, \beta_j \neq 0.
\]

Now

\[
g(z) := \sum_{j=1}^{k} m_j(z - \alpha_j)^{-2} = e^{-2i\theta} \sum_{j=1}^{k} m_j \beta_j^{-2} = Be^{-2i\theta},
\]

where \( B > 0 \). This completes the proof.

COROLLARY 2.4. If \( f(z) \) has only simple, real zeros then \( Wf(z) \) has no real zeros.

THEOREM 2.5. If all the zeros of \( f(z) \) are real and lie in \([-1, 1]\), then the zeros of \( Wf(z) \) lie inside or on the unit circle.

Proof. Let \( \alpha_1, \ldots, \alpha_n \) be the zeros of \( f(z) \), set \( \alpha_0 := -1, \alpha_{n+1} := 1 \), and denote \( \theta_j := \text{arg}(z - \alpha_j) \), \( j = 0, 1, \ldots, n+1 \). Since \( f(z) \) must have real coefficients, we may restrict our attention to the upper half plane; so \( \theta_0 < \theta_{n+1} \).

Now suppose that \( |z| > 1 \). Then the origin lies outside the unit circle centered at \( z \), and therefore \( \theta_{n+1} - \theta_0 < \pi/2 \), by a well-known result in classical geometry. Also \( \theta_0 \leq \theta_j \leq \theta_{n+1} \) for \( j = 1, 2, \ldots, k \). Now \( \text{arg}((z - \alpha_j)^{-2}) = -2\theta_j \) and \( -2\theta_{n+1} \leq \text{arg}((z - \alpha_j)^{-2}) \leq -2\theta_0 \), and therefore all the complex
numbers $m_j/(z - a_j)^2$ lie to one side of a straight line through the origin since $|2\theta_{n+1} - 2\theta_0| < \pi$. Hence $\sum_{j=1}^k m_j(z - a_j)^{-2} \neq 0$ and this proves the theorem.

Remark. We will see that Theorem 2.5 is sharp. In fact, for $m, n \in \mathbb{N}$, let $f_{m, n}(z) := (z - 1)^m (z + 1)^n$. Then it follows from Lemma 2.1 that

$$Wf_{m, n}(z) = -(z - 1)^{2m-2} (z + 1)^{2n-2} [m(z + 1)^2 + n(z - 1)^2].$$

The nontrivial zeros of $Wf_{m, n}(z)$ are the zeros of the term in brackets, i.e., the roots of

$$z^2 + 2 \frac{m-n}{m+n} z + 1 = 0,$$

namely

$$z_0 = \frac{n-m \pm 2i \sqrt{mn}}{n+m}.$$

Note that $|z_0| = 1$ and that $z_0 = i$ if $n = m$. Every point on the unit circle with rational real part is therefore a zero of $Wf_{m, n}(z)$, for some positive integers $m$ and $n$.

**Theorem 2.6.** If the zeros of $f(z)$ lie inside or on the unit circle then the zeros of $Wf(z)$ lie inside or on the circle of radius $\sqrt{2}$ centered at the origin.

**Proof.** We proceed as in the proof of Theorem 2.5. Let $\alpha_1, \ldots, \alpha_n$ be the zeros of $f(z)$, and fix $z$. Then the circle $C_z$ of radius $1$, centered at $z$, contains all $z - \alpha_j$. Let $\theta$ be an angle, based at the origin, whose interior contains the interior of $C_z$. As before we can conclude that $Wf(z) \neq 0$ if $0 < \pi/2$. But this is equivalent to $|z| > \sqrt{2}$, by an easy geometric argument.

**Remark.** Theorem 2.6 is also best possible, as the following example shows. For $n = 1, 2, \ldots$, let

$$f_n(z) := (z + 1)(z - (1 + i)/\sqrt{2})^n (z - (1 - i)/\sqrt{2})^n.$$

Then

$$Wf_n(z) = -(z^2 - \sqrt{2}z + 1)^{2n-2} \left[ (z^2 - \sqrt{2}z + 1)^2 + 2nz(z + 1)^2 (z - \sqrt{2}) \right].$$

and

$$\frac{1}{2n} (z^2 - \sqrt{2}z + 1)^{2-2n} Wf_n(z) \to z(z + 1)^2 (z - \sqrt{2})$$
as $n \to \infty$, uniformly on compact subsets of $\mathbb{C}$. Hence by a theorem of Hurwitz (see, e.g., [10, p. 4]) one zero of $W f_n(z)$ lies arbitrarily close to $z = \sqrt{2}$ if $n$ is sufficiently large. Note that the zeros of $f_n(z)$ have modulus 1.

Remark. Theorems 2.5 and 2.6 are in fact special cases of Theorem (8.1) in Marden's book [10].

The next theorem gives a quantitative version of Corollary 2.4; it shows that the zeros of $W f(z)$ cannot lie too close to the real axis if $f(z)$ has only real simple zeros.

**Lemma 2.7.** Let $f(z)$ have only real zeros. Then the circles that have the line segments connecting adjacent zeros as diameters do not contain zeros of $W f(z)$ in their interiors.

**Proof.** Let $\alpha_1 \leq \cdots \leq \alpha_n$ be the zeros of $f(z)$, counting multiplicities, and denote $\theta_j := \arg(z - \alpha_j), j = 1, \ldots, n$. We may restrict our attention to the case where $z$ lies in the upper half plane. Then

$$0 < \theta_1 \leq \cdots \leq \theta_j \leq \theta_{j+1} \leq \cdots \leq \theta_n < \pi.$$ 

If $\theta_j = \theta_{j+1}$ then $\alpha_j = \alpha_{j+1}$ and $W f(z)$ has a zero at $\alpha_j$, by the remark preceding Lemma 2.3; the lemma then holds trivially. Now assume that $\theta_j < \theta_{j+1}$, and let $z$ lie in the interior of the circle $C_j$ that has the line segment connecting $\alpha_j$ and $\alpha_{j+1}$ as diameter. Then

$$\theta_j + (\pi - \theta_{j+1}) < \frac{\pi}{2}, \quad \text{i.e.,} \quad \theta_{j+1} - \theta_j > \frac{\pi}{2}$$

since the triangle $\alpha_j, \alpha_{j+1}, z$ has an obtuse angle at $z$. Now let

$$\phi_i := \arg((z - \alpha_i)^{-2}) = -2\theta_i, \quad i = 1, 2, \ldots, n.$$ 

Then

$$-2\pi < \phi_n \leq \cdots \leq \phi_{j+1} < \phi_j \leq \cdots \leq \phi_1 < 0$$

and

$$\phi_j - \phi_{j+1} > \pi$$

which means that all the $(z - \alpha_i)^{-2}$ lie strictly to one side of a straight line passing through the origin. Hence, by Lemma 2.2, $W f(z) \neq 0$.

**Example.** If we apply Lemma 2.7 and Theorem 2.5 to the polynomials $f_{m,n}(z)$ of the example after Theorem 2.5, we see immediately that the zeros of $W f_{m,n}(z)$ must have modulus 1.
THEOREM 2.8. Let \( f(z) \) have \( n \) real zeros \( \alpha_1, -1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq 1 \), and let \( d := \min \{ \alpha_{j+1} - \alpha_j \mid j = 1, 2, \ldots, n - 1 \} \). Then \( Wf(z) \) has no zeros in the strip \( |\text{Im} \ z| < \sqrt{3d}/4 \).

Proof. Let \( z \) lie inside or on the circle \( \hat{C}_j \) around \( \alpha_j \) with radius \( d/2 \), but not inside the circles \( C_{j-1} \) or \( C_j \) defined in the previous proof. By definition of \( d \), \( C_{j-1} \) and \( C_j \) have radius at least \( d/2 \). By Lemma 2.2. \( Wf(z) \) is not zero if we can show that

\[
|z - \alpha_j|^{-2} \geq \sum_{i \neq j} |z - \alpha_i|^{-2}.
\]  

Since the distance between two consecutive \( \alpha_i \) is at least \( d \), we have

\[
\sum_{i \neq j} |z - \alpha_i|^2 \leq \sum_{k = 1}^{\infty} (|\bar{z} - kd|^2 + |\bar{z} + kd|^2),
\]

where \( \bar{z} := z - \alpha_j \). Note that \( |\bar{z}| \leq d/2 \). Without loss of generality we may assume \( \text{Re}(\bar{z}) \leq 0 \). Then

\[
|\bar{z} - kd|^2 \leq (kd)^2, \quad k = 1, 2, \ldots,
\]

and also, since \( z \in \hat{C}_j \) and \( z \notin C_j \), \( |\bar{z} + kd|^2 \geq (\sqrt{3d}/4)^2 + (\frac{\sqrt{3}}{4}d + (k - 1)d)^2 \). The right-hand side above is at least \( \sqrt{3kd}/2 \), with equality only if \( k = 1 \). Therefore

\[
|\bar{z} + kd|^2 \leq (\sqrt{3kd}/2)^2, \quad k = 1, 2, \ldots.
\]

Hence

\[
\sum_{i \neq j} |z - \alpha_i|^2 \leq \sum_{k = 1}^{\infty} (kd)^{-2} + \sum_{k = 1}^{\infty} (\sqrt{3kd}/2)^{-2} = \left( d^{-2} + \frac{4}{3} d^{-2} \right) \frac{\pi^2}{6} = \frac{7}{18} \pi^2 d^{-2} < 4d^{-2}.
\]

On the other hand,

\[
|z - \alpha_j|^2 \geq \left( \frac{d}{2} \right)^2
\]

since \( z \) lies inside or on \( \hat{C}_j \); this implies (2.1). Now the circles \( C_j \) and \( \hat{C}_j \) cover the strip \( |\text{Im} \ z| < \frac{1}{4} \sqrt{3d} \) inside the unit circle, so the result follows from Lemma 2.7 and Theorem 2.5.
3. A TRUNCATED PARTIAL FRACTION EXPANSION

Our aim is to get results on the zeros of $Wf(z)$ when those of $f(z)$ are evenly spaced. Lemma 2.2 suggests the study of the sections

$$g_n(z) := \sum_{k = -n}^{n} \frac{1}{(z - k)^2}, \quad n = 1, 2, \ldots$$

of the Mittag-Leffler (or partial fraction) expansion

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{k = -\infty}^{\infty} \frac{1}{(z - k)^2}$$

(see, e.g., [2, p. 187]). We use the same method as in [4] where the zero distribution of the sections of the Mittag-Leffler expansion of $\pi/\sin \pi z$ was examined.

**Theorem 3.1.** Let $k$ be a fixed integer. If $n \geq k$ is sufficiently large then $g_n(x + iy)$ has exactly one zero in each of the rectangles given by

$$\frac{1}{2\pi} (2 + \log n) < |y| < \frac{1}{2\pi} (3 + \log n),$$

and

$$k + \frac{1}{2} < x < k + \frac{1}{2} + \frac{(k + 1) \log n}{18n^2} \quad \text{if } k \geq 0,$$

$$k + \frac{1}{2} + \frac{k \log n}{18n^2} < x < k + \frac{1}{2} \quad \text{if } k \leq -1.$$

Next, fix any $B > 0$. The zeros $z_j = z_j(n)$ with $|\Re z_j| \leq B$ satisfy

$$\Im z_j - \frac{1}{2\pi} \log(2\pi^2 n) \to 0 \quad \text{as } n \to \infty.$$

The next result gives a uniform bound on the imaginary parts of the zeros of $g_n(z)$.

**Theorem 3.2.** The zeros of $g_n(x + iy)$ lie in the region defined by

$$|x| < n, \quad |y| < \frac{1}{2\pi} (4 + \log n).$$

We begin by proving several lemmas.
LEMMA 3.3. Let \( z = x + iy \). Then

(a) \( \Re g_n(z) = \Re(\pi^2/\sin^2 \pi z) - (2/n)(1 + \delta(n)) \),

(b) \( \Im g_n(z) = \Im(\pi^2/\sin^2 \pi z) - (4xy/n^3)(1 + \varepsilon(n)) \),

where \( \delta(n) \to 0 \) and \( \varepsilon(n) \to 0 \) as \( n \to \infty \).

Proof. By (3.2) we can rewrite (3.1) as

\[
g_n(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{k=n+1}^{\infty} \left( \frac{1}{(z-k)^2} + \frac{1}{(z+k)^2} \right). \tag{3.3}
\]

Since

\[
\frac{1}{(z-k)^2} = \frac{(x-k)^2 - y^2 - i2y(x-k)}{[(x-k)^2 + y^2]^2},
\]

we get

\[
\Re g_n(z) = \Re \left( \frac{\pi^2}{\sin^2 \pi z} - \sum_{k=n+1}^{\infty} \left\{ \frac{(k-x)^2 - y^2}{[(k-x)^2 + y^2]^2} + \frac{(k+x)^2 - y^2}{[(k+x)^2 + y^2]^2} \right\} \right), \tag{3.4}
\]

and

\[
\Im g_n(z) = \Im \left( \frac{\pi^2}{\sin^2 \pi z} - 2y \sum_{k=n+1}^{\infty} \left\{ \frac{k-x}{[(k-x)^2 + y^2]^2} - \frac{k+x}{[(k+x)^2 + y^2]^2} \right\} \right). \tag{3.5}
\]

For fixed \( x \) and \( y \), the functions

\[
\frac{(t-x)^2 - y^2}{[(t-x)^2 + y^2]^2}, \quad \frac{t-x}{[(t-x)^2 + y^2]^2}
\]

are decreasing when \( t \geq x + \sqrt{3}y \) and \( t \geq x + y/\sqrt{3} \), respectively. Now

\[
\int_{n}^{\infty} \frac{(t-x)^2 - y^2}{[(t-x)^2 + y^2]^2} \, dt = \frac{n-x}{(n-x)^2 + y^2} \tag{3.6}
\]

and

\[
\int_{n}^{\infty} \frac{t-x}{[(t-x)^2 + y^2]^2} \, dt = \frac{1/2}{(n-x)^2 + y^2}. \tag{3.7}
\]

If \( n \) is sufficiently large, (3.6) yields

\[
\frac{n-x}{(n-x)^2 + y^2} - \frac{(n-x)^2 - y^2}{[(n-x)^2 + y^2]^2} \leq \sum_{k=n+1}^{\infty} \frac{(k-x)^2 - y^2}{[(k-x)^2 + y^2]^2} \leq \frac{n-x}{(n-x)^2 + y^2}.
\]

Therefore the infinite sum in (3.4) is \((2/n)(1 + \delta(n))\), where \( \delta(n) \to 0 \) as \( n \to \infty \). This proves part (a).
To estimate the infinite sum in (3.5), we first observe that for fixed $x$ and $y$ the function
\[
\frac{t-x}{[(t-x)^2+y^2]^2} - \frac{t+x}{[(t+x)^2+y^2]^2}
\]
is decreasing if $t$ is sufficiently large. Hence we find with (3.7) that the infinite sum in (3.5) is close to
\[
\frac{1/2}{(n-x)^2+y^2} - \frac{1/2}{(n+x)^2+y^2} = \frac{2nx}{[(n-x)^2+y^2][(n+x)^2+y^2]}, \tag{3.8}
\]
with an error of at most
\[
\frac{n-x}{[(n-x)^2+y^2]^2} - \frac{n+x}{[(n+x)^2+y^2]^2} \sim \frac{6x}{n^4}
\]
as $n \to \infty$. Therefore, with (3.8), the infinite sum in (3.5) is $2n^{-3}(1 + \varepsilon(n))$, where $\varepsilon(n) \to 0$ as $n \to \infty$. This completes the proof.

**Lemma 3.4.** Let $k$ be a fixed integer and $k + \frac{1}{2} \leq x \leq k + \frac{3}{2}$. If $n$ is sufficiently large, then
\begin{enumerate}
\item[(a)] $\text{Re } g_n(z) < 0$ if $y > (1/2\pi)(\log n + 3)$;
\item[(b)] $\text{Re } g_n(z) > 0$ if $y = (1/2\pi)(\log n + 2)$.
\end{enumerate}

**Proof:** With the addition formula
\[
\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)
\]
we get
\[
\frac{1}{\sin^2 \pi z} = \frac{\sin^2 \pi x \cosh^2 \pi y - \cos^2 \pi x \sinh^2 \pi y}{(\sin^2 \pi x \cosh^2 \pi y + \cos^2 \pi x \sinh^2 \pi y)^2} - \frac{2i \sin \pi x \cos \pi y \sinh \pi y \cosh \pi y}{(\sin^2 \pi x \cosh^2 \pi y + \cos^2 \pi x \sinh^2 \pi y)^2}.
\]
Using the identities $\cosh^2(\pi y) = 1 + \sinh^2(\pi y)$, $\cos^2(\pi x) = 1 - \sin^2(\pi x)$, and $2 \sin(\pi x) \cos(\pi x) = \sin(2\pi x)$, we get
\[
\text{Re } \frac{\pi^2}{\sin^2 \pi z} = -\pi^2 \frac{\sin^2(\pi x) + \sinh^2(\pi y)(2 \sin^2(\pi x) - 1)}{(\sin^2(\pi x) + \sinh^2(\pi y))^2}, \tag{3.9}
\]
\[
\text{Im } \frac{\pi^2}{\sin^2 \pi z} = -\pi^2 \frac{\sin(2\pi x) \sinh(\pi y) \cosh(\pi y)}{(\sin^2(\pi x) + \sinh^2(\pi y))^2}. \tag{3.10}
\]
If $x$ is such that $k + \frac{1}{2} \leq x \leq k + \frac{3}{2}$, then $\frac{3}{4} \leq \sin^2(\pi x) \leq 1$ and therefore by (3.9)

$$\pi^2 \frac{\frac{3}{4} + \frac{1}{2} \sinh^2(\pi y)}{(1 + \sinh^2(\pi y))^2} \leq \text{Re} \frac{\pi^2}{\sin^2 \pi z} \leq \pi^2 \frac{1 + \sinh^2(\pi y)}{(\frac{3}{4} + \sinh^2(\pi y))^2}.$$ 

We note that the numerator of the left-hand term is bigger than $(\cosh^2(\pi y))/2$, and find

$$\text{Re} \frac{\pi^2}{\sin^2 \pi z} > 2\pi^2 \frac{1}{1 + 2e^{-2\pi y} + e^{-4\pi y}} e^{-2\pi y} \quad (3.11)$$

and

$$\text{Re} \frac{\pi^2}{\sin^2 \pi z} < 4\pi^2 \frac{1 + 2e^{-2\pi y} + e^{-4\pi y}}{(1 + e^{-2\pi y} + e^{-4\pi y})^2} e^{-2\pi y} \quad (3.12)$$

$$< 4\pi^2 e^{-2\pi y}.$$ 

To establish part (a), it now suffices to show, by (3.12) and Lemma 3.3(a), that

$$\frac{2}{n} (1 - |\delta(n)|) \geq 4\pi^2 e^{-2\pi y},$$

which is equivalent to

$$y \geq \frac{1}{2\pi} \log \frac{2\pi^2 n}{1 - |\delta(n)|},$$

and this holds for

$$y \geq \frac{1}{2\pi} (3 + \log n)$$

whenever $n$ is sufficiently large.

For part (b) we have to show, by (3.11) and Lemma 3.3(a), that

$$\frac{2}{n} (1 + |\delta(n)|) \leq \frac{2\pi^2}{1 + 2e^{-2\pi y} + e^{-4\pi y}} e^{-2\pi y}.$$ 

With $y = (2 + \log n)/2\pi$, this is equivalent to

$$1 + |\delta(n)| \leq \left(1 + \frac{2}{e^2 n} + \frac{1}{e^4 n^2}\right)^{-1} \pi^2 e^{-2\pi y},$$

which is true if $n$ is sufficiently large. This proves Lemma 3.4.
Lemma 3.5. Let \((2 + \log n)/2\pi \leq y \leq (3 + \log n)/2\pi\). If \(n\) is sufficiently large then for fixed \(k = 0, 1, \ldots\),

(a) \(\text{Im } g_n(z) < 0\) if \(x = k + \frac{1}{2}\);
(b) \(\text{Im } g_n(z) > 0\) if \(x = k + 1/2 + (k + 1) \log n/18n^2\).

Proof: (a) follows directly from Lemma 3.3(b) if we note that \(\text{Im}(\pi^2/\sin^2 \pi z) = 0\) for \(x = k + \frac{1}{2}\), by (3.10).

To prove (b), we first note that \(\sin 2\pi(k + \frac{1}{2} + \theta) = -\sin(2\pi\theta)\) and that \(\sin(2\pi\theta) \geq 6\theta\) if \(\theta \leq \frac{1}{12}\). Hence if \(n\) is sufficiently large then

\[-\sin(2\pi x) \geq \frac{(k + 1) \log n}{3n^2},\]  

(3.13)

where \(x = k + 1/2 + (k + 1) \log n/18n^2\). Furthermore, since \(e^2n \leq e^{2\pi y} \leq e^{3n}\),

\[
\frac{\sinh(\pi y) \cosh(\pi y)}{(\sin^2 \pi x + \sinh^2 \pi y)^2} \geq \frac{\sinh(\pi y) \cosh(\pi y)}{(1 + \sinh^2(\pi y))^2} = \frac{\sinh(\pi y)}{\cosh^3(\pi y)} = 4 \frac{1 - e^{-2\pi y}}{1 + e^{-2\pi y}} e^{-2\pi y}
\]

\[\geq (1 - \gamma(n)) \frac{4}{n} e^{-3},\]  

(3.14)

where \(\gamma(n) \to 0\) as \(n \to \infty\). On the other hand,

\[
\frac{4\pi y}{n^3} \leq \frac{2}{\pi n^3} (3 + \log n)(k + 1) = \frac{2}{\pi} \frac{(k + 1) \log n}{n^3} (1 + \gamma'(n)),
\]

where \(\gamma'(n) \to 0\) as \(n \to \infty\). Now, with Lemma 3.3(b), (3.10), (3.13), and (3.14) we see that \(\text{Im } g_n(z) > 0\) when

\[
\frac{4\pi^2}{3e^3} \geq \frac{2}{\pi} (1 + \tilde{\gamma}(n)),
\]

where \(\tilde{\gamma}(n) \to 0\) as \(n \to \infty\); this holds if \(n\) is sufficiently large.

Proof of Theorem 3.1. Since the numerator polynomial of \(g_n(z)\) is an even function, the zeros are symmetric to the real and imaginary axes. Hence it suffices to consider the case \(k \geq 0\) and \(y > 0\). Define the points \(A, B, C,\) and \(D\) by

\[
A := k + \frac{1}{2} + \frac{(k + 1) \log n}{18n^2} + \frac{i}{2\pi} (2 + \log n);
\]
Now we apply Lemmas 3.4 and 3.5, and find that

\[ \text{Im } g_n(z) > 0 \text{ on } AB, \quad \text{Re } g_n(z) < 0 \text{ on } BC, \]
\[ \text{Im } g_n(z) < 0 \text{ on } CD, \quad \text{Re } g_n(z) > 0 \text{ on } DA. \]

This means that the argument of \( g_n(z) \) increases by \( 2\pi \) as \( z \) traverses the boundary of the rectangle with corners \( A, B, C, \) and \( D. \) Hence by the argument principle, there is exactly one zero in the interior of the rectangle.

We get the last statement if in the proof of Lemma 3.4 we restrict our attention to the interval \( k + 1/2 \leq x \leq k + 1/2 + (k + 1) \log n/18n^2, \) instead of \( k + 1 \leq x \leq k + 3/2. \) Then the imaginary parts of \( A \) and \( B \) will get arbitrarily close to \( \log(2\pi n), \) from below and above, respectively. We omit the details. This completes the proof.

**Proof of Theorem 3.2.** By Theorem 2.5 we may restrict our attention to \( |z| \leq n. \) We shall use (3.4) to show that \( \text{Re } g_n(z) < 0 \) if \( y > y_n := (\log 50n)/2\pi. \) Because of the symmetry of the zeros we may further restrict our attention to \( x \geq 0 \) and \( y > 0. \)

Denote the infinite sum in (3.4) by \( S. \) It is easy to see that for fixed \( x \) and \( y, \) the function

\[ f(t) := \frac{(t-x)^2 - y^2}{[(t-x)^2 + y^2]^2} \]

is increasing for \( -x \leq t \leq x + y \sqrt{3} \) and decreasing for \( t \geq x + y \sqrt{3}. \) Also, \( g(t) := f(-t) \) is increasing for \( -x \leq t \leq -x + y \sqrt{3} \) and decreasing for \( t \geq -x + y \sqrt{3}. \) Hence we can make the integral estimate

\[ S = \sum_{k=n+1}^{\infty} f(k) + \sum_{k=n+1}^{\infty} g(k) \geq \int_{n}^{a} f(t) \, dt + \int_{a}^{b} f(t) \, dt + \int_{n}^{b} g(t) \, dt + \int_{b+1}^{\infty} g(t) \, dt, \]

where \( a := [x + y \sqrt{3}] \) and \( b := [-x + y \sqrt{3}]. \) Now

\[ \int_{a}^{a+1} f(t) \, dt \leq f(x + y \sqrt{3}) = \frac{1}{8y^2}, \]
and
\[ \int_b^{b+1} g(t) \, dt \leq g(-x + y \sqrt{3}) = \frac{1}{8y^2}. \] (3.15)

Hence
\[ S > \int_n^x \left[ f(t) + g(t) \right] \, dt - \frac{1}{4y^2} \]
\[ = \frac{n - x}{(n - x)^2 + y^2} + \frac{n + x}{(n + x)^2 + y^2} - \frac{1}{4y^2}. \]

The sum of the first two terms in the last line is
\[ \frac{2n}{(n^2 - x^2 + y^2) + 4x^2y^2/(n^2 - x^2 + y^2)} \geq \frac{n}{2xy}. \]

This last inequality is obtained by minimizing the left-hand side with respect to \( z := n^2 - x^2 + y^2 \); the minimum occurs at \( z = 2xy \). Hence
\[ S > \frac{n}{2xy} - \frac{1}{4y^2}. \] (3.16)

Since we may restrict our attention to \(|z| \leq n\), we have
\[ \frac{n}{2xy} \geq \frac{n}{2y \sqrt{n^2 - y^2}} \geq \frac{1}{n}, \] (3.17)

where the second inequality is true since the middle expression is minimal for \( y = n/\sqrt{2} \). Now let \( y \geq \sqrt{5n}/2 \). Then with (3.16) and (3.17) we get
\[ S \geq \frac{4}{5n}. \] (3.18)

If \( y \leq \frac{1}{2} \sqrt{5n} \) then \( -x + y \sqrt{3} \leq \sqrt{15n}/2 \leq n - 1 \) for \( n \geq 6 \). Hence \( b + 1 \leq n \), and therefore the term (3.15) can be omitted in the integral estimate for \( S \). Hence we get, with \( x \leq n \),
\[ S \geq \frac{n}{2xy} - \frac{1}{8y^2} \geq \frac{1}{2y} - \frac{1}{8y^2}. \]

Let \( h(y) \) denote the right-hand term of this inequality. It is increasing for \( 0 < y \leq \frac{1}{2} \) and decreasing for \( y \geq \frac{1}{2} \). By Theorem 2.8 (appropriately scaled)
we may restrict our attention to \( y > \sqrt{3}/4 \). Noting that \( h(\sqrt{3}/4) \approx 0.488 \) and using the restriction \( y \leq \sqrt{5n}/2 \), we find that

\[
S \geq \frac{1}{\sqrt{5n}} - \frac{1}{10n} \geq \frac{4}{5n},
\]

for \( n \geq 5 \). This shows that (3.18) holds for all \( y \) with \( \sqrt{3}/4 \leq y \leq n \), provided \( n \geq 6 \).

On the other hand, by (3.9) and \( y \geq y_n \) we see that for all \( x \),

\[
\Re \frac{\pi^2}{\sin^2 \pi z} \leq \pi^2 \frac{1 + \sinh^2 \pi y}{\sinh^4 \pi y} - 4\pi^2 \frac{(1 + e^{-2\pi i})^2}{(1 - e^{-2\pi i})^4} e^{-2\pi i} \\
\leq 4\pi^2 \frac{(1 + 1/50n)^2}{(1 - 1/50n)^4} e^{-2\pi i}.
\]

Now for \( n \geq 10 \) we get

\[
\Re \frac{\pi^2}{\sin^2 \pi z} < 40e^{-2\pi y}.
\]

Finally, we see with (3.4), (3.18), and (3.19) that \( \Re g_n(z) < 0 \) when

\[
40e^{-2\pi i} \leq \frac{4}{5n},
\]

that is,

\[
y \geq \frac{1}{2\pi} \log 50n.
\]

This proves the theorem for \( n \geq 10 \) since \( \log 50 < 4 \). For \( n < 9 \), the theorem was verified by computing the zeros of \( g_n(z) \), \( n = 1, 2, ..., 9 \).

Remark. The proof of Theorem 3.2 can be adapted to show that the zeros of \( g_n(x + iy) \) lie in the region bounded by the lines \( x = \pm n \) and

\[
y = \pm \frac{1}{2\pi} \left[ c + \log(\max\{\sqrt{n^2 - x^2}, d\}) \right],
\]

where \( c \) and \( d \) are easily computable constants.

4. POLYNOMIALS WITH EVENLY SPACED ZEROS

In this section we return to our original normalization where all the zeros of the given polynomial lie in the interval \([-1, 1]\). The following is an immediate consequence of Theorem 3.2.
**Corollary 4.1.** Let \( f \) be the polynomial with \( 2n + 1 \) zeros \( k/n \), where \( -n \leq k \leq n \). Then the zeros of \( Wf(z) \) lie in the strip

\[
|y| < \frac{1}{2\pi n} (4 + \log n), \quad \text{for} \quad n \geq 1.
\]

Corollary 4.1 is illustrated by Fig. 1, for \( n = 10 \). To indicate the scale, we note that the four "corner" zeros are approximately \( \pm 0.9582 \pm 0.05901i \), while the four zeros closest to the imaginary axis are approximately \( \pm 0.05012 \pm 0.08482i \).

We show that \( y \) retains this order of magnitude if the zeros of \( f \) are perturbed within certain small bounds, and that \( y \) remains small even for larger perturbations. This is shown from corollaries of the following result.

**Theorem 4.2.** Let \( -n \leq \alpha_{-n} \leq \alpha_{-n+1} \leq \cdots \leq \alpha_{n-1} \leq \alpha_n \leq n \), and \( d := \max_{-n \leq k \leq n} |\alpha_k - k|, \quad n \geq 11 \). Then

\[
G_n(z) := \sum_{k=-n}^{n} \frac{1}{(z - \alpha_k)^2} \neq 0
\]

\((z = x + iy)\) provided that

\[
\frac{2 + d/|y|}{(1 - d/|y|)^2} \left( 1 + \frac{1}{|y|} \right) \frac{d}{y^2} + 20e^{-\gamma |y|} \leq 2 \frac{2}{5n}. \tag{4.1}
\]

**Proof:** In order to adapt the proof of Theorem 3.2 to the present situation, we estimate the size of

\[
R_n(z) := G_n(z) - g_n(z). \tag{4.2}
\]

We note that

\[
a_k(z) := \frac{1}{(z - \alpha_k)^2} - \frac{1}{(z - k)^2} = \frac{(\alpha_k - k)}{(z - \alpha_k)^2} \frac{2z - k - \alpha_k}{(z - k)^2}.
\]

We may assume \( y > 0 \) since the zeros are symmetric about the real axis. Since

\[
|\alpha_k - k| \leq d, \quad |z - k| = |x - k + iy| \geq y.
\]

FIG. 1. Zeros of \( Wf(z) \) for \( n = 10 \).
we have

\[ |2z - k - \alpha_k| \leq 2 |z - k| + |\alpha_k - k| \leq |z - k| \left(2 + \frac{d}{y}\right), \]

\[ |z - \alpha_k| = |(z - k) - (\alpha_k - k)| \geq |z - k| - d \geq |z - k| \left(1 - \frac{d}{y}\right). \]

Hence

\[ |a_k(z)| \leq \frac{dD}{|z - k|^3}, \quad D := \frac{2 + d/y}{(1 - d/y)^2}. \]

We now have

\[ |R_n(z)| \leq \sum_{k = -n}^{n} |a_k(z)| \leq dD \sum_{k = -\infty}^{\infty} \frac{1}{((x - k)^2 + y^2)^{3/2}} \]

\[ = dD \left\{ \sum_{k = 0}^{\infty} \frac{1}{((k + x - \lfloor x \rfloor)^2 + y^2)^{3/2}} + \sum_{k = 0}^{\infty} \frac{1}{((k + \lfloor x \rfloor + 1 - x)^2 + y^2)^{3/2}} \right\} \]

\[ \leq 2dD \sum_{k = 0}^{\infty} \frac{1}{(k^2 + y^2)^{3/2}} \]

\[ \leq 2dD \left(\frac{1}{y^3} + \int_{0}^{\infty} \frac{dt}{(t^2 + y^2)^{3/2}} \right). \]

By substituting \( t = \frac{1}{y} \tan \theta \) we find that the integral is \( 1/y^2 \). Hence

\[ |R_n(z)| \leq 2dD \left(1 + \frac{1}{y^2}\right) \frac{1}{y^2}. \quad (4.3) \]

By (4.2),

\[ \text{Re } G_n(z) = \text{Re } g_n(z) + \text{Re } R_n(z) \leq \text{Re } g_n(z) + |R_n(z)|. \]

With (3.4), (3.15), and (3.16) we have

\[ \text{Re } g_n(z) < 40e^{-2\pi y} - \frac{4}{5n} \]

and therefore, with (4.3),

\[ \text{Re } G_n(z) < -\frac{4}{5n} + 2dD \left(1 + \frac{1}{y^2}\right) \frac{1}{y^3} + 40e^{-2\pi y}. \]

Hence (4.1) implies \( \text{Re } G_n(z) < 0 \), and the proof is complete.
We now apply Theorem 4.2 to three different cases, namely the cases where the discrepancy of the zeros from equidistribution between $-1$ and $1$ is (i) small with respect to $1/n$, (ii) of the same order as $1/n$, and (iii) large with respect to $1/n$.

**Corollary 4.3.** Let $f(z)$ be a polynomial of degree $2n + 1$ with zeros $\alpha_k := k/n + \varepsilon_k$, $|\varepsilon_k| \leq (\log n)^2/396n^2$, where $k = -n, -n + 1, \ldots, n$. Then the zeros of $Wf(z)$ lie inside the strip

$$|y| < \frac{1}{2\pi n} \left(6 + \log n\right) \quad (n \geq 11).$$

**Proof.** Here and in the proofs of the following corollaries we scale everything up by the factor $n$, to make Theorem 4.2 applicable. We can also assume $y > 0$. If we suppose that $y \geq (6 + \log n)/2\pi$, then

$$20e^{-2\pi y} \leq 20e^{-6} \frac{1}{n} < \frac{1}{20n}. \quad (4.4)$$

Hence it remains to show that

$$\frac{2 + d/y}{(1 - d/y)^2} \left(1 + \frac{1}{y}\right) \frac{d}{y^2} \leq \frac{7}{20n}. \quad (4.5)$$

Now, for $n \geq 11$,

$$1 + \frac{1}{y} \leq 1 + \frac{2\pi}{6 + \log n} < \frac{7}{4}. \quad (4.6)$$

By the restriction placed on the $\varepsilon_k$, we have $d = (\log n)^2/396n$. Hence

$$\frac{d}{y} \leq \frac{2\pi}{396} \frac{\log^2 n}{n(6 + \log n)} \leq \frac{\pi}{198} \frac{\log^2 11}{11(6 + \log 11)} < \frac{1}{1012},$$

for $n \geq 11$. So

$$\frac{2 + d/y}{(1 - d/y)^2} < \frac{401}{200},$$

and therefore, with (4.6), (4.5) holds when

$$d \leq \frac{40}{401} \frac{y^2}{n}. \quad (4.7)$$
But
\[ y^2 \geq \frac{\log^2 n}{4\pi^2} \geq \frac{401 \log^2 n}{40 \cdot 396} = \frac{401}{40} nd. \]

This completes the proof.

Remark. By keeping track of various constants in the proofs of Theorems 3.2 and 4.2 and of Corollary 4.3, one can see that the constant 396 could be replaced by \(8\pi^2 + \varepsilon\) if \(n\) is sufficiently large (depending on the size of \(\varepsilon > 0\)).

**Corollary 4.4.** Let \(d \leq 1\) and let \(f(z)\) be a polynomial of degree \(2n + 1\) with zeros \(z_k := (k + \varepsilon_k)/n, |\varepsilon_k| \leq d, k = -n, -n + 1, \ldots, n\). Then the zeros of \(Wf(z)\) lie inside the strip
\[ |y| < \max \left\{ \frac{5 \sqrt{d/2}}{\sqrt{n}}, \frac{1}{2\pi n} (6 + \log n) \right\} \quad (n \geq 11). \]

**Proof.** Since (4.4) holds again, it suffices to verify (4.5). After scaling, as before, we consider \(y \geq 5 \sqrt{dn/2}\). Since \(d \leq 1\), we have for \(n \geq 11\) that
\[ \frac{d}{y} \leq \frac{\sqrt{d}}{5 \sqrt{n/2}} \leq \frac{1}{5 \sqrt{11/2}}, \]
and therefore
\[ \frac{2 + d/y}{(1 - d/y)^2} \leq \frac{5}{2}. \]

Since (4.6) still holds, (4.5) is true when
\[ \frac{35}{8} \frac{d}{y^2} \leq \frac{7}{20n}; \]
this is equivalent to \(y \geq 5 \sqrt{dn/2}\), and the proof is complete.

Remark. Corollary 4.4 can easily be extended to cases \(d > 1\); the result remains essentially the same.

The next corollary is a generalization of the previous one; Corollary 4.4 follows (in a slightly weaker form) if we set \(\beta = 0\).

**Corollary 4.5.** Let \(c \leq 1, \beta < 1\), and let \(f(z)\) be a polynomial of degree
2n + 1 with zeros \( \alpha_k = (k + \epsilon_k)/n \), \( |\epsilon_k| \leq cn^\beta \), \( k = -n, -n + 1, \ldots, n \). Then the zeros of \( W_f(z) \) lie in the strip

\[
|y| < \max \left\{ \sqrt{19cn^{(\beta - 1)/2}}, \frac{1}{2\pi n} (6 + \log n) \right\} \quad (n \geq 11).
\]

**Proof.** As before, we scale by the factor \( n \) and assume that \( z \) does not lie in the strip. Again, it suffices to verify (4.5) outside the strip. Since

\[
y \geq \sqrt{19cn^{(\beta + 1)/2}}, \quad d = cn^\beta,
\]

we have

\[
\frac{d}{y} \leq \sqrt{\frac{c}{19}} n^{(\beta - 1)/2} \leq \frac{1}{\sqrt{19}};
\]

therefore

\[
\frac{2 + d/y}{(1 - d/y)^2} < \frac{19}{5}.
\]

Hence (4.5) holds when

\[
\frac{19}{5} \frac{7}{4} \frac{cn^\beta}{y^2} \leq \frac{7}{20n},
\]

which is equivalent to \( y \geq \sqrt{19cn^{(\beta + 1)/2}} \). The proof is complete.

We note that Corollary 4.5 holds for negative \( \beta \). However, if \( \beta \leq -1 \), we only get a weaker form of Corollary 4.3.

5. **Nontrivial Zeros of High Multiplicities**

Lemma 2.2 exhibits an expression for \( W_f(z) \) in terms of the zeros of \( f(z) \). The next lemma gives the coefficients of \( W_f(z) \) in terms of the zeros of \( f(z) \).

**Lemma 5.1.** If \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \), then \( W_f(z) = b_{2n-2} z^{2n-2} + \cdots + b_1 z + b_0 \), where

\[
b_k = \sum_{j=0}^\left\lceil (k+1)/2 \right\rceil \left\{ (k + 2 - 2j)^2 - (k + 2) \right\} a_j a_{k-j+2} - A_k,
\]

\[
A_k := \begin{cases} 
\left( \frac{k}{2} + 1 \right) a_{k/2+1}^2 & \text{if } k \text{ is even}, \\
0 & \text{if } k \text{ is odd},
\end{cases}
\]
and \(a_j := 0\) for \(j > n\). In particular, \(b_0 = 2a_0a_2 - a_1^2\), \(b_1 = 6a_0a_3 - 2a_1a_2\), \(b_2 = 12a_0a_4 - 2a_2^2\), \(b_{2n-2} = -na_n^2\).

**Proof.** With

\[
 f'(x) = \sum_{i=0}^{n-1} (j+1) a_{j+1} x^i, \quad f''(x) = \sum_{i=0}^{n-2} (j+1)(j+2) a_{j+2} x^i,
\]

and (1.2) we get

\[
 Wf(x) = \sum_{k=0}^{2n-2} x^k \left( \sum_{i=0}^{k} (j+1)(j+2) a_{j+2} a_{k-i} - \sum_{j=0}^{k} (j+1)(k-j+1) a_{j+1} a_{k-j+1} \right).
\]

For \(k\) odd, split the first of the two inner sums into “head” and “tail” sums from 0 to \(\lceil (k+1)/2 \rceil - 2\) and from \(\lceil k/2 \rceil\) to \(k\), and the second into such sums from 0 to \(\lceil (k+1)/2 \rceil - 1\) and from \(\lceil k/2 \rceil + 1\) to \(k\). Reverse the order of summation in the tail sums, adjust the indices of summation, and then combine all sums to obtain the \(b_k\) formula. For \(k\) even isolate the \(j = k/2 - 1\) term from the first inner sum, and the \(j = k/2\) term from the second. These provide the \(-A_k\) term; the rest of the formula is deduced as in the case of \(k\) odd.

Lemma 5.1 gives the \(2n-1\) coefficients of \(Wf(z)\) as functions of the \(n+1\) coefficients of \(f\). One consequence of this is that not every polynomial of degree \(2n-2\) with negative leading coefficient is the Wronskian of a polynomial of degree \(n\). This is only the case for \(n = 2\) (\(n = 1\) is trivial), while for \(n = 3\), for example, we get the necessary condition \(3b_2^2 = 8b_2b_4\) for \(b_3x^3 + b_3x^3 + b_2x^2 + b_1x + b_0\) to be the Wronskian of a polynomial of degree 3.

Furthermore, Lemma 5.1 can be used to construct polynomials \(f(z)\) such that \(Wf(z)\) satisfy certain conditions. As an example we take the condition that \(Wf(z)\) have a nontrivial zero of highest possible multiplicity at the origin.

We begin by setting \(b_0 = b_1 = \cdots = b_{n-1} = 0\), \(b_n \neq 0\). We try to find a polynomial \(f(z) = a_0 + a_1z + \cdots\) whose Wronskian has these prescribed coefficients \(b_0, \ldots, b_n\), with \(b_{n+1}, \ldots\) arbitrary. We may assume that \(a_0 \neq 0\) since otherwise \(0 = b_0 = 2a_0a_2 - a_1^2\) (Lemma 5.1) implies \(a_1 = 0\), and 0 would be a trivial zero of \(Wf(z)\).

First, let \(a_1 \neq 0\). By using the recursion implicit in Lemma 5.1, we find

\[
 a_2 = \frac{1}{2} \frac{a_1^2}{a_0^1}, \quad a_3 = \frac{1}{6} \frac{a_1^3}{a_0^1}.
\]
and in general,

\[ a_j = \frac{1}{j!} \frac{a'_j}{a_0^{j-1}}, \quad j = 2, 3, \ldots, n + 1. \]

This is best seen by using the expression

\[ b_k = \sum_{j=0}^{k} (j+1)[(j+2) a_{j+2} a_{k-j} - (k-j+1) a_{k-j+1} a_{j+1}] \]

that follows from the proof of Lemma 5.1.

Now assume that \( a_i = 0 \). Then we find successively that \( a_2 = a_3 = \cdots = a_{n+1} = 0 \) if \( b_j = 0, j = 0, 1, \ldots, n - 1 \). Thus we have proved

**THEOREM 5.2.** The polynomials \( f(z) \) of smallest degree such that \( Wf(z) \) has a nontrivial zero at \( z = 0 \) of multiplicity \( n \) are

(a) \( f(z) = a_0 S_n + (a_1 z / a_0) \), where \( a_0 \) and \( a_1 \) are arbitrary nonzero real numbers, and \( S_n(x) = \sum_{j=0}^{n} x^j / j! \);

(b) \( f(z) = a_0 + a_n z^{n+2} + a_0 \), where \( a_0 \) and \( a_n + 2 \) are arbitrary nonzero real numbers.

**Remark.** One can easily extend the definition of the operator \( W \) to have it act on arbitrary entire functions, or even on wider classes of functions. In connection with Theorem 5.2(a) it is interesting to note that \( W \exp(z) \equiv 0 \). We also have

\[ W \sin(z) \equiv W \cos(z) \equiv W \sinh(z) \equiv W \cosh(z) \equiv -1. \]

Correspondingly, an application of Lemma 5.1, as above, would give us the analogue to Theorem 5.2 concerning the zeros of \( Wf(z) + 1 \).

6. **CONCLUDING REMARKS**

One way of looking at the topic of this paper is to take an inequality of the type \( P(x) > 0 \) for \( x \in I \), where \( P \) is a polynomial and \( I \) is some interval on the real axis. We then ask how far the zeros of \( P \) are away from the real axis (or from \( I \)). We answered this question for the Laguerre inequality (see, e.g., [11; 8, p. 171]), with polynomials having real zeros only that are quite evenly spaced in the interval \([-1, 1]\). Other classes of polynomials, such as certain classical orthogonal polynomials, will be considered elsewhere [5]. Finally, it is of interest to study the Wronskian of polynomials with nonreal zeros; this was done in [6].
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