A Dynamic Inversion of
the Classical Variational Problems

SEIICHI IWAMOTO

Department of Economic Engineering,
Faculty of Economics, Kyushu University 27, Fukuoka 812, Japan

Submitted by E. Stanley Lee

1. INTRODUCTION

In general, the Euler equation for a variational problem determines a
family of stationary curves with two parameters. This result associates the
following problem. For a given family of two-parameter curves, find a
variational problem whose family of stationary curves is the family. This is
what we call the inverse problem of the classical variational problem.
Darboux solved it affirmatively [2, 3]. The inversion appears to be static in
its derivation.

On the other hand, Iwamoto [4] has recently inverted the optimal control
(minimal) process into an equivalent (maximal) process through a dynamic
approach. He establishes an inverse theory of Bellman’s dynamic
programming [1]. Iwamoto’s and Bellman’s ideas are common in the sense
that the treatments are dynamic. Both the classical theory of calculus of
variations and the modern control theory based upon Pontryagin’s maximum
principle are static in approach. However, the functional equation approach
based upon Bellman’s principle of optimality is dynamic in itself [1].
Moreover, the classical variational problem may be regarded as an optimal
control problem.

In this paper we study a dynamic inversion of the classical variational
problems. A variational problem may be parametrized and embedded in a
large family of problems. Thus the dynamic inversion of optimal control
process [4] is applicable to the variational problems. The main results are (i)
dynamic derivation of inverse variational problems, and (2) explicit
representation of the inverse problems of the shortest path problem, the
brachistochrone, the minimal surface of revolution, and quadratic problems.

Section 2 motivates the problem through a pair of the simplest variational
problems—the shortest path problem and its inverse problem. The problems
may be parametrized, embedded in a large family of problems, and analyzed
through dynamic programming (Section 3). We derive the inverse problems
of minimization of $\int_0^T f(t, x, \dot{x}) \, dt$ and $\int_0^T f(t, \dot{x}) \, dt$ in Sections 4 and 5, respectively. Section 6 shows the reflexivity of inverse operation. Illustrating two simple self-invertible problems, we propose a general problem on self-invertibility in Section 7. Three classical variational problems—brachistochrone, minimal surface of revolution, and quadratic problems—are inverted in the last section.

2. THE PROBLEM

In this section we specify a pair of the simplest problems. This illustration suggests an interesting theory underlying a number of pairs of general problems.

Given $T > 0$, let us consider the following pair of minimization and maximization problems

$$(M_0) \quad \text{Min} \int_0^T \sqrt{1 + x^2} \, dt \quad \text{s.t.} \quad \begin{array}{l}
(i) \quad x(0) = c \quad (\geq 0) \\
(ii) \quad x(T) = 0,
\end{array}$$

$$(I_0) \quad \text{Max} \int_0^T \sqrt{y^2 - 1} \, dt \quad \text{s.t.} \quad \begin{array}{l}
(i) \quad y(0) = c \quad (\geq T) \\
(ii) \quad y(T) = 0,
\end{array}$$

where the parameter $c$ ranges over the respective semi-infinite intervals, $x, y : [0, T] \rightarrow \mathbb{R}^1$ are appropriate differentiable functions such that each problem is well-defined. For instance $|\dot{y}(t)| \geq 1$ or more strictly speaking $\dot{y}(t) \leq -1$ on $[0, T]$ is assumed in the problem $(I_0)$. Throughout the paper $\dot{z} = \dot{z}(t)$ means the derivative of $z = z(t)$ with respect to $t$ and $\mathbb{R}^n$ the $n$-dimensional Euclidean space.

The calculus of variations regards $c$ as a fixed value. It shows that the Euler equation yields the family of straight lines with two parameters, respectively. Together with the specified boundary conditions, we obtain the extremal (or in fact optimal) functions

$$x^*(t) = -\frac{c}{T} t + c, \quad \dot{y}(t) = -\frac{c}{T} t + c, \quad (1)$$

and the extremal (or in fact optimal) values of $(M_0)$ and $(I_0)$

$$U(c) = \sqrt{c^2 + T^2} \quad (c \geq 0), \quad V(c) = \sqrt{c^2 - T^2} \quad (c \geq T), \quad (2)$$

respectively.
Note that the maximum value function $U(\cdot)$ is the inverse function of the minimum value function $V(\cdot)$ and that both optimal paths $x^*(\cdot)$ and $y^*(\cdot)$ coincide. Here are a few interesting questions. Why is the problem $(I_0)$ introduced? What is the relation between $(M_0)$ and $(I_0)$? What does $y(t)$ represent? Is the above coincidence a miracle? These questions are mutually connected. They are simultaneously solved in Section 5.

### 3. Dynamic Programming Approach

For a while in this section we solves $(M_0)$ and $(I_0)$ through dynamic programming approach. The following procedure suggests an answer to the questions stated above. Dynamic programming approach regards not only $c$ but also $t$ as dynamic parameters. The original problems $(M_0)$ and $(I_0)$ are embedded in the following large families of optimal control problems, respectively,

$$(M_0(t, x)) \quad \text{Min} \int_t^T \sqrt{1 + u^2} \, ds \quad \text{s.t. (i)} \quad x(t) = x$$

$$(I_0(t, y)) \quad \text{Max} \int_t^T \sqrt{v^2 - 1} \, ds \quad \text{s.t. (i)} \quad y(t) = y$$

where $0 \leq t \leq T, x \geq 0$ and $y \geq T - t$. We have the original problems as special cases

$$(M_0(0, c)) = (M_0), \quad (I_0(0, c)) = (I_0),$$

respectively. The minimum value function $F(t, x)$ of $(M_0(t, x))$ and the maximum value function $G(t, y)$ of $(I_0(t, y))$ satisfy the backward Bellman equations

$$-F_t = \min_{-\infty < u < -\infty} \left[ \sqrt{1 + u^2} + uF_x \right], \quad 0 \leq t \leq T, \quad x \geq 0,$$

$$F(T, x) = 0, \quad x \geq 0,$$

and

$$-G_t = \max_{v < -1} \left[ \sqrt{v^2 - 1} + vG_y \right], \quad 0 \leq t \leq T, \quad y \geq T - t,$$

$$G(T, y) = 0, \quad y \geq 0,$$
respectively ([1, 4]). Making minimization over $u$ in (3), we get

$$-F_t = \sqrt{1 - F_x^2}, \quad u^* = - \frac{F_x}{\sqrt{1 - F_x^2}}.$$ 

Thus we obtain the optimal solution (in a sense of dynamic programming) of $(M_0(t, x))$

$$F(t, x) = \sqrt{x^2 + (T - t)^2},$$

$$u^*(t, x) = - \frac{x}{T - t},$$

$$x^*(s) = \frac{x}{T - t} (T - s), \quad t \leq s \leq T,$$

where $0 \leq t \leq T$ and $x \geq 0$. If in particular $t = 0$, $x = c$ and $s = t$, then the resulting $x^*(\cdot)$ coincides with $x^*(\cdot)$ of (1).

Similarly we obtain for (4)

$$G_t = \sqrt{G_y^2 - 1}, \quad \hat{v} = - \frac{G_y}{\sqrt{G_y^2 - 1}}.$$ 

Therefore $(I_0(t, y))$ has the optimal solution

$$G(t, y) = \sqrt{y^2 - (T - t)^2},$$

$$\hat{v}(t, y) = - \frac{y}{T - t},$$

$$\hat{y}(s) = \frac{y}{T - t} (T - s), \quad t \leq s \leq T,$$

where $0 \leq t \leq T$ and $y \geq T - t$. If $t = 0$, $y = c$, and $s = t$, then the corresponding $\hat{y}(\cdot)$ reduces to $\hat{y}(\cdot)$ of (2). From inverse theorem [4], it holds that

$$G(t, y) = F^{-1}(t, y), \quad F(t, x) = G^{-1}(t, x),$$

and

$$\hat{v}(t, y) = u^*(t, F^{-1}(t, y)), \quad u^*(t, x) = \hat{v}(t, G^{-1}(t, x)),$$

where $H^{-1}(t, \cdot)$ is the inverse function of $H(t, \cdot)$. 

4. Dynamic Inversion of $\min \int_0^T f(t, x, \dot{x}) \, dt$

Let $f(t, x, \dot{x})$ be a suitable continuous real-valued function of $(t, x, \dot{x})$, where $0 \leq t \leq T$, $x \in I(t)$, and $\dot{x} \in U(t, x)$. Throughout the paper we assume that $I(t)$ and $U(t, x)$ are appropriate nonempty intervals of $\mathbb{R}^1$.

With a given nonparametric problem

$$(M_1) \quad \min \int_0^T f(t, x, \dot{x}) \, dt \quad \text{s.t.} \quad \begin{align*}
(i) & \quad x(0) = c \\
(ii) & \quad x(T) = 0
\end{align*}$$

we associate the family of parametric problems

$$(M_1(t, x)) \quad \min \int_t^T f(s, x, \dot{x}) \, ds \quad \text{s.t.} \quad \begin{align*}
(i) & \quad x(t) = x \\
(ii) & \quad x(T) = 0
\end{align*}$$

or more definitely in optimal control form

$$(M_1(t, x)) \quad \min \int_t^T f(s, x, u) \, ds \quad \text{s.t.} \quad \begin{align*}
(i) & \quad x(t) = x \\
(ii) & \quad \dot{x}(s) = u(s) \\
(iii) & \quad u(s) \in U(s, x(s)), \quad t \leq s \leq T \\
(iv) & \quad x(T) = 0,
\end{align*}$$

where $0 \leq t \leq T$, $x \in I(t)$. Then the minimum value function $F = F(t, x)$ satisfies the backward Bellman equation ([1, 4])

$$(\text{BBE}) \quad -F_t = \min_{u \in U(t, x)} \left[ f(t, x, u) + uF_x \right], \quad 0 \leq t < T, \quad x \in I(t),$$

$$F(T, x) = 0, \quad x \in I(T).$$

(The reader should also refer the forward Bellman equation (FBE) in [1, 4].)

For the sake of simplicity we set

**Assumption (A).** $I(t) \subset [0, \infty)$ and $U(t, x) \subset (-\infty, 0]$ for $0 \leq t \leq T$, $x \in I(t)$ and $f(t, x, \dot{x})$ is strictly increasing in $x$ and strictly decreasing in $\dot{x}$.

Let $f^{-1}(t, x, \cdot)$ be the inverse function of $f(t, x, \cdot)$. Then we have
**Lemma 1.** Under Assumption (A), $f^{-1}(t, x, -\dot{y})$ is strictly increasing in $x$.

**Proof.** Let $x_1 < x_2$. Then

$$f^{-1}(t, x_1, -\dot{y}) = z_1, \quad f^{-1}(t, x_2, -\dot{y}) = z_2$$

implies

$$f(t, x_1, z_1) = f(t, x_2, z_2) = -\dot{y}. \quad (6)$$

If $z_1 = z_2$, then (6) contradicts the strict increasingness of $f(t, \cdot, z_1)$. If $z_1 > z_2$, then (6) couples with the strict decreasingness of $f(t, x_2, \cdot)$ to give

$$f(t, x_1, z_1) > f(t, x_2, z_1).$$

This contradicts the strict increasingness of $f(t, \cdot, z_1)$. Therefore we have $z_1 < z_2$. This completes the proof.

**Lemma 2.** Under Assumption (A), $F(t, x)$ is strictly increasing in $x$.

**Proof.** Let $0 < x_1 < x_2$ and

$$F(t, x_2) = \int_t^T f(s, x^*, \dot{x}^*) \, ds$$

for $x^*(t) = x_2$, $x^*(T) = 0$. Then the function $x = x(\cdot)$ defined by

$$x(s) = \frac{x_1}{x_2} x^*(s), \quad s \in [t, T]$$

satisfies

$$x(t) = x_1, \quad x(T) = 0$$

and

$$x^*(x) > x(s), \quad \dot{x}^*(s) < \dot{x}(s).$$

The last two inequalities together with the assumed monotonicity of $f(t, \cdot, \cdot)$ imply

$$f(s, x^*(s), \dot{x}^*(s)) > f(s, x(s), \dot{x}(s)) \quad \text{on} \quad [t, T].$$
Therefore we have

\[
F(t, x_t) = \int_t^T f(s, x^*, \dot{x}^*) \, ds \\
> \int_t^T f(s, x, \dot{x}) \, ds \\
\geq F(t, x_t).
\]

This completes the proof.

Let us now show how the main problem \((M_1)\) under Assumption \((A)\) is inverted into an equivalent problem. First for any \(x = x(t), 0 \leq t \leq T\) with \(x(T) = 0\), let us define backward \(y = y(\cdot)\) by

\[
y(t) = \int_t^T f(s, x(s), \dot{x}(s)) \, ds, \quad 0 \leq t \leq T.
\]

The backward integral transformation \(x(\cdot) \rightarrow y(\cdot)\) is a keystone throughout this paper (The reader should also consider the forward integral transformation.) Then differentiating (7), we have

\[
y(t) = -f(s, x(s), \dot{x}(s)), \quad y(T) = 0.
\]

This together with the invertibility of \(f(t, x, \cdot)\) implies

\[
\dot{x}(s) = f^{-1}(s, x(s), -y(s)).
\]

On the other hand, from the minimality of \(F\), we have

\[
y(s) \geq F(s, x(s)).
\]

This together with the strict increasingness of \(F(t, \cdot)\) [by Lemma 2] implies

\[
x(s) \leq F^{-1}(s, y(s)).
\]

The strict increasingness of \(f^{-1}\) [by Lemma 1] couples with (9) and (11) to give

\[
\dot{x}(s) \leq f^{-1}(s, F^{-1}(s, y(s)), y(s)).
\]

Integrating both sides on \([t, T]\) and substituting \(x(T) = 0\), we obtain

\[
-\int_t^T f^{-1}(s, F^{-1}(s, y(s)), -y(s)) \, ds \leq x(t).
\]
This together with (11) yields
\[ -\int_t^T f^{-1}(s, F^{-1}(s, y(s)), -\dot{y}(s)) \, ds \leq F^{-1}(t, y(t)). \] (14)

Thus we see that (14) holds for any feasible \( y = y(\cdot) \) defined by (7).

Second, let \( x_c^\# = x_c^*(t), \ 0 \leq t \leq T, \) with \( x_c^*(T) = 0 \) be the optimal path starting from \( x_c^*(0) = c, \) where \( c \in \mathbf{I}(0) \). Let us define \( \hat{y}_c = \hat{y}_c(t), \ 0 \leq t \leq T, \) by (7) from \( x_c^\# = x_c^*(\cdot) \). Then from Bellman's principle of optimality [1] we have
\[ F(t, x_c^\#(t)) = \hat{y}_c(t), \quad 0 \leq t \leq T. \] (15)

Of course for each \( c \in \mathbf{I}(0) \) the pair \( (x_c^*(\cdot), \hat{y}_c(\cdot)) \) satisfies (8). Furthermore, we should remark that all the "equalities" in the above reasoning hold for the paired process \( (x_c^*(\cdot), \hat{y}_c(\cdot)) \).

Therefore we have obtained the maximum problem
\[ (I_1(t, y)) \quad \text{Max} \int_t^T g(s, y, \dot{y}) \, ds \quad \text{s.t.} \quad \begin{align*}
\text{(i)} & \quad y(t) = y \\
\text{(ii)} & \quad y(T) = 0,
\end{align*} \] where
\[ g(t, y, \dot{y}) = -f^{-1}(t, F^{-1}(t, y), -\dot{y}). \] (16)

The deduction stated above simultaneously yields the maximum value of \((I_1(t, y))\)
\[ F^{-1}(t, y) \] (17)
and the optimal path
\[ \hat{y}_d = \hat{y}_d(s), \quad t \leq s \leq T, \] (18)
where \( d \) is determined by the property
\[ x_d^*(t) = F^{-1}(t, y). \] (19)

Finally we have for the original problem \((M_1)\) the desired inverse problem
\[ (I_1) \quad \text{Max} \int_0^T g(t, y, \dot{y}) \, dt \quad \text{s.t.} \quad \begin{align*}
\text{(i)} & \quad y(0) = c \\
\text{(ii)} & \quad y(T) = 0.
\end{align*} \]
and simultaneously its optimal solution (value and point)

\[ F^{-1}(0, c), \quad \hat{y}_d = \hat{y}_d(t), \quad 0 \leq t \leq T, \quad (20) \]

where

\[ d = F^{-1}(0, c). \]

Now we shall consider another situation. Let us in turn put

**Assumption (A').** \( I(t) \subset (-\infty, 0] \) and \( U(t, x) \subset [0, \infty) \) for \( 0 \leq t \leq T, x \in I(t) \) and \( f(t, x, \dot{x}) \) is strictly decreasing in \( x \) and strictly increasing in \( \dot{x} \).

Then we have the following preliminary results.

**Lemma 1'.** Under Assumption (A'), \( f^{-1}(t, x, -\dot{y}) \) is strictly decreasing in \( x \).

**Lemma 2'.** Under Assumption (A'), \( F(t, x) \) is strictly decreasing in \( x \).

These results lead us the minimum inverse problem

\[
(I'_1(t, y)) \quad \text{Min} \int_{y}^{T} g(s, y, \dot{y}) \, ds \quad \text{s.t.} \quad (i) \quad y(t) = y \\
(ii) \quad y(T) = 0.
\]

\[
(I'_2) \quad \text{Min} \int_{0}^{T} g(t, y, \dot{y}) \, dt \quad \text{s.t.} \quad (i) \quad y(0) = c \\
(ii) \quad y(T) = 0.
\]

Similarly, the optimal solution of \( (I'_1) \) is determined from that of \( (M_1) \).

Concluding this section, we emphasize that Assumption (A) negates the optimizer. However, under Assumption (A') the optimizer remains as it is.

5. **Dynamic Inversion of Min \( \int_{0}^{T} f(t, \dot{x}) \, dt \)**

In this section let us consider two-variable function \( f = f(t, \dot{x}) \) independent of \( x \), where \( 0 \leq t \leq T, x \in I(t) \) and \( \dot{x} \in U(t, x) \).

First we put

**Assumption (B).** \( I(t) \subset [0, \infty) \) and \( U(t, x) \subset (-\infty, 0] \) for \( 0 \leq t \leq T, x \in I(t) \) and \( f(t, \dot{x}) \) is strictly decreasing in \( x \).
Then we have

**Lemma 3.** Under Assumption (B), $F(t, x)$ is strictly increasing in $x$.

**Proof.** Easier than that of Lemma 2.

For a given main problem

\[(M_2) \quad \text{Min} \int_0^T f(t, \dot{x}) \, dt \quad \text{s.t.} \ (i) \quad x(0) = c \]
\[\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (ii) \quad x(T) = 0,\]

the similar procedure as in Section 4 enables us to introduce its inverse problem

\[(I_2) \quad \text{Max} \int_0^T g(t, \dot{y}) \, dt \quad \text{s.t.} \ (i) \quad y(0) = c \]
\[\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (ii) \quad y(T) = 0,\]

where

\[g(t, \dot{y}) = -f^{-1}(t, -\dot{y}) \quad (22)\]

and $f^{-1}(t, \cdot)$ is the inverse function of $f(t, \cdot)$. We have the maximum value

\[F^{-1}(0, c) \quad (23)\]

and the optimal path

\[\hat{y}_d = \hat{y}_d(t), \quad 0 \leq t \leq T, \quad (24)\]

where

\[d = F^{-1}(0, c). \quad (25)\]

It turns out that the preceding simple pair $(M_0), (I_0)$ is a special case of $(M_2), (I_2)$ with

\[f(t, \dot{x}) = \sqrt{1 + \dot{x}^2}, \quad 0 \leq t \leq T, \quad \dot{x} \geq 0, \quad \dot{x} \leq 0, \]
\[g(t, \dot{y}) = \sqrt{\dot{y}^2 - 1}, \quad 0 \leq t \leq T, \quad \dot{y} \geq T - t, \quad \dot{y} \leq -1, \]

and

\[y(t) = \int_t^T \sqrt{1 + \dot{x}^2} \, ds. \]
In this case \( y(t) \) represents the length of the curve \( x = x(\cdot) \) between \((t, x(t))\) and \((T, 0)\). Furthermore we have

\[
x^*_c(t) = -\frac{c}{T} t + c, \quad \hat{y}^*_c(t) = (T - t) \sqrt{1 + \left(\frac{c}{T}\right)^2},
\]

\[
F(t, x) = \sqrt{x^2 + (T - t)^2}, \quad F^{-1}(t, y) = \sqrt{y^2 - (T - t)^2}.
\]

Thus the main problem \((M_0)\) has the minimum value

\[
U(c) = F(0, c) = \sqrt{c^2 + T^2}
\]

and the optimal path

\[
x^*_c(t) = -\frac{c}{T} t + c, \quad 0 \leq t \leq T.
\]

Therefore, the inverse problem \((I_0)\) has the maximum value

\[
V(c) = F^{-1}(0, c) = \sqrt{c^2 - T^2}
\]

and the optimal path \(\hat{y}^*_d = \hat{y}^*_d(t)\) with \(d = F^{-1}(0, c)\), namely,

\[
\hat{y}^*_d(t) = (T - t) \sqrt{1 + \left(\frac{d}{T}\right)^2} = (T - t) \sqrt{1 + \left(\frac{\sqrt{c^2 - T^2}}{T}\right)^2} = -\frac{c}{T} t + c.
\]

We should note that

\[
x^*_c = x^*, \quad \hat{y}^*_d = \hat{y},
\]

where \(x^*, \hat{y}\) are specified in (1), (2), respectively.

On the other hand, let us assume that

**Assumption (B')**. \(I(t) \subset (-\infty, 0]\) and \(U(t, x) \subset [0, \infty)\) for \(0 \leq t \leq T\), \(x \in I(t)\), and \(f(t, x)\) is strictly increasing in \(\dot{x}\).

Then we have

**Lemma 3'.** Under Assumption (B'), \(F(t, x)\) is strictly decreasing in \(x\).
Thus Assumption (B') yields the minimum inverse problem

\[(I_2') \quad \text{Min} \int_0^T g(t, \dot{y}) \, dt \quad \text{s.t.} \quad (i) \quad y(0) = c \]
\[(ii) \quad y(T) = 0.\]

For example we have the following pair

\[(M'_0) \quad \text{Min} \int_0^T \sqrt{1 + \dot{x}^2} \, dt \quad \text{s.t.} \quad (i) \quad x(0) = c \quad (\leq 0) \]
\[(ii) \quad x(T) = 0,\]

\[(I'_0) \quad \text{Min} - \int_0^T \sqrt{\dot{y}^2 - 1} \, dt \quad \text{s.t.} \quad (i) \quad y(0) = c \quad (\geq 1) \]
\[(ii) \quad y(T) = 0.\]

The optimal solutions are straightforward from those of \((M_0), (I_0), \) respectively.

Finally we remark that each of Assumptions (A) and (B) negates the optimizer but that each of (A') and (B') does keep it.

6. Reflexibility

Let us reconsider the pair of main problem \((M_1)\) and its inverse problem \((I_1)\) under Assumption (A). Let \((M_1)\) have the strictly increasing minimum value function \(F = F(t, x)\) and the optimal path \(x^*_1 = x^*_1(t).\) Then \((I_1)\) has the strictly increasing maximum value function \(F^{-1} = F^{-1}(t, y)\) and the optimal path \(\dot{y}_d = \dot{y}_d(t),\) where \(d = F^{-1}(0, c).\) Then we have the inverse problem of \((I_1)\)

\[(I'_1) \quad \text{Min} \int_0^T h(t, z, \dot{z}) \, dt \quad \text{s.t.} \quad (i) \quad z(0) = c \]
\[(ii) \quad z(T) = 0,\]

where

\[h(t, z, \dot{z}) = -g^{-1}(t, G^{-1}(t, z), -\dot{z}).\]

Since

\[G(t, y) = F^{-1}(t, y),\]
\[g(t, y, \dot{y}) = -f^{-1}(t, F^{-1}(t, y), -\dot{y}),\]
it holds that
\[ h(t, z, \dot{z}) = f(t, z, \dot{z}). \]
Thus the reinverse problem \((I_1^2)\) becomes the original problem \((M_1)\).
Therefore, the inverse operation satisfies the reflexivity.

7. SELF-INVERTIBILITY

First let us consider the maximization (main) problem

\[
(M_3) \quad \text{Max} \int_0^T \sqrt{1 - x^2} \, dt \quad \text{s.t.} \quad (i) \quad x(0) = c \quad (0 < c < T) \\
(ii) \quad x(T) = 0
\]

and its \((t, x)\)-subproblem

\[
(M_3(t, x)) \quad \text{Max} \int_t^T \sqrt{1 - x^2} \, ds \quad \text{s.t.} \quad (i) \quad x(t) = x \quad (0 < x < T - t) \\
(ii) \quad x(T) = 0.
\]

The calculus of variations in Section 2 and dynamic programming method in Section 3 yield the maximum value function

\[ F(t, x) = \sqrt{(T - t)^2 - x^2}, \quad 0 < x < T - t. \]

We note that \(F\) is strictly decreasing in \(x\) and self-invertible

\[ F^{-1}(t, z) = F(t, z), \quad 0 < z < T - t. \]

Let us transform \(x(\cdot)\) into \(y(\cdot)\) as usual

\[ y(t) = \int_t^T \sqrt{1 - \dot{x}^2} \, ds. \]

Then the same reasoning as in the latter half-part of Section 4 leads in turn the inverse problem

\[
(I_3(t, y)) \quad \text{Max} \int_t^T \sqrt{1 - y^2} \, ds \quad \text{s.t.} \quad (i) \quad y(t) = y \quad (0 < y < T - t) \\
(ii) \quad y(T) = 0,
\]

\[
(I_3) \quad \text{Max} \int_0^T \sqrt{1 - y^2} \, dt \quad \text{s.t.} \quad (i) \quad y(0) = c \quad (0 < c < T) \\
(ii) \quad y(T) = 0.
\]
We should remark that both main and inverse problems not only have the same optimizer "Max" but also exactly coincide in expression

\[(M_3) = (I_3).\]

Therefore the problem \((M_3)\) is called self-invertible.

Similarly, the minimization (main) problem

\[
(M_4) \quad \text{Min} \int_0^T \sqrt{1 - x^2} \, dt \quad \text{s.t.} \quad (i) \quad x(0) = c \quad (0 \leq c \leq T) \\
(ii) \quad x(T) = 0
\]
together with its self-invertible strictly decreasing minimum value function

\[G(t, x) = T - t - x, \quad 0 \leq x \leq T - t,\]
leads the inverse problem

\[
(I_4) \quad \text{Min} \int_0^T \sqrt{1 - y^2} \, dt \quad \text{s.t.} \quad (i) \quad y(0) = c \quad (0 \leq c \leq T) \\
(ii) \quad y(T) = 0,
\]
where minimum is taken over functions which are differentiable except for one point in \([0, T]\). Thus we have obtained another self-invertible problem \((M_4)\).

Let us consider the general problem. What is a sufficient condition for the self-invertibility? The answer is as follows. We reconsider \((M_1)\) and \((M_2)\) under Assumptions \((A')\) and \((B')\), or under the following Assumptions \((A'')\) and \((B'')\), respectively.

**Assumption \((A'')\).** \(I(t) \subseteq [0, \infty)\) and \(U(t, x) \subseteq (-\infty, 0]\) for \(0 \leq t \leq T, x \in I(t), \) and \(f(t, x, \dot{x})\) is strictly decreasing in \(x\) and strictly increasing in \(\dot{x}\).

**Assumption \((B'')\).** \(I(t) \subseteq [0, \infty)\) and \(U(t, x) \subseteq (-\infty, 0]\) for \(0 \leq t \leq T, x \in I(t), \) and \(f(t, \dot{x})\) is strictly decreasing in \(\dot{x}\).

Then we have

**Lemma 2''.** Under Assumption \((A'')\), \(f^{-1}(t, x, -y)\) is strictly decreasing in \(x\).

**Lemma 3''.** Under Assumption \((B'')\), \(F(t, x)\) is strictly decreasing in \(x\).
These results lead the following pair of main and inverse problems under Assumption (A') (resp. (A''))

\[(M_1) \quad \text{Min} \int_0^T f(t, x, \dot{x}) \, dt \quad \text{s.t.} \quad (i) \quad x(0) = c \]
\[(iI) \quad \text{Min} \int_0^T [-f^{-1}(t, F^{-1}(t, y), -\dot{y})] \, dt \quad \text{s.t.} \quad (i) \quad y(0) = c \]

and the pair under Assumption (B') (resp. (B''))

\[(M_2) \quad \text{Min} \int_0^T f(t, \dot{x}) \, dt \quad \text{s.t.} \quad (i) \quad x(0) = c \]
\[(iI) \quad \text{Min} \int_0^T [-f^{-1}(t, -\dot{y})] \, dt \quad \text{s.t.} \quad (i) \quad y(0) = c \]

Then the equality

\[f(t, z, \dot{z}) = -f^{-1}(t, F^{-1}(t, z), -\dot{z}) \quad (26)\]

implies the self-invertibility of (M_1) under Assumption (A') or (A''). The equality

\[f(t, \dot{z}) = -f^{-1}(t, -\dot{z}) \quad (27)\]

implies the self-invertibility of (M_2) under Assumption (B') or (B''). Thus we have introduced new functional equations (26) for \(f(t, x, \dot{x})\) and (27) for \(f(t, x)\). The pair \((M_3), (I_3)\) is a special case of \((M_2), (I_2')\). That is,

\[f(t, x) = \sqrt{1 - \dot{x}^2}, \quad 0 \leq \dot{x} \leq 1,\]

is a solution of the functional equation (27).

The above discussion and result remain valid provided that \(\text{"Min"}\) is replaced by \(\text{"Max"}\) and that the minimum value function \(f(t, x)\) is replaced by the maximum value function \(G(t, x)\).
In this section we shall essentially apply the inversion idea to brachistochrone, minimal surface of revolution, and quadratic problems. The application is sometimes regional and sometimes global.

8.1. Brachistochrone $f(t, \dot{x}) = \sqrt{(1 + \dot{x}^2)/t}$

Since minimization of $\int_{x_0}^{x_1} \sqrt{((1 + z^2(x))/z(x))} \, dx$ reduces that of $\int_{x_0}^{x_1} \sqrt{((1 + (dx/dz)^2)/z)} \, dz$, we rather consider (main) brachistochrone problem

\[
(MB) \quad \text{Min} \int_{0}^{T} \sqrt{\frac{1 + \dot{x}^2}{t}} \, dt \quad \text{s.t. (i)} \quad x(0) = 0
\]

\[
(ii) \quad x(T) = c,
\]

where $c$ is sufficiently large relative to $T$ (Fig. 1). The restriction to a class of pairs $(c, T)$ with large $c$ relative to $T$ will enable us to invert the problem. Then the forward subproblem

\[
(MB(t, x)) \quad \text{Min} \int_{0}^{t} \sqrt{\frac{1 + \dot{x}^2}{s}} \, ds \quad \text{s.t. (i)} \quad x(0) = 0
\]

\[
(ii) \quad x(t) = x
\]

is considered for sufficiently large $x$ relative to $t$. Then the Euler equation together with this restriction yields the cycloid with $\dot{x}(t) > 0$ for all $t$ considered. Therefore the forward minimum value function $F(t, x)$ is strictly
increasing in \( x \) (recall the proof of Lemma 2'). This property together with the forward integral transformation \( x(\cdot) \mapsto y(\cdot) \) defined by

\[
y(t) = \int_0^t \sqrt{\frac{1 + x^2}{s}} \, ds, \quad 0 \leq t \leq T,
\]

yields the inverse brachistochrone problem

\[
\text{(IB}(t, y)) \quad \text{Max} \int_0^t \sqrt{sy^2 - 1} \, ds \quad \text{s.t.} \quad (i) \quad y(0) = 0
\]

\[
(\text{ii}) \quad y(t) - y,
\]

\[
(\text{IB}) \quad \text{Max} \int_0^T \sqrt{ty^2 - 1} \, dt \quad \text{s.t.} \quad (i) \quad y(0) = 0
\]

\[
(\text{ii}) \quad y(T) = c,
\]

where \( y \) (resp. \( c \)) is also sufficiently large relative to \( t \) (resp. \( T \)). We remark that from (28) the feasible path \( y(\cdot) \) satisfies

\[
y(t) \geq 1/\sqrt{t}
\]

and therefore

\[
y(t) \geq 2 \sqrt{t}.
\]

8.2. Minimal Surface of Revolution \( f = x \sqrt{1 + x^2} \) or \( t \sqrt{1 + t^2} \)

First we consider the main problem

\[
(\text{MC}_1) \quad \text{Min} \int_0^T x \sqrt{1 + x^2} \, dt \quad \text{s.t.} \quad (i) \quad x(0) = c \quad (\gg 1)
\]

\[
(\text{ii}) \quad x(T) = 1,
\]

\[
(\text{MC}_1(t, x)) \quad \text{Min} \int_0^T x \sqrt{1 + x^2} \, ds \quad \text{s.t.} \quad (i) \quad x(t) = x \quad (\gg t \vee 1)
\]

\[
(\text{ii}) \quad x(T) = 1,
\]

where \( a \vee b \) is the larger of two and \( a \gg b \) means that \( a \) is sufficiently large relative to \( b \). It is well-known that the extremals are catenaries. The physical interpretation evaluates the (backward) minimum value function \( F(t, x) \) as follows (Fig. 2): \( T - t < F(t, x) < ((x + 1)/2) \sqrt{(T - t)^2 + (x - 1)^2} \). The restriction yields the minimal solution \( \dot{x}(t) < 0 \) for all \( t \) considered. From Lemma 2, \( F(t, x) \) is strictly increasing in \( x \). Thus the backward integral transformation yields the inverse problem
where

\[ \frac{\gamma^2}{F^{-2}(s, y)} - 1 \, ds + 1 \]

\( \text{s.t.} \quad (i) \quad y(t) = y(T-t) \)

\( \text{and} \quad (ii) \quad y(T) = 0, \)

\[ \text{(IC)} \quad \text{Max} \int_0^T \frac{\gamma^2}{F^{-2}(t, y)} - 1 \, dt + 1 \quad \text{s.t.} \quad (i) \quad y(0) = c(T) \quad (\geq T) \]

\( \text{and} \quad (ii) \quad y(T) = 0, \)

where

\[ F^{-2}(t, y) = (F^{-1}(t, y))^2. \]

Second, we consider the (main) problem on \([1, T]\) as follows.

\[ \text{(MC)} \quad \text{Min} \int_1^T \sqrt{1 + x^2} \, dt \quad \text{s.t.} \quad (i) \quad x(1) = c(\geq 0) \]

\( \text{and} \quad (ii) \quad x(T) = 0, \)

\[ \text{(MC2)} \quad \text{Min} \int_t^T \sqrt{1 + x^2} \, ds \quad \text{s.t.} \quad (i) \quad x(t) = x(\geq 0) \]

\( \text{and} \quad (ii) \quad x(T) = 0, \)

where \(1 \leq t \leq T, \) and \(c\) (resp. \(x\)) is sufficiently small relative to \(T\) (resp. \(t\)) (Fig. 3). The physical interpretation gives us the following rough estimate of the (backward) minimum value function \(F(t, x)\)

\[ tx < F(t, x) < \frac{T + t}{2} \sqrt{(T-t)^2 + x^2}. \]
The restriction on \([1, T]\) and the relative smallness of \(x\) with respect to \(t\) assure the strict increasingness of \(F(t, x)\) in \(x\) (see Lemma 3). Therefore the usual backward integral transformation yields the inverse problem

\[
(\text{IC}_2(t, y)) \quad \text{Max} \int_t^T \sqrt{\frac{y^2}{s^2} - 1} \, ds \quad \text{s.t.} \quad \begin{align*}
(i) & \quad y(t) = y \quad (\geq 0) \\
(ii) & \quad y(T) = 0,
\end{align*}
\]

\[
(\text{IC}_2) \quad \text{Max} \int_1^T \sqrt{\frac{y^2}{t^2} - 1} \, dt \quad \text{s.t.} \quad \begin{align*}
(i) & \quad y(1) = c \quad (\geq 0) \\
(ii) & \quad y(T) = 0.
\end{align*}
\]

Of course, the feasible \(y(.)\) satisfies

\[
\dot{y}(t) \leq -t
\]

and therefore

\[
y(t) \geq \frac{1}{2}(T^2 - t^2).
\]

8.3. Quadratic Problems \(f = x^2 + \dot{x}^2\) or \((t + 1)^2 \dot{x}^2\)

Finally we illustrate two typical quadratic problems with their inverse ones and enumerate the complete analytic optimal solutions. The first problem is the linear equation quadratic criteria control process ([4])

\[
(M_1) \quad \text{Min} \int_0^T (x^2 + \dot{x}^2) \, dt \quad \text{s.t.} \quad \begin{align*}
(i) & \quad x(0) = c \quad (\geq 0) \\
(ii) & \quad x(T) = 0,
\end{align*}
\]
\begin{align*}
\text{(Is)} \quad \max \int_0^T \sqrt{-\frac{y}{k(t)} - \dot{y}} \, dt \quad \text{s.t.} \quad & (i) \quad y(0) = c \quad (\geq 0) \\
& (ii) \quad y(T) = 0,
\end{align*}

where

\[ k(t) = \coth(T - t). \]

We have the following optimal solutions.

\begin{align*}
F(t, x) &= k(t) x^2, \\
x_*(t) &= c \frac{\sinh(T - t)}{\sinh T}, \\
U(c) &= F(0, c) = (\coth T) c^2, \\
G(t, y) &= \sqrt{\frac{y}{k(t)}}, \\
\dot{y}(t, y) &= -\sqrt{k(t)} y, \\
\dot{y}_*(t) &= \frac{\cosh(T - t) \sinh(T - t)}{\sinh^2 T} c, \\
V(c) &= G(0, c) = \sqrt{\frac{c}{\coth T}}.
\end{align*}

The optimal path of (Is) is

\[ \dot{y}_d(t) = \frac{\cosh(T - t) \sinh(T - t)}{\cosh T \sinh T} c \]

since

\[ d = \sqrt{\frac{c}{\coth T}}. \]

The second is the time-variant problem on \([0, 1]\)

\begin{align*}
\text{(M6)} \quad \min \int_0^1 (t + 1)^2 \dot{x}^2 \, dt \quad \text{s.t.} \quad & (i) \quad x(0) = c \quad (\geq 0) \\
& (ii) \quad x(1) = 0, \\
\text{(Is)} \quad \max \int_0^1 \sqrt{\frac{-\dot{y}}{t + 1}} \, dt \quad \text{s.t.} \quad & (i) \quad y(0) = c \quad (\geq 0) \\
& (ii) \quad y(T) = 0.
\end{align*}

The optimal solutions are

\begin{align*}
F(t, x) &= \frac{x^2}{1/(t + 1) - 1/2}, \\
u^*(t, x) &= -\frac{1}{(t + 1)^2} \frac{x}{1/(t + 1) - 1/2}, \\
x_*(t) &= 2 \left( \frac{1}{t + 1} - \frac{1}{2} \right) c, \\
U(c) &= F(0, c) = 2c^2.
\end{align*}
\[ G(t, y) = \sqrt{y} \left( \frac{1}{t+1} - \frac{1}{2} \right), \quad \dot{v}(t, y) = -\frac{\sqrt{y}}{(t+1)^2 \sqrt{1/(t+1) - 1/2}}, \]
\[ \dot{y}_c(t) = 4 \left( \frac{1}{t+1} - \frac{1}{2} \right) c^2, \quad V(c) = G(0, c) = \sqrt{\frac{c}{2}}. \]

The optimal path of \((I_0)\) is
\[ \dot{y}_d(t) = \frac{2c}{t+1} - c \quad \left( d = \sqrt{\frac{c}{2}} \right). \]

Here is also a miracle
\[ \dot{y}_d(\cdot) = x^*_c(\cdot). \]

Recall the first coincidence stated at the end of Section 2 (see (1)). It holds that
\[ F(t, x^*_{\mu - 1}(0, c))(t) = x^*_c(t) \]

or
\[ \dot{y}_d(0, c)(t) = x^*_c(t) \]

for both the first pair \((M_0), (I_0)\) and the last \((M_6), (I_6)\). However, in general, this does not hold.

REFERENCES