Projections of $f$-Vectors of Four-Polytopes

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1. Introduction

A $d$-polytope is the nonempty convex hull of a finite number of points in $R^d$. For each $d$-polytope $P$, let $f_i(P)$ denote the number of $i$-dimensional faces (called $i$-faces) of $P$. The $f$-vector of $P$ is $(f_0(P), f_1(P), \ldots, f_{d-1}(P)) \in R^d$. In particular we will call $f_0(P)$, $f_1(P)$ and $f_{d-1}(P)$ the number of vertices, edges, and facets, respectively. While many partial results are known, it is still an open problem to characterize the complete set of $f$-vectors for all $d$-dimensional polytopes if $d \geq 4$. In view of Euler's formula, an equivalent problem is the characterization of the orthogonal projections of all $d$-dimensional $f$-vectors onto any $(d-1)$-dimensional subspace which does not contain a normal to the Euler hyperplane. In particular, if all triples involving some three of $f_0$, $f_1$, $f_2$, $f_3$ were known (a triple for each four-polytope), then Euler's formula $f_0 - f_1 + f_2 - f_3 = 0$ would allow us to determine all $f$-vectors of four-polytopes.

A more tractable problem in $R^4$ appears to be the determination of all projections of $f$-vectors of four-polytopes onto two of the four coordinates $f_0$, $f_1$, $f_2$, $f_3$. Specifically, for $i < j$ and $i, j \in \{0, 1, 2, 3\}$, let $\pi(i, j) = \{(f_i(P), f_j(P)) \in R^2 \mid P$ is a four-polytope$\}$. Grünbaum [1] determines $\pi(0, 1)$ and $\pi(0, 3)$ (and, hence, by duality, $\pi(2, 3)$) and conjectures the characterization of $\text{conv } \pi(1, 2)$. The purpose of this paper is to settle the one remaining case not considered by Grünbaum, namely, $\pi(0, 2)$. That is, determine which pairs $(v, t)$ represent the number of vertices and two-faces of some four-dimensional polytope. In general, our terminology and notation are those of Grünbaum [1].

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2. Preliminary Results

To simplify notation for the special case $R^4$, let $v, e, t, f$ denote, respectively, the number of vertices, edges, two-faces, and facets of a four-polytope $P$. In this section we first establish asymptotic bounds on the values $(v, t)$ in $\pi(0, 2)$. We then characterize all pairs $(v, t)$ which may be realized as pyramids, bipyramids, or cylinders (i.e., prisms) in $R^4$. The complete characterization of $\pi(0, 2)$ appears in the next section.

**Theorem 1.** If $(v, t) \in \pi(0, 2)$ then 
\[
(2v + 3 + (8v + 9)^{1/2})/2 \leq t \leq v^2 - 3v.
\]

**Proof.** Clearly $4f \leq 2t$ for all four-polytopes, with equality holding for simplicial four-polytopes. This, together with Euler's formula, implies $t \leq 2e - 2v$ and $f \leq e - v$ for all polytopes. It is also clear that there are at most $\left(\frac{v^2}{2} - v\right)$ edges in any polytope, so 
\[
t \leq 2e - 2v \leq v^2 - 3v,
\]
with equality holding for the neighborly 4-polytopes.

Also, $e \leq \left(\frac{v}{2}\right)$ implies $v \geq (1 + (1 + 8e)^{1/2})/2$, so
\[
f \leq e - v \leq (2e - 1 - (1 + 8e)^{1/2})/2.
\]
Passing to the dual we get $v \leq (2t - 1 - (1 + 8t)^{1/2})/2$, or equivalently, 
\[
(2v + 3 + (8v + 9)^{1/2})/2 \leq t,
\]
with equality holding for the duals of the neighborly four-polytopes.

**Theorem 2.** There exists a four-dimensional pyramid with $v$ vertices and $t$ two-faces, if and only if
\begin{itemize}
  \item[(a)] $2v \leq t \leq 5v - 15$,
  \item[(b)] $t + v \equiv 1 \pmod{2}$, and
  \item[(c)] $v \geq 5$.
\end{itemize}

Furthermore, each allowable pair $(v, t)$ may be represented by a four-pyramid with at least one tetravalent vertex and at least one tetrahedral facet.

**Proof.** In 1906 Steinitz [2] showed that the set of all $f$-vectors of three-polytopes is \{$(f_0, f_0 + f_2 - 2, f_2) \mid 4 \leq f_0 \leq 2f_2 - 4$ and $4 \leq f_2 \leq 2f_0 - 4$\}. Furthermore, each $f$-vector of this set is realized by some three-polytope with at least one trivalent vertex and at least one triangular facet. Every four-pyramid over such a three-polytope has at least one tetravalent vertex and at least one tetrahedral facet.

Now $P$ is a four-pyramid with $v$ vertices and $t$ two-faces if and only if $v = f_0 + 1$ and $t = f_2 + f_1$, where $(f_0, f_1, f_2)$ is the $f$-vector of the three-
dimensional base of \( P \). Thus, \( t = f_2 + (f_0 + f_2 - 2) = f_0 + 2f_2 - 2 \) and \( (f_0 + 4)/2 \leq f_2 \leq 2f_0 - 4 \). Combining these gives

\[
2v = 2f_0 + 2 \leq t \leq 5f_0 - 10 = 5v - 15.
\]

For given values of \( v \) and \( t \) within these bounds it follows that \( 2f_2 = t - v + 3 \), and, thus, \( t + v \) must be odd. The restriction \( v \geq 5 \) and the converse are clear.

\textbf{Theorem 3.} \textit{There exists a four-dimensional bipyramid with \( v \) vertices and \( t \) two-faces if and only if (a) \((7v/2) - 5 \leq t \leq 8v - 32\),

(b) \( t + v = 1 \) (mod 3), and

(c) \( v \geq 6 \).

Furthermore, each allowable pair \((v, t)\) may be represented by a four-bipyramid with a least one tetrahedral facet. (Clearly no four-bipyramid has a tetravalent vertex if \( v > 6 \).)

\textbf{Proof.} If \( v \) and \( t \) are the numbers of vertices and two-faces of a four-bipyramid over a three-dimensional base with \( f \)-vector \((f_0, f_1, f_2)\), then \( v = f_0 + 2 \) and \( t = f_2 + 2f_1 = 3f_2 + 2f_0 - 4 \). By the characterization of the \( f \)-vectors of three-polytopes, it follows that \((7v/2) - 5 \leq t \leq 8v - 32\), and \( 3f_2 = t - 2v + 8 \). Thus, \( (t + v) \equiv (t - 2v) = 1 \) (mod 3). The restriction \( v \geq 6 \) and the converse are clear.

The dual of every pyramid is again a pyramid, so no new polytopes are generated by considering the duals of those formed in Theorem 2. This is not the case with bipyramids, however, as the following theorem shows. The dual of a bipyramid is called a cylinder here, rather than a prism.

\textbf{Theorem 4.} \textit{There exists a four-dimensional cylinder with \( v \) vertices and \( t \) two-faces if and only if (a) \((5v/4) + 4 \leq t \leq (7v/2) - 14\),

(b) \( t + v = 1 \) (mod 3), and

(c) \( v \geq 8 \) is even.

Furthermore, each allowable pair \((v, t)\) may be represented by a four-cylinder with at least one tetravalent vertex. (Clearly no cylinder has a tetrahedral facet if \( v > 8 \).)

\textbf{Proof.} If \( v \) and \( t \) are the number of vertices and two-faces of a four-cylinder over a three-dimensional base with \( f \)-vector \((f_0, f_1, f_2)\), then \( v = 2f_0 \) and \( t = 2f_2 + f_1 = 3f_2 + f_0 - 2 \). The proof proceeds as in the last two theorems.

The following two lemmas allow us to generate new polytopes from given ones by either adding a pyramidal cap to a facet which is a simplex,
or dually, by "slicing off" a vertex of minimal valence. These well known processes will then allow a complete characterization of which pairs \((v, t)\) represent the number of vertices and two-faces of some four-polytope, provided \(v\) or \(t\) is sufficiently large.

**Lemma 5.** If there exists a four-polytope \(P\) with \(v\) vertices and \(t\) two-faces and at least one tetrahedral facet, then there exists a four-polytope \(Q\) with \(v + 1\) vertices and \(t + 6\) two-faces. Furthermore, \(Q\) will have a tetrahedral facet and a tetravalent vertex.

*Proof.* If \(V\) is a point beyond only one tetrahedral facet \(F\) of \(P\), then the desired polytope will be the convex hull of \(\{V\} \cup P\).

**Lemma 6.** If there exists a four-polytope \(P\) with \(v\) vertices and \(t\) two-faces and at least one tetravalent vertex, then there exists a four-polytope \(Q\) with \(v + 3\) vertices and \(t + 4\) two-faces. Furthermore, \(Q\) will have a tetrahedral facet and a tetravalent vertex.

*Proof.* If hyperplane \(H\) strictly separates a tetravalent vertex \(V\) of \(P\) from the remaining vertices, then the desired polytope \(Q\) will be formed from \(P\) by "cutting off" vertex \(V\) with hyperplane \(H\). That is, \(Q = P \cap H^+\), where \(H^+\) is the closed half space not containing \(V\) whose boundary is \(H\).

**Lemma 7.** If \((v, t)\) satisfies 
\[
\frac{(2v + 3 + (8v + 9)^{1/2})/2}{2} \leq t \leq v^2 - 3v
\]
and \(t \neq v^2 - 3v - 1\), then \((v, t)\) is in \(\pi(0, 2)\) provided \(v\) or \(t\) is sufficiently large \((v > 19\) or \(t > 29\)).

*Proof.* We will establish this lemma by considering three separate cases of values of \(t\);

- **Case A:** \(5v/4 + 4 \leq t \leq 8v - 34\),
- **Case B:** \(8v - 34 < t \leq v^2 - 3v\),
- **Case C:** \((2v + 3 + (8v + 9)^{1/2})/2 \leq t \leq 5v/4 + 4\).

*Proof of Case A.* Let \(\mathcal{P}, \mathcal{B}\), and \(\mathcal{C}\) denote, respectively, the set of all integer pairs \((v, t)\) such that some pyramid, bipyramid or cylinder, respectively, in \(\mathbb{R}^4\) has \(v\) vertices and \(t\) two-faces. Let 
\[
\Psi(\mathcal{P}) = \{(v + 1, t + 6) \mid (v, t) \in \mathcal{P}\}
\]
and 
\[
\Theta(\mathcal{P}) = \{(v + 3, t + 4) \mid (v, t) \in \mathcal{P}\}.
\]
Thus \(\Psi\) and \(\Theta\) are the functions which generate pairs \((v, t)\) associated with the 4-polytopes described in Lemmas 5 and 6. From Theorems 3 and 4 it follows that the sets \(\mathcal{B}, \Psi(\mathcal{B}), \Psi^2(\mathcal{B}), \mathcal{C}, \Theta(\mathcal{C})\) and \(\Theta^2(\mathcal{C})\) are pairwise disjoint, while \(\Psi^2(\mathcal{B}) \subset \mathcal{B}\), and \(\Theta^2(\mathcal{C}) \subset \mathcal{C}\). Easy calculations
show that the union of these sets together with $\mathcal{P}$, $\Psi(\mathcal{P})$ and $\Theta(\mathcal{P})$ provides all pairs $(v, t)$ within the bounds given in Case A.

**Proof of Case B.** For a fixed $v \geq 5$, the $f$-vector of a neighborly 4-polytope $N_v$ with $v$ vertices is $(v, (v^2 - v)/2, v^2 - 3v, (v^2 - 3v)/2)$, so $(v, v^2 - 3v) \in \pi(0, 2)$.

Now suppose $f_2(P) = v^2 - 3v - 1$ for some four-polytope $P$. Simplicial polytopes have an even number of two-faces (since $t = 2f'$), so $P$ is not simplicial and, hence, not neighborly. Thus, $f_0(P) = f_0(N_v)$, $f_1(P) \leq f_1(N_v) - 1$, and $f_2(P) = f_2(N_v) - 1$. But $f_i(P) \leq f_i(N_v)$ and Euler's formula then implies that $f_1(P) = f_1(N_v) - 1$ and $f_2(P) = f_2(N_v)$. This gives $v^2 - 3v - 1 = f_2(P) \geq 2f_2(N_v) = v^2 - 3v$, a contradiction. Thus, $(v, v^2 - 3v - 1) \notin \pi(0, 2)$.

We next show that for each $v \geq 6$, $(v, t) \in \pi(0, 2)$ provided $[(v - 1)^2 - 3(v - 1)] + 5 \leq t \leq v^2 - 3v - 2$. This is sufficient to establish Case B, since repeated application of Lemma 5 will give all pairs $(v, t)$ within the given bounds. Note that $[(v - 1)^2 - 3(v - 1)] = f_2(N_{v-1})$. For each $n = 1, ..., v - 4$, let $P(v, n)$ denote a polytope $\text{conv}(N_{v-1} \cup \{V\})$, where $V$ is a point of $R^4$ beyond $n$ facets of $N_{v-1}$ taken from some set $S$ of $v - 3$ facets of $N_{v-1}$ having an edge in common, see [1, Theorem 4.8.22]. Thus, $P(v, n)$ is a simplicial polytope with $v$ vertices whose $f$-vector is $(v, e' + n + 3, t' + 2n + 4, f' + n + 2)$, where $(v - 1, e', t', f')$ is the $f$-vector of $N_{v-1}$. Note that $P(v, v - 4) = N_v$. If $P(v, v - 3)$ is defined similarly, then it is also interesting to note that $P(v, v - 5)$ and $P(v, v - 3)$ have the same $f$-vector. Also for $n = 1, ..., v - 5$, let $P(v, n')$ denote a polytope $\text{conv}(N_{v-1} \cup \{V'\})$, where $V'$ is a point of $R^4$ beyond $n$ facets of $S$ and lying in the hyperplane determined by an additional facet of $S$. Thus, $P(v, n')$ is not simplicial, but it always has at least one tetrahedral facet. The $f$-vector of $P(v, n')$ for $n = 1, ..., v - 5$ is $(v, e' + n + 3, t' + 2n + 3, f' + n + 1)$. Note that $P(v, (v - 6)'')$ and $P(v, (v - 4)'')$ have the same $f$-vector, while $P(v, (v - 5)'')$ has $(v^2 - 3v - 3)$ two-faces. Thus, $(v, t) \in \pi(0, 2)$ provided $[(v - 1)^2 - 3(v - 1)] + 5 \leq t \leq v^2 - 3v - 2$.

**Proof of Case C.** The duals of the polytopes $P(v, n)$ and $P(v, n')$ described in Case B, together with the polytopes obtained from then by repeated application of Lemma 6 give the desired examples.

**3. Characterization of $\pi(0, 2)$**

Lemma 7 and Theorem 1 characterize $\pi(0, 2)$ provided $v$ is sufficiently large, but exceptional cases occur for small values of $v$ and $t$, as is shown
in Fig. 3. In that figure, the asymptotes of Theorem 1 are shown as solid lines, while the dotted asymptotes distinguish the three cases of Lemma 7. All pairs \((v, t)\) which may be represented by a bipyramid or a cylinder are shown by a \(B\) or \(C\), respectively. The remaining pairs which may be represented by a pyramid, or a neighborly polytope with \(v\) vertices, or by \(P(v, n)\), or by \(P(v, n')\) are denoted by \(P\), \(N\), \(n\), and \(n'\), respectively. An asterisk denotes the dual of one of these. (For example, \(N_6^*\) is the dual of the neighborly four-polytope with six vertices.) The polytopes \([P(7, 1')]^*\), \([P(7, 2')]^*\), and \([P(8, 1')]^*\) represent the pairs \((11, 19)\), \((12, 20)\), and \((16, 25)\), respectively, and are all denoted on Fig. 3 as \(D\). The remaining pairs, which may be represented by a polytope obtained with the method of Lemmas 5 or 6, or both, are denoted by |, -, and +, respectively. These are sufficient to characterize all but a finite number of the pairs \((v, t)\) within the asymptotic bounds. These few exceptional cases are considered in the next two lemmas.

**Lemma 8.** Each of the following pairs \((v, t)\) can be represented by a four-polytope with \(v\) vertices and \(t\) two-faces: \((7, 17)\) \((8, 18)\) \((9, 21)\) \((10, 19)\) \((14, 24)\) \((17, 28)\).

**Proof.** A four-polytope with \(7 = d + 3\) vertices has a two-dimensional affine or Gale diagram, see Grünbaum [1, Section 6.3]. The affine diagrams shown in Fig. 1 (or their duals) establish the lemma for the pairs \((7, 17)\) \((8, 18)\) \((10, 19)\) \((14, 24)\) \((17, 28)\). The complete \(f\)-vector is given under each diagram. The pairs thus represented are denoted in Fig. 3 by \(A\).

(9, 21) \((14, 24)\) \((17, 28)\). These may be represented by polytopes built by a process called "face splitting." Recall that we may split a facet of a three-polytope \(P\) by drawing a segment across the facet and then bending one side of the facet along this segment. This can be done as long as all vertices on the half of the facet that is bent, and not on the segment, are trivalent. The first two parts of Fig. 2 show this process for a cube in \(\mathbb{R}^3\). Doing the analogous thing in four dimensions, we divide a (three-dimensional) facet by passing a plane through it, and then "bending" the hyperplane of the facet along the (two-dimensional) plane through the facet.
This can also be described more formally as truncating part of the polytope by intersecting it with a closed halfspace whose bounding hyperplane meets the hyperplane of the facet at points of the plane.

For each of the three cases under consideration we start with the four-polytope which is a pyramid over a three-dimensional pyramid over a square. Its $f$-vector is $(6, 13, 13, 6)$. In each case the facet that is split is the three-dimensional pyramid over a square. Figure 2 shows only these three-dimensional base pyramids together with the appropriate splitting planes. The third part of Fig. 2 represents the base polytope split twice. One splitting produces the $f$-vector $(7, 17, 17, 7)$, after the second splitting the $f$-vector is $(9, 22, 21, 8)$. In the fourth part of Fig. 2 the base has been split three times. After the first splitting the $f$-vector is $(10, 21, 18, 7)$, after the second splitting it is $(14, 30, 24, 8)$ and after the third it is $(17, 36, 28, 9)$. Pairs $(v, t)$ represented by polytopes using this splitting process are denoted in Fig. 3 by $S$. This completes the proof of Lemma 8.

**Lemma 9.** None of the following pairs $(v, t)$ can be presented by a four-polytope with $v$ vertices and $t$ two-faces: $(6, 12)$ $(6, 14)$ $(7, 13)$ $(7, 15)$ $(8, 15)$ $(8, 16)$ $(9, 16)$ $(10, 17)$ $(11, 20)$ $(13, 21)$.

**Proof.** $(6, 12)$ $(6, 14)$. There are only four four-polytopes with six vertices, none with these pairs, see [1, Chapter 6].

$(7, 13)$ $(7, 15)$. Using affine diagrams it is possible to compute the $f$-vectors of all four-polytopes with seven vertices and, thus, show $13 \neq t \neq 15$ when $v = 7$, see [1, Chapter 6].

$(8, 15)$ $(8, 16)$ $(9, 16)$. Assume $P$ is any polytope with at most 16 two-faces. Then since $32 \geq 2t \geq 4f$ it follows that there are at most seven facets (so its dual has a two-dimensional affine diagram), or else $32 = 2t = 4f$ and each facet is a simplex. The $f$-vectors in this case would then be $(8, 16, 16, 8)$ or $(9, 17, 16, 8)$. In the first case $P$ would be simplicial and simple, in contradiction to the fact that the simplex is the only such polytope. In the second case some vertex is at most 3 valent, an impossibility. Thus, none of these three pairs may be realized by a four-polytope.

$(10, 17)$. Similar reasoning with $34 = 2t \geq 4f$ shows that $P$ could have at most eight facets. Hence, the only $f$-vector not previously considered...
would be (10, 19, 17, 8). But this is impossible since $2e \geq 4v$ must always hold. Hence, (10, 17) is nonrealizable.

(13, 21). Similar reasoning with $2t \geq 4f$ and $2e \geq 4v$ shows $f = 8$. Hence, any such polytope has $f$-vector (13, 26, 21, 8) and, thus, is simple. This is impossible because the lower bound theorem for 4-polytopes (see [3]) implies that any simple 4-polytope with eight facets has at least 14 vertices.

(11, 20). No 4-polytope has the $f$-vector (11, 31 − $n$, 20, $n$) for $n \geq 10$ because for all four-polytopes $e \geq 2v$. No 4-polytope has that $f$-vector.
for \( n \leq 7 \) because it does not appear in the enumeration of all four-polytopes with seven or fewer facets, see [1, Chapter 6].

If some polytope has the \( f \)-vector \((11, 22, 20, 9)\) then it is a simple polytope with at least one simplicial facet. But this would be a polytope isomorphic to one obtained from a simple four-polytope with \( f \)-vector \((8, 16, 16, 8)\) by truncation. This is a contradiction because we have shown that no four-polytope has eight vertices and 16 two-faces.

Suppose some four-polytope \( P \) has \( f \)-vector \((11, 23, 20, 8)\). Let \( p_i \) be the number of facets with \( i \) two-faces in the dual polytope \( P^* \). We then have the following equations:

\[
\sum i p_i = 46,
\sum 4p_i = 44.
\]

Thus,

\[
\sum (i - 4) p_i = 2,
\]

and we conclude that either \( p_6 = 2 \) or \( p_6 = 1 \) with all other \( p_i = 0 \) except \( i = 4 \).

Case I. \( p_6 = 1 \). All other facets of \( P^* \) are tetrahedra; thus, the one facet with six two-faces is simplicial and is a bipyramid over a triangle. If we span the equator of this facet by a triangle we will change the boundary of \( P^* \) to a triangulation of the 3-sphere with \( f \)-vector \((8, 20, 24, 12)\), which contradicts the lower bound theorem [3].

Case II. \( p_5 = 2 \). In this case the two facets with five two-faces are pyramids over a quadrilateral and they meet on this quadrilateral. We may triangulate this pair of facets by splitting each into two tetrahedra, producing a triangulation of the 3-sphere with \( f \)-vector \((8, 21, 26, 13)\), again contradicting the lower bound theorem.

This completes the proof of Lemma 9.

The foregoing theorems and lemmas give the following complete characterization of \( \pi(0, 2) \).

**Theorem 10.** There exists a four-polytope with exactly \( v \) vertices and \( t \) two-dimensional faces if and only if

\[
(2v + 3 + (8v + 9)^{1/2})/2 \leq t \leq v^2 - 3v,
\]

\( t \neq v^2 - 3v - 1 \), and \((v, t)\) is not in the following set of 10 exceptional pairs:

\{(6, 12) (6, 14) (7, 13) (7, 15) (8, 15) (8, 16) (9, 16) (10, 17) (11, 20) (13, 21)\}. 

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