Europ. J. Combinatorics (2001) **22**, 687–698
doi:10.1006/eujc.2000.0488
Available online at http://www.idealibrary.com on IDEAL®

Combinatorial and Algebraic Structure in Orlik–Solomon Algebras

MICHAEL FALK

The Orlik–Solomon algebra $\mathcal{A}(G)$ of a matroid G is the free exterior algebra on the points, modulo the ideal generated by the circuit boundaries. On one hand, this algebra is a homotopy invariant of the complement of any complex hyperplane arrangement realizing G. On the other hand, some features of the matroid G are reflected in the algebraic structure of $\mathcal{A}(G)$.

In this mostly expository article, we describe recent developments in the construction of algebraic invariants of $\mathcal{A}(G)$. We develop a categorical framework for the statement and proof of recently discovered isomorphism theorems which suggests a possible setting for classification theorems. Several specific open problems are formulated.

1. INTRODUCTION: THE ORLIK–SOLOMON ALGEBRA OF A MATROID

Let G be a simple matroid with ground set $[n] := \{1, \ldots, n\}$. The Orlik–Solomon (OS) algebra of G is defined as follows. Let $\mathcal{E} = \Lambda(e_1, \ldots, e_n)$ be the graded exterior algebra on elements e_i of degree one corresponding to the points of G. For simplicity we will assume the ground field is $\mathbb{C}$. Except where noted, all of the results will hold for coefficients in an arbitrary commutative ring.

Define the linear mapping $\partial : \mathcal{E}^p \longrightarrow \mathcal{E}^{p-1}$ by

$$\partial(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p},$$

where $\widehat{\ }$ indicates an omitted factor.

If $S = (i_1, \ldots, i_p)$ is an ordered p-tuple we denote the product $e_{i_1} \wedge \cdots \wedge e_{i_p}$ by e_S. Let $\mathcal{I}$ denote the ideal of $\mathcal{E}$ generated by $\{\partial e_S \mid S \text{ is dependent}\}$.

DEFINITION 1.1. The *Orlik–Solomon algebra* $\mathcal{A} = \mathcal{A}(G)$ of G is the quotient $\mathcal{E}/\mathcal{I}$.

Since $\mathcal{I}$ is generated by homogeneous elements, both $\mathcal{I}$ and $\mathcal{A}$ inherit gradings from $\mathcal{E}$. We will denote the image of e_S in $\mathcal{A}$ by a_S.

The OS algebra has both combinatorial and topological significance, as demonstrated by these two results from [21]. Recall that a projective realization of G gives rise to a linear hyperplane arrangement. Throughout the paper A will denote a hyperplane arrangement arising from a complex projective realization of G, and M will denote the *complement* of A, $M = \mathbb{C}^\ell - \bigcup_{H \in A} H$.

THEOREM 1.2. *The OS algebra $\mathcal{A}(G)$ is isomorphic to the cohomology algebra $H^*(M)$.*

The Whitney numbers of the second kind are defined in terms of the Möbius function $\mu : L(G) \longrightarrow \mathbb{Z}$ of the lattice of flats $L(G)$. Specifically,

$$w_p(G) = \sum_{X \in L, \mathrm{rk}(X)=p} (-1)^p \mu(0_L, X).$$

THEOREM 1.3. *The dimension of $\mathcal{A}^p(G)$ is equal to the pth Whitney number $w_p(G)$ of G.*

0195–6698/01/050687 + 12 $35.00/0

Theorem 1.2 motivates what is for us the main problem concerning OS algebras: to classify $\mathcal{A}(G)$ up to isomorphism of graded algebras. This type of problem is more familiar in topology than combinatorics, but the classification in this instance will be purely matroidal. Theorem 1.3 provides one line along which a classification could proceed, that is to extract combinatorial features of the matroid G from algebraic invariants of $\mathcal{A}(G)$. In this regard we note that there are many sets of matroids with identical Whitney numbers while, on the other hand, the betti numbers $\dim(\mathcal{A}^p(G))$ in a sense take no account of the ring structure of $\mathcal{A}(G)$.

These observations set the tone for the exposition to follow. We will construct multiplicative invariants of $\mathcal{A}(G)$ and attempt to extract combinatorial structure from them. The most delicate of these are the resonance varieties, discussed in Section 2. In Section 3 we show how 'stabilized' parallel connection and direct sum of matroids yield isomorphic OS algebras. We also show that truncations of matroids with isomorphic OS algebras will have the same property. We make sense of these results using the categories of pointed matroids and affine OS algebras, indicating a framework for the eventual classification. In Section 4 we describe recent work relating the k-adic closure of $\mathcal{A}(G)$ to the 'k-closure' of the matroid G.

We close this introduction by recalling the oldest multiplicative invariant of $\mathcal{A}(G)$, termed 'the global invariant' ϕ_3 in [11]. Consider the multiplication map

$$d : \mathcal{E}^1 \otimes \mathcal{I}^2 \longrightarrow \mathcal{E}^3.$$

This linear map can be shown to be an invariant of $\mathcal{A}(G)$. The nullity of d is denoted by $\phi_3(\mathcal{A})$. This quantity has a topological interpretation in terms of the fundamental group of the complement M. Indeed, the definition of ϕ_3 comes directly out of the study of the rational homotopy type of hyperplane complements [14]. And of course $\phi_3(\mathcal{A})$ can be thought of as an invariant of the matroid G. But the following problem remains open, even for graphic matroids.

PROBLEM 1.4. Give a combinatorial interpretation of $\phi_3(G)$.

We will return to this problem in Section 4.

The reader is referred to [22] for background material on complex hyperplane arrangements and Orlik–Solomon algebras, and to [25] for matroid theory. Section 2 is largely based on [8], and much of Section 3 is a reformulation of part of [7]. Section 4 is a brief report on work in progress; details and proofs will appear in [6, 9].

2. RESONANCE VARIETIES

To answer questions concerning generalized hypergeometric functions, we began studying the OS algebra as a differential complex in [16], and then realized that our work could be used to define algebraic invariants [8].

Fix an element $a_\lambda = \sum_{i=1}^n \lambda_i a_i$ in $\mathcal{A}^1$. Then left multiplication by a_λ defines a map $\mathcal{A}^p \longrightarrow \mathcal{A}^{p+1}$, which squares to zero. Thus we have a cochain complex

$$0 \longrightarrow \mathcal{A}^0 \xrightarrow{a_\lambda} \mathcal{A}^1 \xrightarrow{a_\lambda} \cdots \xrightarrow{a_\lambda} \mathcal{A}^{\ell-1} \xrightarrow{a_\lambda} \mathcal{A}^\ell \longrightarrow 0.$$

The cohomology of this complex determines a stratification of the parameter space $\mathbb{C}^n$. The pth *resonance variety* of $\mathcal{A}$ is defined by

$$R_p(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid H^p(\mathcal{A}, a_\lambda) \neq 0\}.$$

It is shown in [8] that $R_p(\mathcal{A})$, up to ambient linear isomorphism, is an invariant of $\mathcal{A}$.

Basic properties of resonance varieties follow from the main results of [27]. Let Δ denote the diagonal hyperplane $\sum_{i=1}^n \lambda_i = 0$. Then

- $0 \in R_p(\mathcal{A})$ for $0 \leq p \leq \ell$.
- $R_0(\mathcal{A}) = \{0\}$.
- $R_p(\mathcal{A}) \subseteq \Delta$ for all p.
- $R_\ell(\mathcal{A}) \subseteq R_{\ell-1}(\mathcal{A})$.
- if G is connected, then $R_\ell(\mathcal{A}) = R_{\ell-1}(\mathcal{A}) = \Delta$.
- $R_p(\mathcal{A})$ is a proper subvariety of Δ for $0 \leq p \leq \ell - 2$.

Under some genericity conditions on λ, the cohomology $H^*(\mathcal{A}, a_\lambda)$ is isomorphic to the cohomology of M with coefficients in a rank-one complex local system $Ł_\lambda$ with monodromy determined by λ. This local system cohomology plays a role in the definition of generalized (multivariate) hypergeometric integrals. In a sense made precise in recent work of D. Cohen and P. Orlik [4], the complex $(\mathcal{A}, a_\lambda)$ is the derivative at the identity of a cochain complex $(\mathcal{A}, \Delta_\lambda)$ that computes the local system cohomology. The resonance variety $R_p(A)$ is then the tangent cone at the identity to the 'jumping locus' for the local system cohomology, the set of local systems for which the cohomology $H^p(M, Ł_\lambda)$ is non-vanishing. For $p = 1$ the jumping locus for local system cohomology coincides with the character variety in $(\mathbb{C}^*)^n$ associated with the Alexander invariant of the fundamental group. For any p, a theorem of D. Arapura asserts that these jumping loci are subtori of $(\mathbb{C}^*)^n$, possibly translated by elements of finite order. This gives an indication of the proof of the following result, originally conjectured for $p = 1$ in [8], proved in that special case in [5, 19], and finally established for arbitrary p in [4, 18]. See those papers for complete references.

THEOREM 2.1. *The resonance variety $R_p(\mathcal{A})$ is a union of linear subspaces of $\mathbb{C}^n$.*

By Theorem 2.1, $R_p(\mathcal{A})$ can be thought of as a subspace arrangement, and as such, realizes a polymatroid $poly_p(\mathcal{A})$, which in essence records the dimension of the span of each subcollection of irreducible components of $R_p(\mathcal{A})$. Because $R_p(\mathcal{A})$ is invariant up to linear change of coordinates, the polymatroid $poly_p(\mathcal{A})$ is indeed an invariant of $\mathcal{A}$, powerful enough (at least for $p = 1$) to distinguish OS algebras of matroids which are almost identical in other respects [8].

The first cohomology $H^1(\mathcal{A}, a_\lambda)$ can be computed directly, yielding a description of $R_1(\mathcal{A})$. The following lemma reduces the calculation to an analysis of elements of $\mathcal{I}^2$.

LEMMA 2.2. $\lambda \in R_1(\mathcal{A})$ *if and only if* e_λ *is one factor of a non-zero elementary tensor in* $\mathcal{I}^2$.

Proof of this lemma and the results to follow can be found in [8].

Irreducible components of $R_1(\mathcal{A})$ are contained in intersections of Δ with hyperplanes H_X defined by $\sum_{i \in X} \lambda_i = 0$, where X runs over certain flats of G. The flats which occur in these intersections are determined by so-called 'neighborly partitions' of G.

DEFINITION 2.3. A *neighborly partition* of G is a partition Π of $[n]$ such that $|\pi \cap X| \neq |X| - 1$ for all blocks $\pi \in \Pi$ and flats X of rank two in L.

We say a flat X is 'multi-colored' if X meets more than one block of Π. Given a neighborly partition Π of a submatroid $S \subseteq [n]$ of G, set

$$L_\Pi = \Delta \cap \bigcap_{i \notin S} H_i \cap \bigcap_{X \in \mathrm{mc}(\Pi)} H_X,$$

where the last intersection runs over the set $\mathrm{mc}(\Pi)$ multi-colored rank-two flats of Π. Note that $H_i = \{\lambda \in \mathbb{C}^n | \lambda_i = 0\}$. The *support* $\mathrm{supp}(\lambda)$ of λ is $\{i \in [n] | \lambda_i \neq 0\}$, considered as a submatroid of G. Let $\sim$ denote the equivalence relation associated with Π. Finally, for $\tau \in \mathcal{E}^2$ write $\tau = \sum_{i<j} \tau_{ij} e_i \wedge e_j$. Here then is a description of $R_1(\mathcal{A})$, from [8], to which the reader is referred for the proof, examples and consequences.

THEOREM 2.4. *$\lambda \in R_1(\mathcal{A})$ if and only if* supp(λ) *affords a neighborly partition Π such that (i) $\lambda \in L_\Pi$, and (ii) there exists $\mu \in L_\Pi$ not proportional to λ such that $(\lambda \wedge \mu)_{ij} = 0$ for every $i < j$ with $i \sim j$ under Π.*

The second condition will be replaced with a simpler criterion below.

If X is a flat of rank two with $|X| \geq 3$, then $\Pi = \{\{i\} \mid i \in X\}$ is a neighborly partition of X, and $L_\Pi = \Delta \cap \bigcap_{i \notin X} H_i$ has dimension $|X| - 1 \geq 2$. Thus condition (ii) is satisfied, and indeed L_Π is a component of $R_1(\mathcal{A})$ [8]. The components which arise in this way are called the *local components* of $R_1(\mathcal{A})$. Here is a sample result from [8] showing how combinatorial structure may be extracted from $R_1(\mathcal{A})$.

COROLLARY 2.5. *Suppose every non-local component of $R_1(\mathcal{A})$ has dimension two. Then $R_1(\mathcal{A})$ determines the number of rank-two flats of G of each cardinality. In particular, if G has rank three, $R_1(\mathcal{A})$ determines the Tutte polynomial of G.*

D. Cohen informs us that he and J. Oxley have found examples for which the hypothesis fails. We will see in the next section that $\mathcal{A}(G)$ does not generally determine the Tutte polynomial of G for matroids of high rank.

In [19] A. Libgober and S. Yuzvinsky base a study of the resonance variety $R_1(\mathcal{A})$ on the Vinberg classification of Cartan matrices for affine Kac-Moody Lie algebras. Their approach yields substantial additional detail about $R_1(\mathcal{A})$ and the associated neighborly partitions. We state some of their more general conclusions in the following theorem.

THEOREM 2.6 ([19]).

(i) The irreducible components of $R_1(\mathcal{A})$ are precisely the L_Π of dimension at least two.
(ii) If L_Π and $L_{\Pi'}$ are two irreducible components of $R_1(\mathcal{A})$, then $L_\Pi \cap L_{\Pi'} = \{0\}$.
(iii) For any component L_Π of $R_1(\mathcal{A})$, each multi-colored flat of G meets every block of Π.

Theorem 2.6(i) effectively replaces condition (ii) of Theorem 2.4 with the much simpler requirement $\dim(L_\Pi) \geq 2$. Theorem 2.1 for $p = 1$ is an immediate corollary.

Matroids of rank greater than two which support neighborly partitions Π for which L_Π has dimension at least two are quite rare. Some examples appear in [8]. The classification theory used in [19] imposes some restrictions, and also yields a method of constructing examples as a kind of inverse problem. The first part of the following problem is solved in some special cases in [19].

PROBLEM 2.7.

(i) Characterize those matroids which support neighborly partitions Π satisfying $\dim(L_\Pi) \geq 2$.
(ii) Describe the polymatroid $poly_1(G)$ associated with the arrangement of subspaces $\{L_\Pi | \Pi \text{ is neighborly and } \dim(L_\pi) \geq 2\}$.

Libgober and Yuzvinsky [19] also uncover a connection between non-local components of $R_1(\mathcal{A})$, for arrangements of rank three, and pencils of curves $\mathbb{C}P^2 \longrightarrow \mathbb{C}P^1$ which include the arrangement in their singular locus. The existence of such pencils imposes further restrictions on the structure of matroids supporting non-trivial ($\dim(L_\pi) \geq 2$) neighborly partitions.

In addition, these pencils of curves bear a relationship to the $K(\pi, 1)$ problem for complex hyperplane arrangements, and were studied in that vein in [12]. So a solution to Problem 2.7(i) might have some implications for the $K(\pi, 1)$ problem [15].

In another direction, D. Matei and A. Suciu [20] discovered deep connections between the resonance varieties of $\mathcal{A}(G) \otimes \mathbb{Z}_p$ and the structure of the second nilpotent quotient of the fundamental group $\pi_1(M)$. This work leads to some other interesting open questions. We briefly summarize.

Write $R_1(\mathcal{A}, \mathbb{Z}_p)$ for the first resonance variety of $\mathcal{A}(G) \otimes \mathbb{Z}_p$, and let

$$R_{1,d}(\mathcal{A}, \mathbb{Z}_p) = \{\lambda \in R_1(\mathcal{A}, \mathbb{Z}_p) \mid \dim H^1(\mathcal{A} \otimes \mathbb{Z}_p, a_\lambda) \geq d\}.$$

These are subvarieties of $(\mathbb{Z}_p)^n$, easily seen to be homogeneous. Let $\widehat{R}_{1,d}(\mathcal{A}, \mathbb{Z}_p)$ denote the projective image of $R_{1,d}(\mathcal{A}, \mathbb{Z}_p)$. Finally, let $\pi = \pi^1 \supseteq \pi^2 \supseteq \pi^3 \supseteq \cdots$ denote the descending central series of $\pi = \pi_1(M)$, and $\Gamma = \pi/\pi^3$ the second nilpotent quotient. Let $\nu_{p,d}$ denote the number of normal subgroups K of Γ of index p, such that the abelianization of Γ/K has p-torsion of rank d.

THEOREM 2.8 ([20]).

$$\nu_{p,d} = \sharp(\widehat{R}_{1,d}(\mathcal{A}, \mathbb{Z}_p) - \widehat{R}_{1,d+1}(\mathcal{A}, \mathbb{Z}_p))$$

The quantity on the right-hand side is also an invariant of $\mathcal{A}(G)$.

The proof uses a relationship between the resonance varieties and the Alexander invariant of the fundamental group, similar to the observations used to prove Theorem 2.1 in [5]. In this case, the (linearized) Alexander matrix (mod p) is used to count normal subgroups of index p in the second nilpotent quotient of $\pi_1(M)$, on one hand, and to define the resonance variety of $\mathcal{A}(G) \otimes \mathbb{Z}_p$ on the other.

Theorem 2.8 leads to the study of resonance varieties of OS algebras over finite fields. Because the variety $R_1(\mathcal{A})$ is defined over $\mathbb{Z}$, we can reduce mod p. But there are matroids G which have 'exceptional primes' p, for which the reduction $R_1(\mathcal{A}) \otimes \mathbb{Z}_p$ does not coincide with $R_1(\mathcal{A}, \mathbb{Z}_p)$. The basic results of this section, from [8], will hold over an arbitrary ground field, but the techniques of [5, 19], for instance, and thus Theorems 2.1 and 2.6, require complex coefficients. In [20] the authors give examples of matroids for which

(i) $R_1(\mathcal{A}, \mathbb{Z}_p)$ has non-local components while $R_1(\mathcal{A})$ has none.
(ii) $R_1(\mathcal{A}, \mathbb{Z}_p)$ has a non-local components of dimension greater than two, while all non-local components of $R_1(\mathcal{A})$ are 2-dimensional.
(iii) $R_{1,d}(\mathcal{A}, \mathbb{Z}_p)$ has components which are not $(d+1)$-dimensional. By contrast, the components of the analogous variety $R_{1,d}(\mathcal{A})$ over $\mathbb{C}$ always have dimension $d+1$ [19].

This suggests a variation of Problem 2.7, suggested by A. Suciu.

PROBLEM 2.9. Given a matroid G, determine the exceptional primes for G, that is, the primes p for which $\mathbb{R}_1(\mathcal{A}, \mathbb{Z}_p) \ncong \mathbb{R}_1(\mathcal{A}) \otimes \mathbb{Z}_p$.

3. ISOMORPHISMS: AFFINE OS ALGEBRAS AND POINTED MATROIDS

In [7] we showed how one could construct, from an arbitrary pair of (realizable) matroids G_0 and G_1, a pair of non-isomorphic matroids G and G' for which $\mathcal{A}(G) \cong \mathcal{A}(G')$. The matroids G and G' are, respectively, the direct sum $G_0 \oplus G_1$, and any parallel connection $P(G_0, G_1)$, stabilized by adding an isthmus (so G and G' have the same number of points).

In this section we cast this result in a simpler conceptual framework, motivated by the fact that parallel connection is the categorical direct sum of base-pointed matroids [3, 25].

We will also prove that, for two matroids G and G', if $\mathcal{A}(G) \cong \mathcal{A}(G')$, then $\mathcal{A}(\overline{G}) \cong \mathcal{A}(\overline{G'})$, where the bar denotes truncation. Together with the equivalences involving direct sum, this result explains all known instances of isomorphisms of OS algebras, and so we are led to a possible formulation for a classification result.

We start with some fundamental observations. The elementary proofs are left to the reader.

PROPOSITION 3.1.

(i) If $i \in S$ then $e_i \partial e_S = \pm e_S$.
(ii) If S is dependent then $e_S \in \mathcal{I}$.
(iii) The ideal $\mathcal{I}$ is generated by $\{\partial e_C | C$ is a circuit$\}$.

Our setup involves generalizing the definition of OS algebra. This is carried out in [22] by giving an algebra presentation associated with an arrangement of affine hyperplanes. We adopt a different approach, so that we can stay in the realm of matroid theory. The combinatorial model for an affine arrangement is a *pointed matroid*, that is, a matroid with a specified base point. Given an arrangement A of affine hyperplanes, the underlying pointed matroid will be the matroid of the cone cA of A [22], with the hyperplane at infinity as base point. Conversely, given a central arrangement A realizing the matroid G, the effect of choosing a base point in G will yield the pointed matroid associated with the decone dA of A relative to the hyperplane corresponding to the chosen base point. In keeping with the notation of [22], we will write dG to denote a pointed matroid, with underlying unpointed matroid G. Our convention will be that G has ground set $\{0, \ldots, n\}$, and that dG has 0 as base point. More generally, the pointed matroid on G with base point i will be denoted $d_i G$.

DEFINITION 3.2. The OS algebra of the pointed matroid dG is the subalgebra $\mathcal{A}_d(dG)$ of the OS algebra $\mathcal{A}(G)$ generated by $\{a_1 - a_0, \ldots, a_n - a_0\}$.

The reader will find that this definition agrees with the definition of [22] of the OS algebra of an affine arrangement dA with underlying pointed matroid dG. In particular we have [22, Corollary 3.58]

$$\sum_p \dim(\mathcal{A}^p(G))t^p = (1+t)\sum_p \dim(\mathcal{A}_d(dG))t^p.$$

We recover the ordinary OS algebra as follows. Given an unpointed matroid G on ground set $[n]$, let cG denote the matroid $\{0\} \oplus G$ of rank $\mathrm{rk}(G) + 1$, with the point 0 marked. Here $\{0\}$ is understood to be the rank-one matroid with one point, an isthmus. The reader is invited to verify the following result.

LEMMA 3.3. $\mathcal{A}_d(cG) \cong \mathcal{A}(G)$.

There are two operations on pointed matroids which have a predictable effect on OS algebras. The first of these will be obvious to those familiar with the topology of hyperplane arrangements. Indeed, the complement M supports an action of $\mathbb{C}^*$, and the induced map $H^*(M/\mathbb{C}^*) \longrightarrow H^*(M)$ is a split injection with image $\mathcal{A}_d(d_i G)$, for any i [22, Proposition 5.1].

THEOREM 3.4. *For any $i, j \in \{0, \ldots, n\}$,*

$$\mathcal{A}_d(d_i G) = \mathcal{A}_d(d_j G).$$

PROOF. This is immediate from the identities $a_k - a_j = (a_k - a_i) - (a_j - a_i)$ for $k \neq i, j$ and $a_i - a_j = -(a_j - a_i)$. $\square$

The parallel connection of pointed matroids dG_0 and dG_1 is the unique (up to isomorphism) pointed matroid $P_d(dG_0, dG_1)$ of largest rank which is a union of pointed submatroids isomorphic to dG_0 and dG_1, whose ground sets intersect only at the base point [25]. The underlying matroid of $P_d(dG_0, dG_1)$ is called a parallel connection of G_0 and G_1, denoted $P(G_0, G_1)$. The following result from [3] motivated the present formulation of the equivalence discovered in [7].

LEMMA 3.5. *Parallel connection is a sum in the category of pointed matroids and pointed strong maps. That is,*

$$\begin{array}{ccc} \{0\} & \longrightarrow & G_0 \\ \downarrow & & \downarrow \\ G_1 & \longrightarrow & P(G_0, G_1) \end{array}$$

is a pushout diagram of pointed strong maps.

LEMMA 3.6. *The assignment $dG \mapsto \mathcal{A}_d(dG)$ yields a functor from the category of pointed matroids and pointed strong maps to the category of connected (i.e., $\mathcal{A}^0 \cong \mathbb{C}$) graded algebras over $\mathbb{C}$.*

PROOF. Let dG and dG' be pointed matroids on $\{0, \ldots, n\}$ and $\{0, \ldots, m\}$ respectively. A pointed strong map $dG \longrightarrow dG'$ arises from a set function $\eta : \{0, \ldots, n\} \longrightarrow \{0, \ldots, m\}$ mapping 0 to 0. This function yields a homomorphism of exterior algebras $\hat{\eta} : \mathcal{E} \longrightarrow \mathcal{E}'$ determined by $\hat{\eta}(e_i) = e_{\eta(i)}$. According to [25, Lemmas 8.1.4 and 8.1.6], the image of each circuit of G is dependent in G'. Using Lemma 3.1 this implies that $\hat{\eta}$ sends $\mathcal{I}$ into $\mathcal{I}'$, inducing a homomorphism $\mathcal{A}(G) \longrightarrow \mathcal{A}(G')$. Since $\hat{\eta}(a_0) = a_0'$, $\hat{\eta}$ restricts to a homomorphism $\mathcal{A}_d(dG) \longrightarrow \mathcal{A}_d(dG')$. $\square$

As a consequence of these observations, the effect of parallel connection on OS algebras becomes natural.

THEOREM 3.7. *The OS algebra of $P_d(dG_0, dG_1)$ is isomorphic to $\mathcal{A}_d(dG_0) \otimes \mathcal{A}_d(dG_1)$.*

PROOF. Let us write dG for $P_d(dG_0, dG_1)$. Using the fact that tensor product is a sum in the category of connected graded algebras, together with Lemma 3.6, we obtain a surjective homomorphism $\mathcal{A}_d(dG_0) \otimes \mathcal{A}_d(dG_1) \longrightarrow \mathcal{A}_d(G)$. Using Theorem 1.3 and [26, Proposition 7.2.9], one can show that the domain and target have the same dimension in each degree. Thus the two algebras are isomorphic. $\square$

As a consequence of Theorems 3.7 and 3.4, we easily obtain the combinatorial/algebraic version of the main topological result of [7].

THEOREM 3.8. *Let G_0 and G_1 be arbitrary matroids. Then $G = G_0 \oplus G_1$ and $G' = \{0\} \oplus P(G_0, G_1)$ have isomorphic OS algebras.*

PROOF. Consider the pointed parallel connection $d\hat{G} = P_d(cG_0, cG_1)$. The underlying matroid $\hat{G}$ is $\{0\} \oplus G_0 \oplus G_1 = \{0\} \oplus G$, which is precisely cG. Then, by Lemma 3.3,

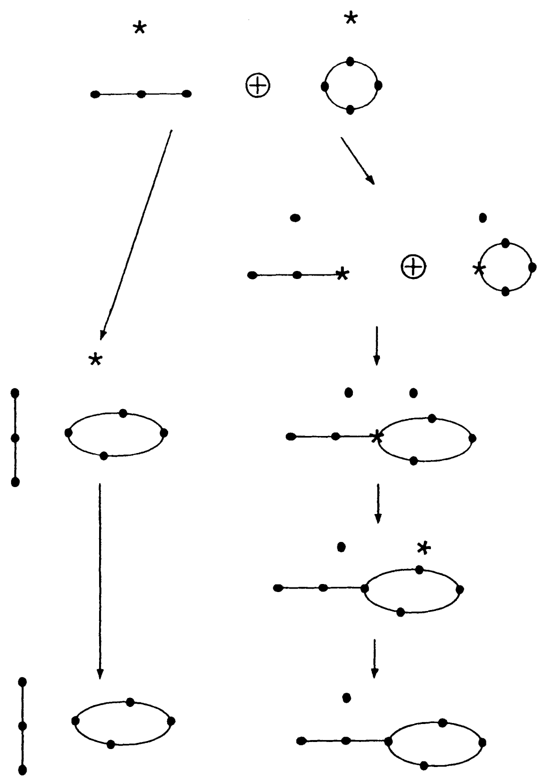

FIGURE 1. The proof of Theorem 3.8.

the $\mathcal{A}_d(dG) \cong \mathcal{A}(G)$. On the other hand, by Theorem 3.7, $\mathcal{A}_d(dG)$ is also isomorphic to $\mathcal{A}_d(cG_0) \otimes \mathcal{A}_d(cG_1)$, which again by Lemma 3.3, is isomorphic to $\mathcal{A}(G_0) \otimes \mathcal{A}(G_1)$.

Now, according to Theorem 3.4, we may change the base points of cG_0 and cG_1 without affecting the affine OS algebras. The pointed parallel connection $d\hat{G}'$ of these new pointed matroids will have underlying matroid $\hat{G}'$ isomorphic to the the sum of two isthmuses (neither marked) with an ordinary parallel connection $P(G_0, G_1)$ of G_0 and G_1 along the new marked points of each. Again, we have $\mathcal{A}_d(d\hat{G}') \cong \mathcal{A}_d(cG_0) \otimes \mathcal{A}_d(cG_1) \cong \mathcal{A}(G_0) \otimes \mathcal{A}(G_1)$. Now we change the base point of $d\hat{G}'$ to one of the isthmuses, and recognize the resulting pointed matroid as cG'. We apply Lemma 3.3 once more to obtain the result. $\square$

We regard the method of proof above as 'diagrammatic', and indeed the argument is easier to follow in pictures than in words. See Figure 1. It should now be clear that these isomorphisms arise from the trivial operations of changing base points and forming sums.

In [7] we proved a stronger result for realizations of G_0 and G_1, by constructing a natural realization of $P(G_0, G_1)$ and proving that the complements of the arrangements realizing G and G' are in fact diffeomorphic. Theorem 3.8 follows in this case by Theorem 1.2.

We state two interesting consequences of Theorem 3.8 from [7]. The first should be compared with Theorems 1.3 and 2.5.

COROLLARY 3.9. *Given an arbitrary matroid G_0, there exist extensions G and G' of G_0 with isomorphic OS algebras but different Tutte polynomials.*

The second corollary results from the indeterminacy in the change of base point in the proof of Theorem 3.8.

COROLLARY 3.10. *For any positive integer n, there exist n non-isomorphic matroids with isomorphic OS algebras.*

The original examples of non-isomorphic matroids with isomorphic OS algebras, which appeared in [10, 11, 22], are truncations of G and G', where the factors G_0 and G_1 both have rank two. In an NSF-sponsored *REU* undergraduate research project directed by the author, C. Pendergrass showed that truncation of matroids always preserves isomorphisms of the associated OS algebra [24].

THEOREM 3.11. *Suppose $\mathcal{A}(G) \cong \mathcal{A}(G')$, and let $\overline{G}$ and $\overline{G'}$ denote the (corank-one) truncations of G and G' respectively. Then $\mathcal{A}(\overline{G}) \cong \mathcal{A}(\overline{G'})$.*

PROOF. Suppose η is an isomorphism of $\mathcal{A}(G)$ to $\mathcal{A}(G')$. To begin with, we can then assume without loss that G and G' have the same ground set. The isomorphism $\eta : \mathcal{A}^1(G) \longrightarrow \mathcal{A}^1(G')$ determines an isomorphism $\hat{\eta} : \mathcal{E}(G) \longrightarrow \mathcal{E}(G')$, and $\hat{\eta}(\mathcal{I}(G)) = \mathcal{I}(G')$. We need only show that $\hat{\eta}(\mathcal{I}(\overline{G})) = \mathcal{I}(\overline{G'})$.

Let $n = \mathrm{rk}(G) = \mathrm{rk}(G')$. Then, for $p < n - 1$,

$$\hat{\eta}(\mathcal{I}^p(\overline{G})) = \hat{\eta}(\mathcal{I}^p(G)) = \mathcal{I}^p(G') = \mathcal{I}^p(\overline{G'}).$$

Since the truncations have rank $n-1$, we also have, for $p \geq n-1$, $\mathcal{I}^p(\overline{G}) = \partial\mathcal{E}^{p+1} = \mathcal{I}^p(\overline{G'})$. Since $\hat{\eta}$ is an algebra homomorphism, it commutes with ∂, and thus $\hat{\eta}(\mathcal{I}^p(\overline{G})) = \mathcal{I}^p(\overline{G'})$ for $p \geq n - 1$. This completes the proof. □

All known examples of isomorphisms of OS algebras arising from non-isomorphic matroids are consequences of Theorems 3.8 and 3.11. So we are led to the following problem. Recall that a matroid which is not a truncation is called *inerectible*.

PROBLEM 3.12. For inerectible parallel-irreducible matroids G and G', $\mathcal{A}(G) \cong \mathcal{A}(G')$ if and only if $G \cong G'$.

We prefer an alternate formulation based on the categorical framework developed earlier.

PROBLEM 3.13. Suppose dG and dG' are inerectible pointed matroids which are irreducible in the category of pointed matroids. Then $\mathcal{A}_d(dG) \cong \mathcal{A}_d(dG')$ if and only if $dG \cong dG'$ up to change of base point.

4. THE k-ADIC CLOSURE OF $\mathcal{A}(G)$

We have recently become interested in quadratic OS algebras, and more generally the quadratic closure of $\mathcal{A} = \mathcal{A}(G)$. This is the first in a series of *k-adic closures* whose dimensions are algebraic invariants of $\mathcal{A}$, and about which little is known. In this section we briefly present these ideas and describe some recent results and work in progress, to appear in [6, 9].

For $k \geq 2$, define the *k-adic OS ideal* $\mathcal{I}_k$ to be the ideal generated by $\sum_{j \leq k} \mathcal{I}^j$ and the *k-adic closure of* $\mathcal{A}$ to be the quotient $\mathcal{A}_k = \mathcal{E}/\mathcal{I}_k$. These algebras form a sort of resolution of $\mathcal{A}$:

$$\mathcal{E} = \mathcal{A}_1 \longrightarrow \mathcal{A}_2 \longrightarrow \mathcal{A}_3 \longrightarrow \cdots \longrightarrow \mathcal{A}_{\ell-1} \longrightarrow \mathcal{A}_\ell = \mathcal{A}.$$

The following problem is wide open, even for $k = 2$.

PROBLEM 4.1. Calculate the dimension of $\mathcal{A}_k^p$ in terms of the underlying matroid G.

Of special interest is the condition $\mathcal{A}_2 = \mathcal{A}$, in which case we say $\mathcal{A}$ is *quadratic*. Examples indicate that this condition is related to the notion of line-closed matroid. The *line-closure* of a set $S \subseteq [n]$ is the smallest subset $\ell c(S)$ of n containing S and containing the entire line in G spanned by any pair of points of $\ell c(S)$. The matroid G is line-closed if and only if every line-closed set is closed. A proof of the following result will appear in [9].

THEOREM 4.2. *If $\mathcal{A}$ is quadratic then G is line-closed.*

This result was originally announced in [13], at which time we conjectured that the converse is also true, that is, that line-closed matroids have quadratic OS algebras. S. Yuzvinsky subsequently found a counterexample to this conjecture, the matroid on eight points with non-trivial lines

$$123,\ 3456,\ 167,\ 258,\ \text{and } 478.$$

Yuzvinsky proposed a different condition for quadraticity of $\mathcal{A}$, which fails for the example above. This condition is also necessary for quadraticity, and is demonstrably stronger than line-closure. G. Denham subsequently found an example (a 9_3 configuration) showing this stronger condition is still not sufficient for quadraticity. The work of Denham and Yuzvinsky is based on a detailed study of the annihilator of the quadratic OS ideal I_2 inside the full tensor algebra, and is reported on in [6]. At this point there seems to be no easily stated matroidal criterion equivalent to quadraticity.

Theorem 4.2 is actually a corollary of a more general result concerning $\mathcal{A}_2$. We define a set $nbb(G)$ of increasing subsets of $[n]$ by $S = (i_1, \ldots, i_p)_< \in nbb(G)$ if and only if $i_j = \min \ell c(\{i_j, \ldots, i_p\})$ for all j. This is an analogue of the set $nbc(G)$ of *nbc* (= 'no-broken-circuit') sets of G [1]. In fact these sets are precisely the *NBB* (= 'not-bounded-below') sets of A. Blass and B. Sagan [2], which generalize *nbc* sets, for the lattice of line-closed sets of G, with a linear ordering of the atoms. It is the case that $nbb(G) = nbc(G)$ if and only if G is line-closed. Then 4.2 follows easily from the next theorem.

THEOREM 4.3. *The set of monomials $\{a_S | S \in nbb(G)\}$ forms a linearly independent subset of $\mathcal{A}_2$.*

This generalizes half of the well-known theorem [1, 17] that $\{a_S | S \in nbc(G)\}$ yields a basis for the OS algebra $\mathcal{A}$. Yuzvinsky's example shows that the set $\{a_S | S \in nbb(G)\}$ cannot form a basis for $\mathcal{A}_2$ in general.

An analogue of Theorem 4.3 holds for $\mathcal{A}_k$ for each $k \geq 2$, giving a partial solution to Problem 4.1 in the form of combinatorial lower bounds. Of course, a formula for the cardinality of $nbb(G)$ has not been found. In fact, this cardinality can change if the linear order of the points is changed.

PROBLEM 4.4. Calculate the maximal cardinality of $nbb(G)$ over all linear orderings of the points of G.

L. Paris has informed us that $\{\partial e_C | C \text{ is a circuit}\}$ can be shown directly to be a Gröbner basis for the OS ideal $\mathcal{I}$. A complete direct proof is seemingly not extant. The fact that *nbc* monomials form a basis for $\mathcal{A}$ is an immediate consequence. In fact the latter assertion implies the former—see [23, Theorem 4.1]. The following problem seems more delicate.

PROBLEM 4.5. Find a Gröbner basis for the quadratic OS ideal $\mathcal{I}_2$.

Our experiments lead us to another interesting question, which seems to be related.

PROBLEM 4.6. Determine conditions on S under which ∂e_S will lie in the k-adic OS ideal $\mathcal{I}_k$.

We close by returning to the invariant ϕ_3 defined in the Introduction. It turns out that a calculation of $\dim(\mathcal{A}_2^3)$ would yield a combinatorial formula for ϕ_3. Indeed, ϕ_3 is the nullity of $\mathcal{E}^1 \otimes \mathcal{I}^2 \longrightarrow \mathcal{E}^3$, while the cokernel of the same map is precisely $\mathcal{A}_2^3$. The dimension of $\mathcal{I}^2$ is just $\dim(\mathcal{E}^2) - \dim(\mathcal{A}^2)$, so we obtain the following formula.

THEOREM 4.7. *Let $n = \operatorname{rk}(G)$ and $w_2 = \dim(\mathcal{A}^2)$, the second Whitney number of G. Then*

$$\phi_3 = 2\binom{n+1}{3} - nw_2 + \dim(\mathcal{A}_2^3).$$

Thus Problem 1.4 is a special case of Problem 4.1.

ACKNOWLEDGEMENTS

I am grateful to Raul Cordovil and Michel Las Vergnas for inviting me to speak at the CIRM conference, and for providing financial support. I thank Sergey Yuzvinsky for his help in studying quadratic OS algebras, and Alex Suciu for his suggestion to include resonance varieties over $\mathbb{Z}_p$, and his help in understanding them. My *REU* students Carrie Eschenbrenner, Cayley Pendergrass, and Samantha Melcher assisted me in sorting out much of the material in the last two sections. Finally I wish to thank Diane MacLagan for a helpful correspondence concerning Gröbner bases.

REFERENCES

1. A. Björner, On the homology of geometric lattices, *Algebr. Universalis*, **14** (1982), 107–128.
2. A. Blass and B. Sagan, Möbius functions of lattices, *Adv. Math.*, **127** (1997), 94–123.
3. T. H. Brylawski, A combinatorial model for series-parallel networks, *Trans. Am. Math. Soc.*, **154** (1971), 1–22.
4. D. Cohen and P. Orlik, Arrangements and local systems. preprint, 1999, available at `http://xxx.lanl.gov/abs/math.AG/9907117`.
5. D. Cohen and A. Suciu, Characteristic varieties of arrangements, *Math. Proc. Camb. Phil. Soc.*, **127** (1999), 33–53.
6. G. Denham and S. Yuzvinsky, Annihilators of ideals of exterior algebras, preprint, 2000, available at `http://xxx./an/.gov/abs/math.CO/0010168`.
7. C. Eschenbrenner and M. Falk, Orlik–Solomon algebras and Tutte polynomials, *J. Algebr. Comb.*, **10** (1999), 189–199.
8. M. Falk, Arrangements and cohomology, *Ann. Comb.*, **1** (1997), 135–157.
9. M. Falk, Formal arrangements, line-closed matroids and quadratic algebras, preprint, 2000, available at `http://xxx./an/.gov/abs/math.CO/0010167`.
10. M. Falk, Homotopy types of line arrangements, *Inventiones Mathematicae*, **111** (1993), 139–150.
11. M. Falk, On the algebra associated with a geometric lattice, *Adv. Math.*, **80** (1989), 152–163.
12. M. Falk, Pencils of curves and $K(\pi, 1)$ arrangements, 1997, invited 20-minute talk, Special session on *Computational Algebraic Geometry*, 918th meeting of the AMS, San Diego.
13. M. Falk, *Quadratic Algebras and Line-closed Matroids*, invited 60-minute lecture, Arrangements in Boston, Northeastern University, Boston, 1999.

14. M. Falk, The minimal model of the complement of an arrangement of hyperplanes, *Trans. Am. Math. Soc.*, **309** (1988), 543–556.
15. M. Falk and R. Randell, On the homotopy theory of arrangements, II, in: *Arrangements in Tokyo, 1998, Advanced Studies in Mathematics*, **27** (2000), 185–215.
16. M. Falk and H. Terao, βnbc bases for cohomology of local systems on hyperplane complements, *Trans. Am. Math. Soc.*, **349** (1997), 189–202.
17. M. Jambu and H. Terao, Arrangements of hyperplanes and broken circuits, in: *Singularities, Contemporary Mathematics*, **90**, American Mathematical Society, 1989, pp. 147–162.
18. A. Libgober, First order deformations of complexes with nonvanishing cohomology, preprint, 1999.
19. A. Libgober and S. Yuzvinsky, Cohomology of the Orlik–Solomon algebras and local systems, *Comp. Math.*, **121** (2000), 337–361.
20. D. Matei and A. Suciu, Cohomology rings and nilpotent quotients of real and complex arrangements, in: *Arrangements in Tokyo, 1998, Advanced Studies in Mathematics*, **27** (2000), 185–215.
21. P. Orlik and L. Solomon, Topology and combinatorics of complements of hyperplanes, *Inventiones Mathematicae*, **56** (1980), 167–189.
22. P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer Verlag, Berlin, 1992.
23. I. Peeva, Hyperplane arrangements and linear strands in resolutions, *Trans. Am. Math. Soc.*, to appear.
24. C. Pendergrass, Isomorphic truncations of isomorphic connections, Technical Report, Northern Arizona University *REU* Program, 1998.
25. N. White (ed.), *Theory of Matroids*, Cambridge University Press, Cambridge, 1986.
26. N. White (ed.), *Combinatorial Geometries*, Cambridge University Press, Cambridge, 1987.
27. S. Yuzvinsky, Cohomology of Brieskorn-Orlik–Solomon algebras, *Commun. Algebra*, **23** (1995), 5339–5354.

Received 1 January 2000 in revised form 31 July 2000

MICHAEL FALK
Department of Mathematics and Statistics,
Northern Arizona University,
Flagstaff, AZ 86011-5717
E-mail: michael.falk@nau.edu