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LOCAL COMPACTNESS AND HEWITT REALCOMPACTIFICATIONS OF PRODUCTS II

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We generalize and refine some results from the author's paper [18]. For a completely regular Hausdorff space X, υX denotes the Hewitt realcompactification of X. It is proved that if $\upsilon(X\times Y)=\upsilon X\times\upsilon Y$ for any metacompact subparacompact (or $\mathfrak{m}$-paracompact) space Y, then X is locally compact. A $P(\mathfrak{n})$-space is a space in which every intersection of less than $\mathfrak{n}$ open sets is open. A characterization of those spaces X such that $\upsilon(X\times Y)=\upsilon X\times\upsilon Y$ for any (metacompact) $P(\mathfrak{n})$-space Y is also obtained.

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Hewitt realcompactification | realcompact | product space
locally compact | metacompact | subparacompact
$\mathfrak{m}$-paracompact | $P(\mathfrak{n})$-space | C-embedding
weakly-$\mathfrak{n}$-compact | non-measurable cardinal

1. Introduction and main theorems

All spaces considered are assumed to be completely regular Hausdorff and all maps are continuous. The Hewitt realcompactification υX of a space X is the unique realcompactification of X to which each real-valued continuous function on X admits a continuous extension. For details of Hewitt realcompactifications, the reader is referred to [9]. The purpose of this paper is to prove a refinement and a generalization of the following theorem due to Comfort [6], Hušek [12, 13], McArthur [17] and the author [18].

1.1. Theorem. *The following conditions on a space X are equivalent*:
(a) *X is locally compact, realcompact and $|X|<\mathfrak{m}_1$.*
(b) *$\upsilon(X\times Y)=\upsilon X\times\upsilon Y$ for any space Y.*

Here, $|X|$ denotes the cardinality of X and $\mathfrak{m}_1$ stands for the first measurable cardinal. Thus the inequality $|X|<\mathfrak{m}_1$ means that the cardinality of X is non-measurable. Following [17], let $\mathcal{R}$ denote the class of all spaces X satisfying 1.1(b).

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In this paper, for a property $\mathscr{P}$ of spaces we denote the class of all spaces X such that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for any $\mathscr{P}$-space Y by $\mathscr{R}(\mathscr{P})$. Let $\mathfrak{l}$, $\mathfrak{m}$, and $\mathfrak{n}$ denote infinite cardinals; $\mathfrak{l}^+$ is the smallest cardinal greater than $\mathfrak{l}$. Recall that a space is *metacompact* (resp. *subparacompact*) if each open cover has a point finite open (resp. σ-locally finite closed) refinement, and that a space is $\mathfrak{m}$*-paracompact* if each open cover of cardinality at most $\mathfrak{m}$ has a locally finite open refinement (cf. [1]). For a space X, $\chi(X)$ and $w(X)$ denote the character and the weight of X, respectively. Our first result shows that $\mathscr{R} = \mathscr{R}$(metacompact and subparacompact) = $\mathscr{R}$($\mathfrak{m}$-paracompact).

1.2. Theorem. *Each of the following conditions on a space X is equivalent to* 1.1(a):

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ *for any metacompact subparacompact space Y with* $w(Y) \leq \chi(\upsilon X) \cdot \aleph_1$.

(d) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ *for any $\mathfrak{m}$-paracompact space Y with* $w(Y) \leq \chi(\upsilon X) \cdot \mathfrak{m}^+$.

A *0-dimensional space* is a space which has a base consisting of open-and-closed sets. As the reader will observe in the proof, we can add 0-dimensionality to the conditions on Y in (c) and (d).

A $P(\mathfrak{n})$*-space* is a space in which every intersection of less than $\mathfrak{n}$ open sets is open. Any space is a $P(\aleph_0)$-space and a $P(\aleph_1)$-space usually is called a P-space. Recall from [10] (or [7]) that a space is *weakly-$\mathfrak{n}$-compact* if each open cover has a subfamily of cardinality less than $\mathfrak{n}$ with dense union. A space is called *locally weakly-$\mathfrak{n}$-compact* if each point has a weakly-$\mathfrak{n}$-compact neighborhood. As far as I know, this notion first appears in [14]. In [14] Hušek proved that if X is a locally weakly-$\mathfrak{n}$-compact, realcompact space with $|X| < \mathfrak{m}_1$ and Y is a $P(\mathfrak{n})$-space, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. Let $\mathfrak{n}^*$ denote the smallest regular cardinal not less than $\mathfrak{n}$. The next theorem generalizes his theorem as well as Theorem 1.1, and gives a characterization of $\mathscr{R}(P(\mathfrak{n}))$ and $\mathscr{R}$(metacompact $P(\mathfrak{n})$).

1.3. Theorem. *For any infinite cardinal* $\mathfrak{n}$, *the following conditions on a space X are equivalent*:

(a′) *Each point of υX has a neighborhood G in υX such that $G \cap X$ is weakly-$\mathfrak{n}^*$-compact and* $|X| < \mathfrak{m}_1$.

(b′) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ *for any $P(\mathfrak{n})$-space Y.*

(c′) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ *for any metacompact $P(\mathfrak{n})$-space Y with* $w(Y) \leq \exp w(\upsilon X)$.

It will be remarked in the final section that if $\mathfrak{n} < \mathfrak{m}_1$ and if each $\mathscr{P}$- and $P(\mathfrak{n})$-space is normal countably paracompact, then $\mathscr{R}(\mathscr{P}$ and $P(\mathfrak{n})) \neq \mathscr{R}(P(\mathfrak{n}))$. Therefore, in case $\aleph_1 \leq \mathfrak{n} < \mathfrak{m}_1$, metacompactness of Y in (c′) cannot be replaced by subparacompactness since a subparacompact $P(\aleph_1)$-space is paracompact [2]. Further, since normal metacompact spaces and normal subparacompact spaces are known [8] to be countably paracompact, we cannot add normality to the conditions on Y in (c), (d) and (c').

Hereafter $C(X)$ denotes the set of all real-valued continuous functions on a space X. For an ordinal α, $W(\alpha)$ denotes the space of all ordinals less than α, topologized with the usual interval topology, and ω_0 denotes the first infinite ordinal. For general terminology, see [1, 9].

2. Preliminaries

We list certain basic facts and definitions that will be used in the sequal. Let X and Y be spaces.

2.1. *If Y is a C-embedded subspace of X, then $\upsilon Y = \mathrm{cl}_{\upsilon X} Y$* [9, Theorem 8.10].

2.2. *If Y is a cozero-set of υX, then $\upsilon(Y \cap X) = Y$* [3, Theorem 5.1].

Recall from [13] that a space is *pseudo-$\mathfrak{m}_1$-compact* if every locally finite family of non-empty open sets is of non-measurable cardinal. A map $f: X \to Y$ is called *z-closed* if the image of each zero-set of X is closed in Y.

2.3. *If the projection $\pi_Y: X \times Y \to Y$ is z-closed, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ if and only if either $|X| < \mathfrak{m}_1$ or Y is pseudo-$\mathfrak{m}_1$-compact* [14, Theorem 2].

2.4. *If X is weakly-$\mathfrak{n}^*$-compact and Y is a $P(\mathfrak{n})$-space, then the projection $\pi_Y: X \times Y \to Y$ is z-closed* [11; 10, Theorem 3.1].

The following result 2.5 is essentially proved by McArthur, and 2.5 follows from [13, Theorem 3] and [15, Theorem 2].

2.5. *If $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for any 0-dimensional paracompact space Y with $w(Y) \leq \chi(\upsilon X)$, then X is realcompact* [17, Theorem 5.2].

2.6. *If $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for any discrete space Y with $|Y| \leq \chi(\upsilon X)$, then $|X| < \mathfrak{m}_1$.*

2.7. Let $\mathfrak{m} \geq \mathfrak{n}$. Let S be a set of cardinality $\mathfrak{m}$, and let $\mathcal{S}$ be the family of all subsets σ of S with $|\sigma| < \mathfrak{n}$. Define $\Sigma(\mathfrak{m}, \mathfrak{n})$ to be the space $\mathcal{S} \cup \{\infty\}$ topologized as follows: Each point of $\mathcal{S}$ is isolated and $\{J(\sigma) \mid \sigma \in \mathcal{S}\}$, where $J(\sigma) = \{\infty\} \cup \{\sigma' \in \mathcal{S} \mid \sigma' \supset \sigma\}$, is a neighborhood base of ∞. For a space Z, let $\Sigma(Z, \mathfrak{m}, \mathfrak{n})$ denote the space obtained from the product space $Z \times \Sigma(\mathfrak{m}, \mathfrak{n})$ by letting each point of $Z \times \mathcal{S}$ be isolated. The following simple facts are listed without proofs.

(a) *$w(\Sigma(Z, \mathfrak{m}, \mathfrak{n})) \leq w(Z) \cdot \exp \mathfrak{m}$ and $w(\Sigma(Z, \mathfrak{m}, \aleph_0)) \leq w(Z) \cdot \mathfrak{m}$.*

(b) *If Z has one of the following properties, then $\Sigma(Z, \mathfrak{m}, \mathfrak{n})$ has the same property: metacompactness, subparacompactness, $\mathfrak{k}$-paracompactness, normality, 0-dimensionality.*

(c) *If Z is a $P(\mathfrak{n})$-space and $\mathfrak{n}$ is regular, then $\Sigma(Z, \mathfrak{m}, \mathfrak{n})$ is a $P(\mathfrak{n})$-space.*

3. Proofs of Theorems 1.2 and 1.3

In proving Theorems 1.2 and 1.3, the central issue is how to find a space Y such that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$ when X is not locally weakly-$\mathfrak{n}$-compact. The following lemma reduces this issue to the problem of finding a space Z which has a certain locally finite family of subsets. We call a space *weakly-$(\mathfrak{m}, \mathfrak{n})$-compact* if each open cover of cardinality $\mathfrak{m}$ has a subfamily of cardinality less than $\mathfrak{n}$ with dense union, and denote the character at a point x in X by $\chi(x, X)$.

3.1. Lemma. *Let X be a space having a point x_0, with $\chi(x_0, X) \leq \mathfrak{k}$, that has no weakly-$(\mathfrak{m}, \mathfrak{n})$-compact neighborhood. Let Z be a space having a locally finite family $\mathcal{F}$ of subsets in Z such that $|\mathcal{F}| = \mathfrak{k}$ and $\bigcap\{\mathrm{cl}_{\upsilon Z} F \mid F \in \mathcal{F}\} \neq \emptyset$. Then $X \times Y$ is not C-embedded in $X \times \upsilon Y$, where $Y = \Sigma(Z, \mathfrak{m}, \mathfrak{n})$.*

Proof. Recall that Y is the space obtained from $Z \times \Sigma(\mathfrak{m}, \mathfrak{n})$ by letting each point of $Z \times \mathcal{S}$ be isolated, and $\{J(\sigma) \mid \sigma \in \mathcal{S}\}$ is a neighborhood base of ∞ in $\Sigma(\mathfrak{m}, \mathfrak{n})$ $(= \mathcal{S} \cup \{\infty\})$. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a neighborhood base of x_0 in X with $|\Lambda| = \mathfrak{k}$. For each $\lambda \in \Lambda$, since $\mathrm{cl}_X G_\lambda$ is not weakly-$(\mathfrak{m}, \mathfrak{n})$-compact, there is an open cover $\mathcal{U}_\lambda$ of X with $|\mathcal{U}_\lambda| = \mathfrak{m}$ such that no subfamily of cardinality less than $\mathfrak{n}$ has dense union in G_λ. By the definition of $\mathcal{S}$, we may denote the collection of all subfamilies of $\mathcal{U}_\lambda$ whose cardinality is less than $\mathfrak{n}$ by $\{\mathcal{U}_{\lambda\sigma} \mid \sigma \in \mathcal{S}\}$, and we may assume that $\sigma \subset \sigma'$ if and only if $\mathcal{U}_{\lambda\sigma} \subset \mathcal{U}_{\lambda\sigma'}$. For each $\sigma \in \mathcal{S}$, let $H_{\lambda\sigma} = G_\lambda - \mathrm{cl}_X(\bigcup\{U \mid U \in \mathcal{U}_{\lambda\sigma}\})$. Then $H_{\lambda\sigma} \neq \emptyset$, so pick $x_{\lambda\sigma} \in H_{\lambda\sigma}$. On the other hand, since $|\mathcal{F}| = \mathfrak{k}$, we may write $\mathcal{F} = \{F_\lambda \mid \lambda \in \Lambda\}$. For each $\lambda \in \Lambda$ and each $\sigma \in \mathcal{S}$, set

$$J_{\lambda\sigma} = \{x_{\lambda\sigma}\} \times (\mathrm{cl}_Z F_\lambda \times \{\sigma\}) \subset X \times Y,$$

$$K_{\lambda\sigma} = H_{\lambda\sigma} \times (\mathrm{cl}_Z F_\lambda \times \{\sigma\}) \subset X \times Y.$$

Since $\mathrm{cl}_Z F_\lambda \times \{\sigma\}$ is open and closed in Y, there is $f_{\lambda\sigma} \in C(X \times Y)$ such that $f_{\lambda\sigma}(J_{\lambda\sigma}) = \{0\}$ and $f_{\lambda\sigma}((X \times Y) - K_{\lambda\sigma}) = \{1\}$. Let us show that $\mathcal{K} = \{K_{\lambda\sigma} \mid \lambda \in \Lambda, \sigma \in \mathcal{S}\}$ is locally finite in $X \times Y$. Let $p = (x, y) \in X \times Y$; then $y = (z, \tau)$ for some $z \in Z$ and some $\tau \in \Sigma(\mathfrak{m}, \mathfrak{n})$. Since $\mathcal{F}$ is locally finite, z has a neighborhood $G(z)$ in Z which meets only finitely many members, say $F_{\lambda_1}, \ldots, F_{\lambda_n}$, of $\mathcal{F}$. In case $\tau \in \mathcal{S}$, $X \times (G(z) \times \{\tau\})$ is a neighborhood of p which meets only $K_{\lambda_1\tau}, \ldots, K_{\lambda_n\tau}$, so suppose $\tau = \infty$. For each $i = 1, \ldots, n$, choose $U_i \in \mathcal{U}_{\lambda_i}$ with $x \in U_i$; then $\{U_i\} = \mathcal{U}_{\lambda_i\sigma_i}$ for some $\sigma_i \in \mathcal{S}$. If we set $G(x) = U_1 \cap \cdots \cap U_n$, then $G(x) \cap H_{\lambda_i\sigma_i} = \emptyset$ for each i. Let $\sigma_0 = \sigma_1 \cup \cdots \cup \sigma_n$, and set $G(p) = G(x) \times (G(z) \times J(\sigma_0))$. Then $G(p)$ is a neighborhood of p which meets no member of $\mathcal{K}$. For, if $G(p) \cap K_{\lambda\sigma} \neq \emptyset$, then $G(x) \cap$

$H_{\lambda\sigma} \neq \emptyset$, $\sigma \supset \sigma_0$ and $\lambda = \lambda_i$ for some i. Since $\mathcal{U}_{\lambda_i\sigma} \supset \mathcal{U}_{\lambda_i\sigma_i}$, $H_{\lambda\sigma} = H_{\lambda_i\sigma} \subset H_{\lambda_i\sigma_i}$, and hence $G(x) \cap H_{\lambda_i\sigma_i} \neq \emptyset$. This is a contradiction, that proves local finiteness of $\mathcal{H}$. Therefore if we define a function f on $X \times Y$ by

$$f(q) = \inf\{f_{\lambda\sigma}(q) \mid \lambda \in \Lambda, \sigma \in \mathcal{S}\}, \quad q \in X \times Y,$$

then f is continuous. To see that f admits no continuous extension over $X \times \upsilon Y$, choose $z_0 \in \bigcap\{\mathrm{cl}_{\upsilon Z} F_\lambda \mid \lambda \in \Lambda\}$. Then $z_0 \in \upsilon Z - Z$. Since $Z \times \{\infty\}$ is C-embedded in Y, it follows from 2.1 that $\upsilon Z = \upsilon(Z \times \{\infty\}) \subset \upsilon Y$, and so we may consider z_0 as an element of $\upsilon Y - Y$. Let $V \times W$ be a given neighborhood of (x_0, z_0) in $X \times \upsilon Y$. Then there is $\lambda \in \Lambda$ with $G_\lambda \in V$. Since $W \cap (F_\lambda \times \{\infty\}) \neq \emptyset$, we can find $z \in F_\lambda$ and $\sigma \in \mathcal{S}$ such that $(z, \infty) \in W$ and $(z, \sigma) \subset W$. Then both $p_1 = (x_0, (z, \infty))$ and $p_2 = (x_{\lambda\sigma}, (z, \sigma))$ belong to $V \times W$ and $f(p_1) = 1$, while $f(p_2) = 0$. This shows that f does not extend continuously to (x_0, z_0). Hence the proof is complete.

3.2. Fact. *For every two infinite cardinals $\mathfrak{k}$ and $\mathfrak{n}$, there exists a 0-dimensional metacompact $P(\mathfrak{n})$-space $Z = Z(\mathfrak{k}, \mathfrak{n})$, with $w(Z) = \mathfrak{k} \cdot (\mathfrak{n}^*)^+$, that has a discrete family $\mathcal{F}$ of closed sets in Z such that $|\mathcal{F}| = \mathfrak{k}$ and $\bigcap\{\mathrm{cl}_{\upsilon Z} F \mid F \in \mathcal{F}\} \neq \emptyset$. Moreover the space $Z = Z(\mathfrak{k}, \aleph_0)$ is subparacompact.*

Proof. Let α_1 (resp. α_2) be the initial ordinal of $(\mathfrak{n}^*)^+$ (resp. $\mathfrak{n}^*$). Define T_i, $i = 1, 2$, to be the subspace of $W(\alpha_i + 1)$ obtained by deleting all non-isolated points except α_i. Let us set $T = (T_1 \times T_2) - \{t_0\}$, where $t_0 = (\alpha_1, \alpha_2)$. Then T is a 0-dimensional metacompact $P(\mathfrak{n})$-space with $w(T) = (\mathfrak{n}^*)^+$. Since T is C-embedded in $T_1 \times T_2$, it follows from 2.1 that

$$\upsilon T \supset T_1 \times T_2. \tag{1}$$

Let us set $E = \{\alpha_1\} \times (T_2 - \{\alpha_2\})$ and $F = (T_1 - \{\alpha_1\}) \times \{\alpha_2\}$; then E and F are disjoint closed subsets of T such that

$$t_0 \in \mathrm{cl}_{\upsilon T} E \cap \mathrm{cl}_{\upsilon T} F. \tag{2}$$

Let Λ be the discrete space of cardinality $\mathfrak{k}$, and let Z' be the quotient space obtained from $T \times \Lambda$ by collapsing the set $\{e\} \times \Lambda$ to a point for each $e \in E$. Let $\phi' : T \times \Lambda \to Z'$ be the quotient map. We denote a base for the topology on T by $\mathcal{B}$. Let Z be the set Z', retopologized by letting $\bigcup\{\mathcal{B}(B) \mid B \in \mathcal{B}\}$ be a base, where

$$\mathcal{B}(B) = \begin{cases} \{\phi'(B \times \Lambda)\}, & \text{if } B \cap E \neq \emptyset, \\ \{\phi'(B \times \{\lambda\}) \mid \lambda \in \Lambda\}, & \text{if } B \cap E = \emptyset. \end{cases}$$

Then the space Z is easily seen to be a 0-dimensional metacompact $P(\mathfrak{n})$-space with $w(Z) = \mathfrak{k} \cdot (\mathfrak{n}^*)^+$, and the natural map $\phi : T \times \Lambda \to Z$ is continuous. In case $\mathfrak{n} = \aleph_0$, Z is subparacompact in addition, since it is the countable union of paracompact closed subspaces. Setting $F_\lambda = \phi(F \times \{\lambda\})$ for each $\lambda \in \Lambda$, we have a discrete family $\{F_\lambda \mid \lambda \in \Lambda\}$ of closed sets in Z. It remains to show that $\bigcap\{\mathrm{cl}_{\upsilon Z} F_\lambda \mid \lambda \in \Lambda\} \neq \emptyset$.

There is a continuous extension $\Phi : \upsilon(T\times\Lambda)\to\upsilon Z$ of ϕ. By (1), $\upsilon(T\times\Lambda)\supset\upsilon T\times\Lambda\supset(T_1\times T_2)\times\Lambda$. For each $\lambda\in\Lambda$, let $z(\lambda)=\Phi((t_0,\lambda))$; then $z(\lambda)\in \mathrm{cl}_{\upsilon Z}F_\lambda$. If $z(\lambda_1)\neq z(\lambda_2)$ for some $\lambda_1,\lambda_2\in\Lambda$, then they have disjoint neighborhoods U_1 and U_2 in υZ, respectively. For $i=1,2$, since $\Phi^{-1}(U_i)$ is a neighborhood of (t_0,λ_i), there is a neighborhood G_i of t_0 in υT such that $G_i\times\{\lambda_i\}\subset\Phi^{-1}(U_i)$. Then, since $\Phi(E\times\{\lambda_1\})=\Phi(E\times\{\lambda_2\})$ and $U_1\cap U_2=\emptyset$, $G_1\cap G_2\cap E=\emptyset$, that contradicts (2). Thus $z(\lambda_1)=z(\lambda_2)$ for each $\lambda_1,\lambda_2\in\Lambda$, and consequently $\bigcap\{\mathrm{cl}_{\upsilon Z}F_\lambda\mid\lambda\in\Lambda\}\neq\emptyset$. Hence Z is proved to be the desired space $Z(\mathfrak{k},\mathfrak{n})$.

3.3. Fact. *For every two infinite cardinals $\mathfrak{k}$ and $\mathfrak{m}$, there exists a 0-dimensional $\mathfrak{m}$-paracompact space $Z=Z_{\mathfrak{m}}(\mathfrak{k})$, with $w(Z)=\mathfrak{k}\cdot\mathfrak{m}^+$, that has a discrete family $\mathcal{F}$ of closed sets in Z such that $|\mathcal{F}|=\mathfrak{k}$ and $\bigcap\{\mathrm{cl}_{\upsilon Z}F\mid F\in\mathcal{F}\}\neq\emptyset$.*

Proof. We utilize a space similar to the space Y constructed by Comfort in [5, p. 99]. Let α be the initial ordinal of $\mathfrak{m}^+$, and let S_0 be the quotient space obtained from the product space

$$S_1=W(\omega_0)\times W(\alpha+1)\times W(\alpha+1)$$

by identifying, for each $n<\omega_0$ and each $\gamma\leq\alpha$, two points (n,α,γ) and $(n+1,\gamma,\alpha)$. Let $f:S_1\to S_0$ be the quotient map. If we set $S=S_0-\{s_0\}$, where $s_0=f((0,\alpha,\alpha))$, then $\upsilon S=S_0$ as he showed in [5]. Let $T=W(\omega_0+1)\times S$. Then, by [16, Theorem 17] T is a 0-dimensional $\mathfrak{m}$-paracompact space with $w(T)=\mathfrak{m}^+$, and it follows from [4, Theorem 5.3] that

$$\upsilon T=W(\omega_0+1)\times\upsilon S. \tag{3}$$

Setting $H_n=f(\{i\mid i\geq n\}\times W(\alpha+1)\times W(\alpha+1))\cap S$ for each $n<\omega_0$, we have a decreasing sequence $\{H_n\}$ of closed sets in S with empty intersection such that $s_0\in\bigcap\{\mathrm{cl}_{\upsilon S}H_n\mid n<\omega_0\}$. Let us set $E=\{\omega_0\}\times S$, $F=\bigcup\{\{n\}\times H_n\mid n<\omega_0\}$ and $t_0=(\omega_0,s_0)$; then E and F are disjoint closed subsets of T such that

$$t_0\in \mathrm{cl}_{\upsilon T}E\cap \mathrm{cl}_{\upsilon T}F. \tag{4}$$

Let Λ be the discrete space of cardinality $\mathfrak{k}$. Let Z be the space obtained from $T\times\Lambda$ by the same way just as in the proof of Fact 3.2, and let $\phi:T\times\Lambda\to Z$ be the natural map. Then the space Z is easily seen to be $\mathfrak{m}$-paracompact since it is no other than the product of S and a metric space with only one non-isolated point. (Use [16, Theorem 5] and the following fact: if Y is a countably paracompact space and X is a metric space, then any closed subset of $X\times Y$ disjoint from $\{x\}\times Y$, where $x\in X$, has a neighborhood whose closure misses $\{x\}\times Y$.) Moreover, Z is 0-dimensional and $w(Z)=\mathfrak{k}\cdot\mathfrak{m}^+$. Let $F_\lambda=\phi(F\times\{\lambda\})$ for each $\lambda\in\Lambda$. Then $\{F_\lambda\mid\lambda\in\Lambda\}$ is a discrete family of closed sets in Z, and a similar argument to the proof of Fact 3.2 shows that $\bigcap\{\mathrm{cl}_{\upsilon Z}F_\lambda\mid\lambda\in\Lambda\}\neq\emptyset$. Hence Z is the desired space $Z_{\mathfrak{m}}(\mathfrak{k})$.

3.4. Remark. Let Ψ be the space described in [9, 5I, p. 79]. The space Ψ is known to be a Moore space which is pseudocompact but not countably compact. Thus Ψ

has a decreasing sequence $\{H_n\}$ of closed subsets with empty intersection such that $\bigcap \mathrm{cl}_{\upsilon\Psi} H_n \neq \emptyset$. If one use Ψ instead of S in the proof of Fact 3.3, then the resulting space Z is a 0-dimensional Moore space with $w(Z) = \mathfrak{k} \cdot \exp \aleph_0$. This space will be used in [19] to characterize the class $\mathfrak{R}$(Moore).

Before proving Theorems 1.2 and 1.3, we establish the following theorem which is a generalization of [6, Theorem 2.1] and [18, Theorem 1].

3.5. Theorem. *Let $\mathfrak{n}$ be an infinite cardinal. Then the following conditions on a space X, with $|X| < \mathfrak{m}_1$, are equivalent*:

(a) *X is locally weakly-$\mathfrak{n}^*$-compact.*

(b) *$X \times Y$ is C-embedded in $X \times \upsilon Y$ for any $P(\mathfrak{n})$-space Y.*

(c) *$X \times Y$ is C-embedded in $X \times \upsilon Y$ for any metacompact $P(\mathfrak{n})$-space Y with $w(Y) \leq \exp w(X)$.*

In case $\mathfrak{n} = \aleph_0$, the following conditions (c′) *and* (d) *are also equivalent to* (a):

(c′) *$X \times Y$ is C-embedded in $X \times \upsilon Y$ for any 0-dimensional metacompact subparacompact space Y with $w(Y) \leq w(X) \cdot \aleph_1$.*

(d) *$X \times Y$ is C-embedded in $X \times \upsilon Y$ for any 0-dimensional $\mathfrak{m}$-paracompact space Y with $w(Y) \leq w(X) \cdot \mathfrak{m}^+$.*

Proof. The implication (a) → (b) is a simple consequence of 2.3 and 2.4, and (b) → (c) is obvious. To prove (c) → (a), suppose on the contrary that X is not locally weakly-$\mathfrak{n}^*$-compact at $x_0 \in X$. Let Z be the space $Z(\mathfrak{k}, \mathfrak{n}^*)$ constructed in Fact 3.2, where $\mathfrak{k} = \chi(x_0, X)$, and let $Y = \Sigma(Z, w(X), \mathfrak{n}^*)$. Then by 2.7 Y is a metacompact $P(\mathfrak{n})$-space, and $w(Y) \leq \mathfrak{k} \cdot (\mathfrak{n}^*)^+ \cdot \exp w(X) = \exp w(X)$. Since x_0 has no weakly-$(w(X), \mathfrak{n}^*)$-compact neighborhood and Z has a locally finite family $\mathcal{F}$ of subsets such that $|\mathcal{F}| = \mathfrak{k}$ and $\bigcap\{\mathrm{cl}_{\upsilon Z} F \mid F \in \mathcal{F}\} \neq \emptyset$, it follows from Lemma 3.1 that $X \times Y$ is not C-embedded in $X \times \upsilon Y$. This contradiction establishes the implication. In case $\mathfrak{n} = \aleph_0$, the implications (a) → (c′) and (a) → (d) follow from [6, Theorem 2.1], and the proof that (c′) → (a) ((d) → (a)) is the same as above if one use $Z(\mathfrak{k}, \aleph_0)$ $(Z_{\mathfrak{m}}(\mathfrak{k}))$ instead of $Z(\mathfrak{k}, \mathfrak{n}^*)$. Hence the proof is complete.

3.6. Proof of Theorem 1.2. Since the implications 1.1(a) → (c) and 1.1(a) → (d) follow from [6, Corollary 2.2], we prove that (c) ((d)) implies 1.1(a). By 2.5 and 2.6, (c) ((d)) implies that X is realcompact and $|X| < \mathfrak{m}_1$. To complete the proof suppose that X is not locally compact at $x_0 \in X$. Then each neighborhood of x_0 is not pseudocompact, because a realcompact pseudocompact space is compact, and so x_0 has no weakly-$(\aleph_0, \aleph_0)$-compact neighborhood. Let $Z = Z(\mathfrak{k}, \aleph_0)$ $(Z = Z_{\mathfrak{m}}(\mathfrak{k}))$, where $\mathfrak{k} = \chi(x_0, X)$, and let $Y = \Sigma(Z, \aleph_0, \aleph_0)$. Then Y satisfies the condition stated in (c) ((d)), and it follows from Lemma 3.1 that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$. Hence the proof is complete.

3.7. Proof of Theorem 1.3. (a′) → (b′). Let Y be a $P(\mathfrak{n})$-space. Since υX is locally weakly-$\mathfrak{n}^*$-compact, $\upsilon(\upsilon X \times Y) = \upsilon X \times \upsilon Y$ by Theorem 3.5, and so it remains to prove that $X \times Y$ is C-embedded in $\upsilon X \times Y$. Let $f \in C(X \times Y)$, and let $x \in \upsilon X - X$. It suffices to find a neighborhood G of x such that f admits a continuous extension over $(X \times Y) \cup (G \times Y)$. Choose a cozero-set neighborhood G of x in υX such that $X \cap \mathrm{cl}_{\upsilon X} G$ is weakly-$\mathfrak{n}^*$-compact. If we set $X_1 = X \cap \mathrm{cl}_{\upsilon X} G$, then it follows from 2.3 and 2.4 that $X_1 \times Y$ is C-embedded in $\upsilon X_1 \times Y$. There is a cozero-set G_1 of υX_1 such that $G_1 \cap X_1 = G \cap X$. Then f can be continuously extended over $(X \times Y) \cup (G_1 \times Y)$. Since $G_1 = \upsilon(G_1 \cap X_1) = \upsilon(G \cap X) = G$ by 2.2, f admits a continuous extension over $(X \times Y) \cup (G \times Y)$, as required.

(b′) → (c′). Obvious.

(c′) → (a′). By 2.6, $|X| < \mathfrak{m}_1$, and it follows from Theorem 3.5 that X is locally weakly-$\mathfrak{n}^*$-compact. To complete the proof, suppose that (a′) is false at $x_0 \in \upsilon X - X$. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a neighborhood base of x_0 in υX with $|\Lambda| = \chi(\upsilon X)$. For each $\lambda \in \Lambda$, $X \cap \mathrm{cl}_{\upsilon X} G_\lambda$ is not weakly-$\mathfrak{n}^*$-compact, and thus there is an open cover $\mathcal{U}_\lambda$ of X such that no subfamily of cardinality less than $\mathfrak{n}^*$ has dense union in $X \cap G_\lambda$. With the notation in 2.7, let $\Sigma(\mathfrak{m}, \mathfrak{n}^*) = \mathcal{S} \cup \{\infty\}$, where $\mathfrak{m} = w(X)$. Since it can be assumed without loss of generality that $|\mathcal{U}_\lambda| = \mathfrak{m}$, we denote the collection of all subfamilies of $\mathcal{U}_\lambda$ whose cardinality is less than $\mathfrak{n}^*$ by $\{\mathcal{U}_{\lambda\sigma} \mid \sigma \in \mathcal{S}\}$, and we may then assume that $\sigma \subset \sigma'$ if and only if $\mathcal{U}_{\lambda\sigma} \subset \mathcal{U}_{\lambda\sigma'}$. For each $\sigma \in \mathcal{S}$, let $H_{\lambda\sigma} = (X \cap G_\lambda) - \mathrm{cl}_X(\bigcup\{U \mid U \in \mathcal{U}_{\lambda\sigma}\})$, and pick $x_{\lambda\sigma} \in H_{\lambda\sigma}$. Topologize Λ with the discrete topology. Let Y be the quotient space obtained from $\Sigma(\mathfrak{m}, \mathfrak{n}^*) \times \Lambda$ by collapsing the set $\{\infty\} \times \Lambda$ to a point $y_0 \in Y$, and let $\phi : \Sigma(\mathfrak{m}, \mathfrak{n}^*) \times \Lambda \to Y$ be the quotient map. Then Y is a 0-dimensional paracompact $P(\mathfrak{n})$-space with $w(Y) \leq \exp w(\upsilon X)$. For each $\lambda \in \Lambda$ and each $\sigma \in \mathcal{S}$, let $y_{\lambda\sigma} = \phi((\sigma, \lambda))$, and set

$$p_{\lambda\sigma} = (x_{\lambda\sigma}, y_{\lambda\sigma}) \in X \times Y,$$

$$K_{\lambda\sigma} = H_{\lambda\sigma} \times \{y_{\lambda\sigma}\} \subset X \times Y.$$

Since $\{y_{\lambda\sigma}\}$ is open in Y, there is $f_{\lambda\sigma} \in C(X \times Y)$ such that $f_{\lambda\sigma}(p_{\lambda\sigma}) = 0$ and $f_{\lambda\sigma}((X \times Y) - K_{\lambda\sigma}) = \{1\}$. We show that $\mathcal{K} = \{K_{\lambda\sigma} \mid \lambda \in \Lambda, \sigma \in \mathcal{S}\}$ is locally finite in $X \times Y$. Let $p = (x, y) \in X \times Y$. In case $y \neq y_0$, $X \times \{y\}$ is a neighborhood of p which meets only one element of $\mathcal{K}$. In case $y = y_0$, choose a weakly-$\mathfrak{n}^*$-compact neighborhood $G(x)$ of x; then for each $\lambda \in \Lambda$ there is $\sigma_\lambda \in \mathcal{S}$ such that $G(x) \subset \mathrm{cl}_X(\bigcup\{U \mid U \in \mathcal{U}_{\lambda\sigma_\lambda}\})$. Let $G(y) = \bigcup\{\phi(J(\sigma_\lambda) \times \{\lambda\}) \mid \lambda \in \Lambda\}$, where $J(\sigma_\lambda) = \{\infty\} \cup \{\sigma \in \mathcal{S} \mid \sigma \supset \sigma_\lambda\}$. Then $G(y)$ is a neighborhood of y, and $G(x) \times G(y)$ meets no member of $\mathcal{K}$. Thus $\mathcal{K}$ is proved to be locally finite in $X \times Y$. Therefore if we define a function f on $X \times Y$ by

$$f(q) = \inf\{f_{\lambda\sigma}(q) \mid \lambda \in \Lambda, \sigma \in \mathcal{S}\}, \quad q \in X \times Y,$$

then f is continuous. Let $V \times W$ be a neighborhood of (x_0, y_0) in $\upsilon X \times Y$. Choose $\lambda \in \Lambda$ with $G_\lambda \subset V$ and $\sigma \in \mathcal{S}$ with $y_{\lambda\sigma} \in W$. Then both $p_1 = (x_{\lambda\sigma}, y_0)$ and $p_2 = (x_{\lambda\sigma}, y_{\lambda\sigma})$ belong to $V \times W$ and $f(p_1) = 1$, while $f(p_2) = 0$. This shows that f does not extend continuously to (x_0, y_0), and thus $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$. Hence the proof is complete.

3.8. Remarks. (1) The condition 1.3(a′) implies that both X and υX are locally weakly-$\mathfrak{n}^*$-compact, but the converse is not true in general. In fact, let $X = W(\alpha)$, where α is the initial ordinal of $\mathfrak{n}^* \cdot \aleph_1$; then both X and υX $(= W(\alpha+1))$ are locally compact, but X does not satisfy 1.3(a′).

(2) The latter half of the proof of Theorem 1.3 tells us that if X is locally weakly-$\mathfrak{n}^*$-compact and if $X \times Y$ is C-embedded in $\upsilon X \times Y$ for any paracompact $P(\mathfrak{n})$-space Y, then each point of $\upsilon X - X$ has a neighborhood G in υX such that $G \cap X$ is weakly-$\mathfrak{n}^*$-compact. It might be interesting to know whether this statement can be proved without assuming local weak-$\mathfrak{n}^*$-compactness of X or not (cf. Problem 4.3).

4. Problems and remarks

4.1. It is reasonable to ask what property $\mathscr{P}$ of spaces satisfies the equality $\mathscr{R}(\mathscr{P}$ and $P(\mathfrak{n})) = \mathscr{R}(P(\mathfrak{n}))$. In view of 2.5, 2.6 and Lemma 3.1, if $\mathscr{P}$ satisfies the following conditions (a)–(c), then $\mathscr{R}(\mathscr{P}$ and $P(\mathfrak{n})) = \mathscr{R}(P(\mathfrak{n}))$.

(a) Every 0-dimensional paracompact space has $\mathscr{P}$.

(b) 2.7(b) holds for $\mathscr{P}$.

(c) For every infinite cardinal $\mathfrak{k}$, there exists a $\mathscr{P}$- and $P(\mathfrak{n})$-space $Z = Z(\mathfrak{k}, \mathfrak{n})$ having a locally finite family $\mathscr{F}$ of subsets such that $|\mathscr{F}| = \mathfrak{k}$ and $\bigcap \{\mathrm{cl}_{\upsilon Z} F \mid F \in \mathscr{F}\} \neq \emptyset$.

Conversely, if $\mathfrak{n} < \mathfrak{m}_1$ and if $\mathscr{R}(\mathscr{P}$ and $P(\mathfrak{n})) = \mathscr{R}(P(\mathfrak{n}))$, then

(d) there exists a $\mathscr{P}$- and $P(\mathfrak{n})$-space Z having a countable locally finite family $\mathscr{F}$ of open sets such that $\bigcap \{\mathrm{cl}_{\upsilon Z} F \mid F \in \mathscr{F}\} \neq \emptyset$. For, if $\mathscr{P}$ does not satisfy (d), then it follows from [19, Remarks 3.5(1)] that any metric space of non-measurable cardinal belongs to $\mathscr{R}(\mathscr{P}$ and $P(\mathfrak{n}))$, while it is easy to find a metric space of non-measurable cardinal which does not belong to $\mathscr{R}(P(\mathfrak{n}))$ (use 1.3(a′)).

Problem. For every two infinite cardinals $\mathfrak{k}$ and $\mathfrak{n}$, do there exist the following spaces?

(1) a normal $P(\mathfrak{n})$-space $Z(\mathfrak{k}, \mathfrak{n})$,

(2) a $\mathfrak{m}$-paracompact $P(\mathfrak{n})$-space $Z_{\mathfrak{m}}(\mathfrak{k}, \mathfrak{n})$ for $\mathfrak{n} > \aleph_0$,

(3) a metacompact subparacompact $\mathfrak{m}$-paracompact space $Z_{\mathfrak{m}}(\mathfrak{k}, \aleph_0)$.

Since normal countable paracompactness does not satisfy (d), $\mathscr{R}$(normal countably paracompact $P(\mathfrak{n})) \neq \mathscr{R}(P(\mathfrak{n}))$ in case $\mathfrak{n} < \mathfrak{m}_1$. Therefore, if there exists (1) for $\mathfrak{n} < \mathfrak{m}_1$, then it must be a Dowker space. In case $\mathfrak{k} = \aleph_0$ and $\mathfrak{n} = \aleph_1$, there exists such a space. In fact, the Dowker space X constructed by Rudin in [20] is a $P(\aleph_1)$-space and has a decreasing sequence $\{D_n\}$ of closed subsets with empty intersection such that $\bigcap \mathrm{cl}_{\upsilon X} D_n \neq \emptyset$ as she essentially proved. The technique used in Facts 3.2 and 3.3 cannot be applied to make (1), because every pairwise disjoint closed subsets of a normal space T have disjoint closures in υT.

4.2. For convenience, we call a space X an $\alpha(\mathfrak{n})$-*space* if each point of υX has a neighborhood G in υX such that $G \cap X$ is weakly-$\mathfrak{n}$-compact.

Problem. Characterize $\mathscr{R}(\alpha(\mathfrak{n}))$ for $\mathfrak{n} > \aleph_0$.

As is easily seen, $\mathscr{R}(\alpha(\mathfrak{n})) = \mathscr{R}$(weakly-$\mathfrak{n}$-compact). This problem has been solved for $\mathfrak{n} = \aleph_0$. In fact, an $\alpha(\aleph_0)$-space is precisely a locally compact, realcompact space, and [13, Theorem 4] shows that $\mathscr{R}$(locally compact, realcompact) is the class of all pseudo-$\mathfrak{m}_1$-compact spaces.

4.3. Problem. Characterize $\mathscr{R}$ (paracompact $P(\mathfrak{n})$) for $\mathfrak{n} > \aleph_0$.

Recently, the author has proved that if X is a realcompact space with $|X| < \mathfrak{m}_1$, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ for any paracompact space Y. This fact combined with 2.5 and 2.6 implies that $\mathscr{R}$(paracompact) is precisely the class of all realcompact spaces X with $|X| < \mathfrak{m}_1$; however, the characterization of $\mathscr{R}$(paracompact $P(\mathfrak{n})$) is not yet known in case $\mathfrak{n} > \aleph_0$.

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