On pairwise Lindelöf bitopological spaces

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Abstract

In this paper, we shall continue the study of bitopological separation axioms begun by Kelly and obtained some results. Furthermore, we introduce two concepts of pairwise Lindelöf bitopological spaces and the properties for them are established. We also show that a pairwise Lindelöf space is not hereditary property.

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1. Introduction

In 1963, J.C. Kelly [1] introduced the notion of bitopological spaces. Such spaces equipped with its two (arbitrary) topologies. The reader is suggested to refer [1] for the detail definitions and notations. Furthermore, Kelly was extended some of the standard results of separation axioms in a topological space to a bitopological space. Such extensions are pairwise regular, pairwise Hausdorff and pairwise normal.

There are several works dedicated to the investigation of bitopologies, i.e., pairs of topologies on the same set; most of them deal with the theory itself but very few with applications. We are concerned in this paper with the idea of pairwise Lindelöfness in bitopological spaces and give some results.

In Section 3, we obtain a result in pairwise regular bitopological spaces. Furthermore, we obtain some results in pairwise normal bitopological spaces. We define a new weaker form of the pairwise normal spaces, i.e., \( p_1 \)-normal spaces, and we also give some examples. One interesting example is every \( p \)-normal space is \( p_1 \)-normal but not the converse.

In Section 4, we define two types of pairwise Lindelöf spaces, i.e., \( p \)-Lindelöf and \( p_1 \)-Lindelöf spaces. We obtain a result about subset of each such space. The main result we are obtain here is every pairwise regular and \( p \)-Lindelöf (or \( p_1 \)-Lindelöf) bitopological space is \( p_1 \)-normal.

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2. Preliminaries

Throughout this paper, all spaces \((X, \mathcal{P})\) and \((X, \mathcal{P}, \mathcal{Q})\) (or simply \(X\)) are always mean topological spaces and bitopological spaces, respectively. In this paper, we shall use \(p\)- to denote pairwise. For instance, \(p\)-compact and \(p\)-Lindelöf stand for pairwise compact and pairwise Lindelöf, respectively. While \(p_1\) is used to denote another type of pairwise.

Let \(F\) be a subset of \((X, \mathcal{P}, \mathcal{Q})\), \(\mathcal{P}\)-cl\((F)\) and \(\mathcal{Q}\)-cl\((F)\) represent the \(\mathcal{P}\)-closure and \(\mathcal{Q}\)-closure of \(F\) with respect to \(\mathcal{P}\) and \(\mathcal{Q}\), respectively. The open (respectively closed) sets in \(X\) with respect to \(\mathcal{P}\) is denoted by \(\mathcal{P}\)-open (respectively \(\mathcal{P}\)-closed), and the open (respectively closed) sets in \(X\) with respect to \(\mathcal{Q}\) is denoted by \(\mathcal{Q}\)-open (respectively \(\mathcal{Q}\)-closed).

**Definition 1.** A bitopological space \((X, \mathcal{P}, \mathcal{Q})\) is said to be \(p\)-compact if the topological space \((X, \mathcal{P})\) and \((X, \mathcal{Q})\) are both compact. Equivalently, \((X, \mathcal{P}, \mathcal{Q})\) is \(p\)-compact if every \(\mathcal{P}\)-open cover of \(X\) can be reduced to a finite \(\mathcal{P}\)-open cover and every \(\mathcal{Q}\)-open cover of \(X\) can be reduced to a finite \(\mathcal{Q}\)-open cover.

In [5], it was mentioned that T. Birsan (1969) has given definitions of pairwise compactness which do allow Tychonoff product theorems. According to Birsan, a bitopological space \((X, \mathcal{P}, \mathcal{Q})\) is said to be pairwise compact (denote \(p_1\)-compact) if every \(\mathcal{P}\)-open cover of \(X\) can be reduced to a finite \(\mathcal{Q}\)-open cover and every \(\mathcal{Q}\)-open cover of \(X\) can be reduced to a finite \(\mathcal{P}\)-open cover. We will generalize it to \(p_1\)-Lindelöf in Section 4.

We shall sometimes say that a bitopological space \((X, \mathcal{P}, \mathcal{Q})\) has a particular topological property, without referring specifically to \(\mathcal{P}\) or \(\mathcal{Q}\), and we shall then mean that both \((X, \mathcal{P})\) and \((X, \mathcal{Q})\) have the property; for instance, \((X, \mathcal{P}, \mathcal{Q})\) is said to satisfy second axiom of countability if both \((X, \mathcal{P})\) and \((X, \mathcal{Q})\) do so.

**Definition 2.** Let \((X, \mathcal{P}, \mathcal{Q})\) be a bitopological space.

(i) A set \(G\) is said to be pairwise open if \(G\) are both \(\mathcal{P}\)-open and \(\mathcal{Q}\)-open in \(X\).

(ii) A set \(F\) is said to be pairwise closed if \(F\) are both \(\mathcal{P}\)-closed and \(\mathcal{Q}\)-closed in \(X\).

3. Bitopological separation axioms

**Definition 3 (Kelly).** In a space \((X, \mathcal{P}, \mathcal{Q})\), \(\mathcal{P}\) is said to be regular with respect to \(\mathcal{Q}\) if, for each point \(x \in X\), there is a \(\mathcal{P}\)-neighbourhood base of \(\mathcal{Q}\)-closed sets, or, as is easily seen to be equivalent, if, for each point \(x \in X\) and each \(\mathcal{P}\)-closed set \(P\) such that \(x \notin P\), there are a \(\mathcal{P}\)-open set \(U\) and a \(\mathcal{Q}\)-open set \(V\) such that

\[ x \in U, \quad P \subseteq V, \quad \text{and} \quad U \cap V = \emptyset. \]

\((X, \mathcal{P}, \mathcal{Q})\) is, or \(\mathcal{P}\) and \(\mathcal{Q}\) are, pairwise regular if \(\mathcal{P}\) is regular with respect to \(\mathcal{Q}\) and vice versa.

**Theorem 1.** In a space \((X, \mathcal{P}, \mathcal{Q})\), \(\mathcal{P}\) is regular with respect to \(\mathcal{Q}\) if and only if for each point \(x \in X\) and \(\mathcal{P}\)-open set \(H\) containing \(x\), there exists a \(\mathcal{P}\)-open set \(U\) such that

\[ x \in U \subseteq \mathcal{Q}\text{-cl}(U) \subseteq H. \]

**Proof.** \((\Rightarrow)\) Suppose \(\mathcal{P}\) is regular with respect to \(\mathcal{Q}\). Let \(x \in X\) and \(H\) is a \(\mathcal{P}\)-open set containing \(x\). Then \(G = X - H\) is a \(\mathcal{P}\)-closed set which \(x \notin G\). Since \(\mathcal{P}\) is regular with respect to \(\mathcal{Q}\), then there are \(\mathcal{P}\)-open set \(U\) and \(\mathcal{Q}\)-open set \(V\) such that \(x \in U\), \(G \subseteq V\), and \(U \cap V = \emptyset\). Since \(U \subseteq X - V\), then \(\mathcal{Q}\text{-cl}(U) \subseteq \mathcal{Q}\text{-cl}(X - V) = X - V \subseteq X - G = H\). Thus, \(x \in U \subseteq \mathcal{Q}\text{-cl}(U) \subseteq H\) as desired.

\((\Leftarrow)\) Suppose the condition holds. Let \(x \in X\) and \(P\) is a \(\mathcal{P}\)-closed set such that \(x \notin P\). Then \(x \in X - P\), and by hypothesis there exists a \(\mathcal{P}\)-open set \(U\) such that \(x \in U \subseteq \mathcal{Q}\text{-cl}(U) \subseteq X - P\). It follows that \(x \in U\), \(P \subseteq X - \mathcal{Q}\text{-cl}(U)\) and \(U \cap (X - \mathcal{Q}\text{-cl}(U)) = \emptyset\). This completes the proof. \(\square\)

**Remark 1.** In other words, Theorem 1 stated that \(\mathcal{P}\) is regular with respect to \(\mathcal{Q}\) if, for each point \(x \in X\), there is a \(\mathcal{P}\)-neighbourhood base of \(\mathcal{Q}\)-closed sets containing \(x\). This is equivalent definition in Definition 3.
If $Q$ is also regular with respect to $P$, we have the similar result as previous theorem and stated in the following corollary. By these reason we obtain a pairwise regular space.

**Corollary 1.** In a space $(X, P, Q)$, $Q$ is regular with respect to $P$ if and only if for each point $x \in X$ and $Q$-open set $H$ containing $x$, there exists a $Q$-open set $U$ such that 

$$x \in U \subseteq P\cdot cl(U) \subseteq H.$$ 

If $Y \subseteq X$, then the collections $P_Y = \{A \cap Y \colon A \in P\}$ and $Q_Y = \{B \cap Y \colon B \in Q\}$ are the relative topology on $Y$. A bitopological space $(Y, P_Y, Q_Y)$ is then called a subspace of $(X, P, Q)$. Moreover, $Y$ is said to be pairwise closed subspace of $X$ if $Y$ is both $P_Y$-closed and $Q_Y$-closed in $X$. The pairwise open subspace is defined in the similar way. The following theorem shows that, pairwise regular spaces satisfy the hereditary property.

**Theorem 2.** Every subspace of a pairwise regular bitopological space is pairwise regular.

**Proof.** Let $(X, P, Q)$ be a pairwise regular space and let $(Y, P_Y, Q_Y)$ be a subspace of $(X, P, Q)$. Furthermore, let $F$ be a $P_Y$-closed set in $Y$, then $F = A \cap Y$ where $A$ is a $P$-closed set in $X$. Now if $y \in Y$ and $y \notin F$, then $y \notin A$, so there are $P$-open set $U$ and $Q$-open set $V$ such that

$$y \in U, \quad A \subseteq V, \quad \text{and} \quad U \cap V = \emptyset.$$ 

But $U \cap Y$ and $V \cap Y$ are $P_Y$-open set and $Q_Y$-open set in $Y$ respectively. Also $y \in U \cap Y, F \subseteq V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y = \emptyset$.

Similarly, let $G$ be a $Q_Y$-closed set in $Y$, then $G = B \cap Y$ where $B$ is a $Q$-closed set in $X$. Now if $y \in Y$ and $y \notin G$, then $y \notin B$, so there are $Q$-open set $U$ and $P$-open set $V$ such that

$$y \in U, \quad B \subseteq V, \quad \text{and} \quad U \cap V = \emptyset.$$ 

But $U \cap Y$ and $V \cap Y$ are $Q_Y$-open set and $P_Y$-open set in $Y$ respectively. Also $y \in U \cap Y, G \subseteq V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \emptyset$. This completes the proof. \(\square\)

**Definition 4 (Kelly).** A bitopological space $(X, P, Q)$ is said to be $p$-normal if, given a $P$-closed set $A$ and a $Q$-closed set $B$ with $A \cap B = \emptyset$, there exist a $Q$-open set $U$ and a $P$-open set $V$ such that $A \subseteq U, B \subseteq V$, and $U \cap V = \emptyset$.

Equivalently, $(X, P, Q)$ is $p$-normal if, given a $Q$-closed set $C$ and a $P$-open set $D$ such that $C \subseteq D$, there are a $P$-open set $G$ and a $Q$-closed set $F$ such that $C \subseteq G \subseteq F \subseteq D$.

We shall prove the equivalent definition above in the following theorem.

**Theorem 3.** A space $(X, P, Q)$ is $p$-normal if and only if given a $Q$-closed set $C$ and a $P$-open set $D$ such that $C \subseteq D$, there are a $Q$-open set $G$ and a $Q$-closed set $F$ such that $C \subseteq G \subseteq F \subseteq D$.

**Proof.** ($\Rightarrow$) Suppose $(X, P, Q)$ is $p$-normal. Let $C$ be a $Q$-closed set and $D$ a $P$-open set such that $C \subseteq D$. Then $K = X - D$ is a $P$-closed set with $K \cap C = \emptyset$. Since $(X, P, Q)$ is $p$-normal, there exists a $Q$-open set $U$ and a $P$-open set $V$ such that $K \subseteq U, C \subseteq G,$ and $U \cap G = \emptyset$. Hence $G \subseteq X - U \subseteq X - K = D$. Thus $C \subseteq G \subseteq X - U \subseteq D$ and the result follows by taking $X - U = F$.

($\Leftarrow$) Suppose the condition holds. Let $A$ be a $P$-closed set and $B$ a $Q$-closed set with $A \cap B = \emptyset$. Then $D = X - A$ is a $P$-open set with $B \subseteq D$. By hypothesis, there are a $P$-open set $G$ and a $Q$-closed set $F$ such that $B \subseteq G \subseteq F \subseteq D$. It follows that $A = X - D \subseteq X - F, B \subseteq G$ and $(X - F) \cap G = \emptyset$ where $X - F$ is $Q$-open set and $G$ is $P$-open set. This completes the proof. \(\square\)

**Theorem 4.** A space $(X, P, Q)$ is $p$-normal if and only if given a $P$-closed set $C$ and a $Q$-open set $D$ such that $C \subseteq D$, there are a $Q$-open set $U$ and a $P$-closed set $F$ such that $C \subseteq U \subseteq F \subseteq D$.

**Proof.** ($\Rightarrow$) Suppose $(X, P, Q)$ is $p$-normal. Let $C$ be a $P$-closed set and $D$ a $Q$-open set such that $C \subseteq D$. Then $K = X - D$ is a $Q$-closed set with $C \cap K = \emptyset$. Since $(X, P, Q)$ is $p$-normal, there exists a $Q$-open set $U$ and a $P$-open
set $V$ such that $C \subseteq U$, $K \subseteq V$, and $U \cap V = \emptyset$. Hence $U \subseteq X - V \subseteq X - F = D$. Thus $C \subseteq U \subseteq X - V \subseteq D$ and the result follows by taking $X - V = F$.

$(\Rightarrow)$ Suppose the condition holds. Let $A$ be a $P$-closed set and $B$ a $Q$-closed set with $A \cap B = \emptyset$. Then $D = X - B$ is a $Q$-open set with $A \subseteq D$. By hypothesis, there are a $Q$-open set $U$ and a $P$-closed set $F$ such that $A \subseteq U \subseteq F \subseteq D$. It follows that $B = X - D \subseteq X - F$, $A \subseteq U$ and $(X - F) \cap U = \emptyset$ where $X - F$ is $P$-open set and $U$ is $Q$-open set. This completes the proof. \square

Now we define a new weaker form of pairwise normal bitopological spaces.

**Definition 5.** A space $(X, P, Q)$ is said to be $p_1$-normal if, given $A$ and $B$ are pairwise closed sets with $A \cap B = \emptyset$, there exist a $Q$-open set $U$ and a $P$-open set $V$ such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

**Theorem 5.** A bitopological space $(X, P, Q)$ is $p_1$-normal if and only if given a pairwise closed set $C$ and a pairwise open set $D$ such that $C \subseteq D$, there are a $P$-open set $G$ and a $Q$-closed set $F$ such that $C \subseteq G \subseteq F \subseteq D$.

**Proof.** $(\Rightarrow)$ Suppose $(X, P, Q)$ is $p_1$-normal. Let $C$ be a pairwise closed set and $D$ a pairwise open set such that $C \subseteq D$. Then $K = X - D$ is a pairwise closed set with $K \cap C = \emptyset$. Since $(X, P, Q)$ is $p_1$-normal, there exists a $Q$-open set $U$ and a $P$-open set $G$ such that $K \subseteq U$, $C \subseteq G$, and $U \cap G = \emptyset$. Hence $G \subseteq X - U \subseteq X - K = D$. Thus $C \subseteq G \subseteq X - U \subseteq D$ and the result follows by taking $X - U = F$.

$(\Leftarrow)$ Suppose the condition holds. Let $A$ and $B$ are pairwise closed sets with $A \cap B = \emptyset$. Then $D = X - A$ is a pairwise open set with $B \subseteq D$. By hypothesis, there are a $P$-open set $G$ and a $Q$-closed set $F$ such that $B \subseteq G \subseteq F \subseteq D$. It follows that $A = X - D \subseteq X - F$, $B \subseteq G$ and $(X - F) \cap G = \emptyset$ where $X - F$ is $Q$-open set and $G$ is $P$-open set. This completes the proof. \square

**Example 1.** Consider $X = \{a, b, c\}$ with topologies $P = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $Q = \{\emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b\}, \{a\}, \{a, b, c\}, X\}$ defined on $X$. Observe that $P$-closed subsets of $X$ are $\emptyset, \{a\}, \{b\}$, $\{b, c\}$ and $X$, and $Q$-closed subsets of $X$ are $\emptyset, \{b, c\}, \{a\}, \{c\}, \{a, b\}$ and $X$. It follows that $(X, P, Q)$ does satisfy the condition in definition of $p$-normal. One of them we can take $A = \{a\}, B = \{b\}, U = \{a\}$ and $V = \{b, c\}$ in the definition, we can checks for the other. Hence $(X, P, Q)$ is $p$-normal, and hence $p_1$-normal.

It is clear from definition that every $p$-normal space is $p_1$-normal. The converse is not true in general as shown in the following counter-example.

**Example 2.** Consider $X = \{a, b, c, d\}$ with topologies $P = \{\emptyset, \{a\}, \{b\}, X\}$ and $Q = \{\emptyset, \{a\}, \{b, c, d\}, \{a, b, c, d\}, X\}$ defined on $X$. Observe that $P$-closed subsets of $X$ are $\emptyset, \{c\}$ and $X$, and $Q$-closed subsets of $X$ are $\emptyset, \{b, c, d\}, \{a\}$ and $X$. Hence $(X, P, Q)$ is $p_1$-normal as we can checks since the only pairwise closed sets of $X$ are $\emptyset$ and $X$. However $(X, P, Q)$ is not $p$-normal since the $P$-closed set $A = \{c\}$ and $Q$-closed set $B = \{a\}$ satisfy $A \cap B = \emptyset$, but do not exist the $Q$-open set $U$ and $P$-open set $V$ such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Naturally, any result stated in terms of $P$ and $Q$ has a ‘dual’, in terms of $Q$ and $P$. The definitions of separation properties of two topologies $P$ and $Q$, such as pairwise regularity, of course reduce to the usual separation properties of one topology, such as regularity, when we take $P = Q$, and the theorems quoted above then yield as corollaries of the classical results of which they are generalizations.

4. Pairwise Lindelöf spaces

According to Definition 1, we generalize $p$-compact spaces to $p$-Lindelöf as the following.

**Definition 6.** A bitopological space $(X, P, Q)$ is said to be $p$-Lindelöf if the topological space $(X, P)$ and $(X, Q)$ are both Lindelöf. Equivalently, $(X, P, Q)$ is $p$-Lindelöf if every $P$-open cover of $X$ can be reduced to a countable $P$-open cover and every $Q$-open cover of $X$ can be reduced to a countable $Q$-open cover.
Recall that, the relation between compactness and Lindelöfness is very strong, where every compact space is Lindelöf, but not the converse. Accordingly, every \( p \)-compact space is \( p \)-Lindelöf but not the converse, and hence the relation between \( p \)-compactness and \( p \)-Lindelöfness is very strong also.

**Theorem 6.** If \((X, \mathcal{P}, \mathcal{Q})\) is second countable space, then \((X, \mathcal{P}, \mathcal{Q})\) is \( p \)-Lindelöf.

**Proof.** Let \(\{P_n\}\) and \(\{Q_n\}\), \(n = 1, 2, \ldots\), be countable bases for \(\mathcal{P}\) and \(\mathcal{Q}\) respectively. Let \(\mathcal{U} = \{U_\alpha\}_{\alpha \in \Delta}\) be a \(\mathcal{P}\)-open covering of \(X\), then for every \(x \in X\), there exists \(U_x \in \mathcal{U}\) such that \(x \in U_x\). Since \((X, \mathcal{P}, \mathcal{Q})\) is second countable, then so is \((X, \mathcal{P})\). Since \(\{P_n\}\) is a base for \(\mathcal{P}\), for each \(x \in U_x\) and \(U_x \in \mathcal{U}\), there is \(P_x \in \{P_n\}\) such that \(x \in P_x \subseteq U_x\). Hence \(X = \bigcup\{P_x: x \in X\}\). But \(\{P_x: x \in X\} \subseteq \{P_n\}\), so it is countable and hence \(\{P_x: x \in X\} = \{P_n: n \in \mathbb{N}\}\). For each \(n \in \mathbb{N}\), choose one set \(P_n \in \{P_n\}\) such that \(P_n \subseteq U_n\). Then \(X = \bigcup\{P_n: n \in \mathbb{N}\} = \{U_n: n \in \mathbb{N}\}\) and so \(\{U_n: n \in \mathbb{N}\}\) is a countable subcover of \(X\). Thus \((X, \mathcal{P})\) is a Lindelöf space.

Similarly \((X, \mathcal{Q})\) is also a Lindelöf space. Therefore \((X, \mathcal{P}, \mathcal{Q})\) is \( p \)-Lindelöf. \( \Box \)

**Lemma 1.** Every pairwise closed subset of a \( p \)-Lindelöf bitopological space is \( p \)-Lindelöf.

**Proof.** Let \((X, \mathcal{P}, \mathcal{Q})\) be a \( p \)-Lindelöf bitopological space and let \(F\) is a pairwise closed subset of \(X\). Then \((X, \mathcal{P})\) and \((X, \mathcal{Q})\) are Lindelöf, and \(\mathcal{F}\) are \(\mathcal{P}\)-closed and \(\mathcal{Q}\)-closed subset of \(X\). If \(\{U_\alpha: \alpha \in \Delta\}\) is a \(\mathcal{P}\)-open cover of \(F\), then \(X = \bigcup\{U_\alpha\}_{\alpha \in \Delta} \cup (X - F)\). Hence the collection \(\{U_\alpha: \alpha \in \Delta\}\) and \(X - F\) form a \(\mathcal{P}\)-open cover of \(X\). Since \((X, \mathcal{P})\) is Lindelöf, there will be a countable subcover, \(\{X - F, U_{\alpha_1}, U_{\alpha_2}, \ldots\}\). But \(F\) and \(X - F\) are disjoint; hence the subcollection of \(\mathcal{P}\)-open set \(\{U_\alpha: i \in \mathbb{N}\}\) also cover \(F\), and so \(\{U_\alpha: \alpha \in \Delta\}\) has a countable subcover.

On the other hand, if \(\{G_\alpha: \alpha \in \Delta\}\) is a \(\mathcal{Q}\)-open cover of \(F\), then \(X = \bigcup\{G_\alpha\}_{\alpha \in \Delta} \cup (X - F)\). Hence the collection \(\{G_\alpha: \alpha \in \Delta\}\) and \(X - F\) form a \(\mathcal{Q}\)-open cover of \(X\). Since \((X, \mathcal{Q})\) is Lindelöf, there will be a countable subcover, \(\{X - F, G_{\alpha_1}, G_{\alpha_2}, \ldots\}\). But \(F\) and \(X - F\) are disjoint; hence the subcollection of \(\mathcal{Q}\)-open set \(\{G_\alpha: i \in \mathbb{N}\}\) also cover \(F\), and so \(\{G_\alpha: \alpha \in \Delta\}\) has a countable subcover. This shows that \(F\) is \( p \)-Lindelöf. \( \Box \)

Arbitrary subsets of \( p \)-Lindelöf bitopological spaces need not be \( p \)-Lindelöf as the following example show. Accordingly, \( p \)-Lindelöf property is not a hereditary property. We need the following lemma [6, 11].

**Lemma 2.** If \(A\) is a countable subset of ordinals \(\Omega\) not containing \(\omega_1\), then \(\sup A < \omega_1\).

**Example 3.** Let \(\Omega\) denotes the set of ordinals which are less than or equal to the first uncountable ordinal \(\omega_1\). This \(\Omega\) is an uncountable well-ordered set with a largest element \(\omega_1\), having the property that if \(\alpha \in \Omega\) with \(\alpha < \omega_1\), then \(\{\beta \in \Omega: \beta \leq \alpha\}\) is countable (see [6, 10–11]). Since \(\Omega\) is a totally ordered space, it can be provided with its order topology. Let us denote this order topology by \(\mathcal{P}\).

Let \(\beta = \{a, b\}\) be a collection of all sets in \(\Omega\) of the form (called interval)

\[(a, b) = \{\beta \in \Omega: a < \beta < b\}, \quad (a_0, \omega_1) = \{\beta \in \Omega: a_0 < \beta \leq \omega_1\}\]

and

\[\{1, b_0\} = \{\beta \in \Omega: 1 \leq \beta < b_0\}\]

Then the collection \(\beta\) is a base for the order topology \(\mathcal{P}\) for \(\Omega\).

Now \((\Omega, \mathcal{P})\) is a Lindelöf space. In fact, given any \(\mathcal{P}\)-open cover of \(\Omega\), find one element \(U\) which contains \(\omega_1\). Then \(U\) contains an interval \((\gamma, \omega_1]\) for some \(\gamma < \omega_1\). But this leaves at most the set \([1, \gamma]\) to be covered, and this set is countable. So at most countably many more elements of the cover will be needed to cover \(\Omega\), and these countably many elements of the cover together with \(U\) form a countable \(\mathcal{P}\)-open cover of \(\Omega\).

The subset \(\Omega_0 = \Omega - \{\omega_1\}\), however is not Lindelöf. If for each \(\alpha \in \Omega_0\), we set \(U_\alpha = [1, \alpha]\), then \(\{U_\alpha: \alpha \in \Omega_0\}\) is a \(\mathcal{P}\)-open cover of \(\Omega_0\) which has no countable subcover. For if \(\{U_\alpha, U_{\alpha_2}, \ldots\}\) is a \(\mathcal{P}\)-open cover of \(\Omega_0\), then \(\sup\{\alpha_1, \alpha_2, \ldots\} = \omega_1\), which is impossible by Lemma 2.

Let us denote another topology for \(\Omega\) distinguish from \(\mathcal{P}\) by \(\mathcal{Q}\). So \((\Omega, \mathcal{P}, \mathcal{Q})\) form a bitopological space. Choose topology \(\mathcal{Q}\) such that \((\Omega, \mathcal{Q})\) is Lindelöf, and therefore \((\Omega, \mathcal{P}, \mathcal{Q})\) is \( p \)-Lindelöf. Since the arbitrary subsets of \((\Omega, \mathcal{P})\) is not Lindelöf, the arbitrary subsets of \((\Omega, \mathcal{P}, \mathcal{Q})\) is not \( p \)-Lindelöf.
Definition 7. In a bitopological space $(X, \mathcal{P}, \mathcal{Q})$, $\mathcal{P}$ is said to be Lindelöf with respect to $\mathcal{Q}$ if, every $\mathcal{P}$-open cover of $X$ can be reduced to a countable $\mathcal{Q}$-open cover.

$(X, \mathcal{P}, \mathcal{Q})$ is, or $\mathcal{P}$ and $\mathcal{Q}$ are, $p_1$-Lindelöf if $\mathcal{P}$ is Lindelöf with respect to $\mathcal{Q}$ and vice versa.

Similar to the previous argument, the relation between $p_1$-compactness and $p_1$-Lindelöfness is very strong, where every $p_1$-compact space is $p_1$-Lindelöf but not the converse by the following counter-example.

Example 4. Let $\mathcal{B}$ be the collection of open-closed intervals in the real line $\mathbb{R}$:

$$\mathcal{B} = \{(a, b]: a, b \in \mathbb{R}, a < b\}.$$ 

Hence $\mathcal{B}$ is a base for the upper limit topology $\mathcal{P}$ on $\mathbb{R}$. Similarly, the collection of closed-open intervals,

$$\mathcal{B}^\circ = \{(a, b): a, b \in \mathbb{R}, a < b\}$$

is a base for the lower limit topology $\mathcal{Q}$ on $\mathbb{R}$. Now it is clear that $(\mathbb{R}, \mathcal{P})$ and $(\mathbb{R}, \mathcal{Q})$ are Lindelöf spaces. Thus $(\mathbb{R}, \mathcal{P}, \mathcal{Q})$ is a $p$-Lindelöf space. But $(\mathbb{R}, \mathcal{P}, \mathcal{Q})$ is not $p$-compact since $\{(n, n + 1]: n \in \mathbb{Z}\}$ is a $\mathcal{P}$-open cover of $\mathbb{R}$ contains no finite subcover.

It is also clear that every $\mathcal{P}$-open cover of $\mathbb{R}$ can be reduced to a countable $\mathcal{Q}$-open cover and every $\mathcal{Q}$-open cover of $\mathbb{R}$ can be reduced to a countable $\mathcal{P}$-open cover (note: for every open-closed interval $(a, b)$, there exist a closed-open interval $[c, d]$ such that $[a, b] \subset (c, d)$ and vice versa). Therefore $(\mathbb{R}, \mathcal{P}, \mathcal{Q})$ is a $p_1$-Lindelöf space. But $(\mathbb{R}, \mathcal{P}, \mathcal{Q})$ is not $p_1$-compact since $\{(n, n + 1]: n \in \mathbb{Z}\}$ is a $\mathcal{P}$-open cover of $\mathbb{R}$ cannot be reduced to a finite $\mathcal{Q}$-open cover.

Lemma 3. In a bitopological space $(X, \mathcal{P}, \mathcal{Q})$, let $\mathcal{P}$ is Lindelöf with respect to $\mathcal{Q}$. Then $\mathcal{P}$-closed subset of $(X, \mathcal{P}, \mathcal{Q})$ is also $\mathcal{P}$ Lindelöf with respect to $\mathcal{Q}$.

Proof. Let $F$ be a $\mathcal{P}$-closed subset of $(X, \mathcal{P}, \mathcal{Q})$. If $\{U_\alpha: \alpha \in \Delta\}$ is a $\mathcal{P}$-open cover of $F$, then $X = (\bigcup_{\alpha \in \Delta} U_\alpha) \cup (X - F)$. Hence the collection $\{U_\alpha: \alpha \in \Delta\}$ and $X - F$ form a $\mathcal{P}$-open cover of $X$. Since $\mathcal{P}$ is Lindelöf with respect to $\mathcal{Q}$, then the $\mathcal{P}$-open cover of $X$ can be reduced to a countable $\mathcal{Q}$-open cover $\{X - F, U_{\alpha_1}, U_{\alpha_2}, \ldots\}$. But $F$ and $X - F$ are disjoint; hence the subcollection of $\mathcal{Q}$-open set $\{U_{\alpha_i}: i \in \mathbb{N}\}$ also cover $F$, and so $\{U_\alpha: \alpha \in \Delta\}$ can be reduced to a countable $\mathcal{Q}$-open cover. This shows that $F$ is $\mathcal{P}$ Lindelöf with respect to $\mathcal{Q}$.

Similarly, if $\mathcal{Q}$ is Lindelöf with respect to $\mathcal{P}$, then $\mathcal{Q}$-closed subset of $(X, \mathcal{P}, \mathcal{Q})$ is $\mathcal{Q}$ Lindelöf with respect to $\mathcal{P}$.

Lemma 4. Every pairwise closed subset of a $p_1$-Lindelöf bitopological space is $p_1$-Lindelöf.

Proof. Let $(X, \mathcal{P}, \mathcal{Q})$ be a $p_1$-Lindelöf bitopological space and let $F$ is a pairwise closed subset of $X$. Then $F$ are $\mathcal{P}$-closed and $\mathcal{Q}$-closed subset of $X$. If $\{U_\alpha: \alpha \in \Delta\}$ is a $\mathcal{P}$-open cover of $F$, then $X = (\bigcup_{\alpha \in \Delta} U_\alpha) \cup (X - F)$. Hence the collection $\{U_\alpha: \alpha \in \Delta\}$ and $X - F$ form a $\mathcal{P}$-open cover of $X$. Since $(X, \mathcal{P}, \mathcal{Q})$ is $p_1$-Lindelöf, then the $\mathcal{P}$-open cover of $X$ can be reduced to a countable $\mathcal{Q}$-open cover $\{X - F, U_{\alpha_1}, U_{\alpha_2}, \ldots\}$. But $F$ and $X - F$ are disjoint; hence the subcollection of $\mathcal{Q}$-open set $\{U_{\alpha_i}: i \in \mathbb{N}\}$ also cover $F$, and so $\{U_\alpha: \alpha \in \Delta\}$ can be reduced to a countable $\mathcal{Q}$-open cover.

On the other hand, if $\{G_\alpha: \alpha \in \Delta\}$ is a $\mathcal{Q}$-open cover of $F$, then $X = (\bigcup_{\alpha \in \Delta} G_\alpha) \cup (X - F)$. Hence the collection $\{G_\alpha: \alpha \in \Delta\}$ and $X - F$ form a $\mathcal{Q}$-open cover of $X$. Since $(X, \mathcal{P}, \mathcal{Q})$ is $p_1$-Lindelöf, then the $\mathcal{Q}$-open cover of $X$ can be reduced to a countable $\mathcal{P}$-open cover $\{X - F, G_{\alpha_1}, G_{\alpha_2}, \ldots\}$. But $F$ and $X - F$ are disjoint; hence the subcollection of $\mathcal{P}$-open set $\{G_{\alpha_i}: i \in \mathbb{N}\}$ also cover $F$, and so $\{G_\alpha: \alpha \in \Delta\}$ can be reduced to a countable $\mathcal{P}$-open cover. This shows that $F$ is $p_1$-Lindelöf.

Theorem 7. Every pairwise regular and $p$-Lindelöf bitopological space $(X, \mathcal{P}, \mathcal{Q})$ is $p_1$-normal.
Proof. Let A and B are pairwise closed sets with $A \cap B = \emptyset$ in X. Then A and B are both $\mathcal{P}$-closed and Q-closed set in X. Since $(X, \mathcal{P}, Q)$ is pairwise regular, then by Theorem 1, for each $x \in B$ and $\mathcal{P}$-open set $X - A$ containing $x$, there is a $\mathcal{P}$-open set $P_x$ such that

$$x \in P_x \subseteq Q\text{-cl}(P_x) \subseteq X - A,$$

i.e., $Q\text{-cl}(P_x) \cap A = \emptyset$. The collection $\{P_x : x \in B\}$ forms a $\mathcal{P}$-open covering of B. Since $(X, \mathcal{P}, Q)$ is $p$-$\text{Lindelöf}$, then $B$ is also $p$-$\text{Lindelöf}$ by Lemma 1. Hence we obtain a countable $\mathcal{P}$-open cover of B, which we denote by $\{P_i : i \in \mathbb{N}\}$.

Similarly, for each $y \in A$ and Q-open set $X - B$ containing $y$, there is a Q-open set $Q_y$ such that

$$y \in Q_y \subseteq \mathcal{P}\text{-cl}(Q_y) \subseteq X - B,$$

i.e., $\mathcal{P}\text{-cl}(Q_y) \cap B = \emptyset$. The collection $\{Q_y : y \in A\}$ forms a Q-open covering of A. Since $(X, \mathcal{P}, Q)$ is $p$-$\text{Lindelöf}$, then A is also $p$-$\text{Lindelöf}$ by Lemma 1. Hence we obtain a countable Q-open cover of A, which we denote by $\{Q_i : i \in \mathbb{N}\}$. Let

$$U_n = Q_n - \bigcup \{Q\text{-cl}(V_i) : i \leq n\}$$

and

$$V_n = P_n - \bigcup \{\mathcal{P}\text{-cl}(U_i) : i \leq n\}.$$

Since $U_n \cap Q\text{-cl}(V_m) = \emptyset$ for $m \leq n$, it follows that $U_n \cap V_m = \emptyset$ for $m \leq n$.

Similarly, $V_m \cap \mathcal{P}\text{-cl}(U_n) = \emptyset$ for $n \leq m$, it follows that $V_m \cap U_n = \emptyset$ for $n \leq m$. Thus $U_n \cap V_m = \emptyset$ for all $m$ and $n$, and consequently $U = \bigcup \{U_n : n \in \mathbb{N}\}$ is disjoint from $V = \bigcup \{V_n : n \in \mathbb{N}\}$. Finally, $Q\text{-cl}(V_i) \cap A$ and $\mathcal{P}\text{-cl}(U_i) \cap B$ are empty set for all $i$ and hence the set $U$ contains $A$ and is Q-open, whilst the set $V$ contains $B$ and is $\mathcal{P}$-open. The proof is complete. □

Theorem 8. Every pairwise regular and $p_1$-$\text{Lindelöf}$ bitopological space $(X, \mathcal{P}, Q)$ is $p_1$-$\text{normal}$. 

Proof. Let A and B are pairwise closed sets with $A \cap B = \emptyset$ in X. Since $(X, \mathcal{P}, Q)$ is pairwise regular, then by Theorem 1, for each $x \in B$ and $\mathcal{P}$-open set $X - A$ containing $x$, there is a $\mathcal{P}$-open set $P_x$ such that

$$x \in P_x \subseteq Q\text{-cl}(P_x) \subseteq X - A,$$

i.e., $Q\text{-cl}(P_x) \cap A = \emptyset$. The collection $\{P_x : x \in B\}$ forms a $\mathcal{P}$-open covering of B. Since $(X, \mathcal{P}, Q)$ is $p_1$-$\text{Lindelöf}$, then B is also $p_1$-$\text{Lindelöf}$ by Lemma 2. Thus $\{P_x : x \in B\}$ can be reduced to a countable Q-open cover, which we denote by $\{P_i : i \in \mathbb{N}\}$.

Similarly, for each $y \in A$ and Q-open set $X - B$ containing $y$, there is a Q-open set $Q_y$ such that

$$y \in Q_y \subseteq \mathcal{P}\text{-cl}(Q_y) \subseteq X - B,$$

i.e., $\mathcal{P}\text{-cl}(Q_y) \cap B = \emptyset$. The collection $\{Q_y : y \in A\}$ forms a Q-open covering of A. Since $(X, \mathcal{P}, Q)$ is $p_1$-$\text{Lindelöf}$, then A is also $p_1$-$\text{Lindelöf}$ by Lemma 4. Thus $\{Q_y : y \in A\}$ can be reduced to a countable $\mathcal{P}$-open cover, which we denote by $\{Q_i : i \in \mathbb{N}\}$. Let

$$U_n = Q_n - \bigcup \{\mathcal{P}\text{-cl}(V_i) : i \leq n\}$$

and

$$V_n = P_n - \bigcup \{Q\text{-cl}(U_i) : i \leq n\}.$$

Since $U_n \cap \mathcal{P}\text{-cl}(V_m) = \emptyset$ for $m \leq n$, it follows that $U_n \cap V_m = \emptyset$ for $m \leq n$.

Similarly, $V_m \cap Q\text{-cl}(U_n) = \emptyset$ for $n \leq m$, it follows that $V_m \cap U_n = \emptyset$ for $n \leq m$. Thus $U_n \cap V_m = \emptyset$ for all $m$ and $n$, and consequently $U = \bigcup \{U_n : n \in \mathbb{N}\}$ is disjoint from $V = \bigcup \{V_n : n \in \mathbb{N}\}$. Finally, $\mathcal{P}\text{-cl}(V_i) \cap A$ and $Q\text{-cl}(U_i) \cap B$ are empty set for all $i$ and hence the set $U$ contains $A$ and is $\mathcal{P}$-open, whilst the set $V$ contains $B$ and is Q-open. The proof is complete. □
References