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EVENT STRUCTURES AND DOMAINS

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Abstract. In the theory of denotational semantics, we study event structures which generalize Kahn and Plotkin's concrete data structures and which model computational processes. With each event structure we associate canonically an event domain (a particular algebraic complete partial order), and conversely we derive a representation result for event domains. For a particular class of event structures, the canonical event structures, we obtain that any two canonical event structures are isomorphic iff they have order-isomorphic canonical domains.

1. Introduction

In the mathematical theory of semantics of programming languages, various kinds of systems of information and associated partial orders (domains) of information have been extensively studied. Scott [7] introduced information systems as consisting of a set of tokens (to be imagined as propositions or units of information), together with consistency and entailment relations. Kahn and Plotkin [4] considered concrete data structures and concrete domains. Sequential algorithms on these structures were studied by Berry and Curien [1]. Winskel [9, 10] (cf. also [6]) introduced a generalization, the event structures and particular associated domains. For a variety of further results, see [2].

In this paper we introduce and study event domains and their relationship with event structures. An *event structure* consists of a set E of tokens together with a consistency relation for finite subsets of E and an enabling relation between finite subsets and elements of E (satisfying certain natural axioms). The elements of E can be thought of, for example, as the units of information which can in principle be computed by a machine, whereas the enabling relation describes the computation possibilities themselves. A *state* or *configuration of events* is a subset X of E such that each finite subset of X is consistent and each element of X can be deduced through finitely many successive applications of the enabling relation from a finite number of elements of X which are "a priori true", i.e. enabled by the empty set. The set $(D(E), \subseteq)$ of all such states of E, partially ordered under inclusion, will be called the *canonical event domain associated with E.*

Event structures as a model of computational processes have been studied in detail in Winskel [9, 10]. They were shown to be related to Petri nets and to Scott domains. Also, they can be used to provide semantics to programming languages for parallel processes like CCS and CSP and to languages with higher types. For details on this and further background, we refer the reader to [10].

We now give a summary of our results. In Section 2, we first define axiomatically the class of *event domains* which are particular algebraic complete partial orders. We show that any canonical event domain $(D(E), \subseteq)$ is an event domain. Conversely, any event domain $(D, \leqslant)$ is order-isomorphic to $(D(E), \subseteq)$ for some canonically chosen event structure $E = E_D$. This partially uses methods and generalizes results of Kahn and Plotkin [4] and Winskel [9, 10], cf. also [2]. Winskel [9, 10] obtained the corresponding characterization theorems under the additional assumption either that E is stable or that the consistency relation on E is induced by a binary conflict relation on E. We also show how to derive Winskel's representation theorem [9] for "conflict event domains" from our present results.

Trivial examples show that two event structures E_1, E_2 with order-isomorphic domains $(D(E_1), \subseteq), (D(E_2), \subseteq)$ themselves need not be isomorphic (i.e. identical up to renaming of events). Therefore in Section 3 we define the class of *canonical event structures*, which are particular event structures. We show for any event structure E that E is a canonical event structure if and only if $E \cong E_D$ for some event domain $(D, \leqslant)$. As a consequence we obtain that two canonical event structures E_1, E_2 are isomorphic if and only if their canonical domains $(D(E_1), \subseteq), (D(E_2), \subseteq)$ are order-isomorphic.

In a sequel to this paper [3], we apply the present results to show that stable injection-projection pairs can also be used to solve recursive domain equations for arbitrary event domains; this generalizes results of Berry and Curien [1] for distributive concrete domains.

2. Event structures and event domains

Let us start with a precise definition of event structures and their canonically associated domains. For any set E, let $\mathrm{Fin}(E)$ denote the system of all finite subsets of E.

Definition 2.1 (cf. [10]). An *event structure* is a triple $\mathscr{E} = (E, \mathrm{Cons}, \vdash)$ satisfying the following conditions:

(a) E is a set (the *events* or units of information);

(b) $\mathrm{Cons} \subseteq \mathrm{Fin}(E)$ is non-empty (the *consistent* sets) and whenever $A \subseteq B$ and $B \in \mathrm{Cons}$, then $A \in \mathrm{Cons}$;

(c) $\vdash \subseteq \mathrm{Cons} \times E$ (the *enabling relation* between consistent subsets and elements of E) and whenever $A \vdash e$, $A \subseteq B$ and $B \in \mathrm{Cons}$, then $B \vdash e$.

If there is no ambiguity, we also denote $(E, \text{Cons}, \vdash)$ simply by E. We say that a set $A \subseteq E$ is *consistent* iff $A \in \text{Cons}$. A subset X of E is a *state* of E, if the following two conditions are satisfied:

(1) $A \subseteq X$, A finite$\Rightarrow A \in \text{Cons}$ (consistency);

(2) $e \in X \Rightarrow \exists e_1, \ldots, e_n \in X$ such that $e_n = e$ and

$$\forall i \leqslant n, \{e_j : j < i\} \vdash e_i \quad \text{(deductibility).}$$

The set of all states of E, partially ordered by inclusion, is denoted by $(D(\mathscr{E}), \subseteq)$ (or simply $(D(E), \subseteq)$) and called the *canonical event domain associated with E.*

First we depict a few simple event structures and their canonical domains (Fig. 1) which provide typical examples for many phenomena occurring below.

(1) $E_1 = \{1, 2, 3\}$, $\text{Cons} = \text{Fin}(E_1) \setminus \{E_1\}$, $A \vdash i$ for each $A \in \text{Cons}$, $i \in E_1$.

(2) $E_2 = E_1$, Cons as in (1), $A \vdash 1$, $A \vdash 2$ for each $A \in \text{Cons}$, and $A \vdash 3$ iff $A \in \text{Cons}$ and $1 \in A$ or $2 \in A$.

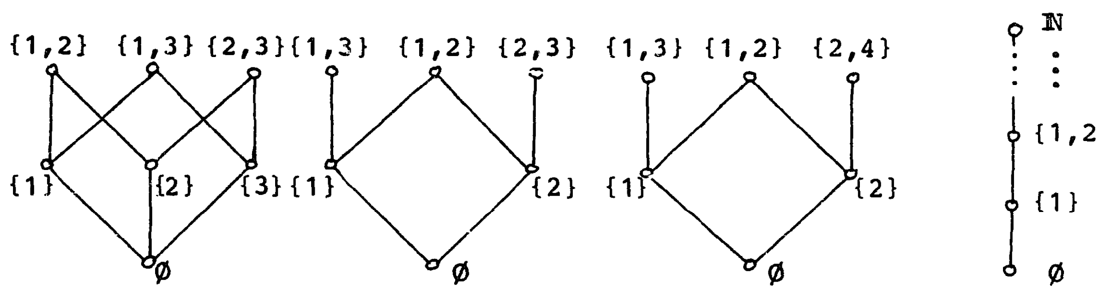

Fig. 1.

(3) $E_3 = \{1, 2, 3, 4\}$, $\text{Cons} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}\}$, $A \vdash 1$, $A \vdash 2$ for each $A \in \text{Cons}$, and $A \vdash 3$ iff $1 \in A$, $A \vdash 4$ iff $2 \in A$ ($A \in \text{Cons}$).

(4) $E_4 = \mathbb{N}$, $\text{Cons} = \text{Fin}(\mathbb{N})$, $A \vdash 1$ for each $A \in \text{Cons}$, and $A \vdash i+1$ iff $\{1, \ldots, i\} \subseteq A$ ($i \in \mathbb{N}$, $A \in \text{Cons}$).

Our first main goal is to characterize the partial orders $(D, \leqslant)$ occurring as canonical event domains $(D(E), \subseteq)$. Our notation needed for this task is standard (cf. [2]); we summarize it here for the convenience of the reader.

Let $(D, \leqslant)$ be a partially ordered set. For $x, y \in D$ we write $x \uparrow y$ if there is $z \in D$ with $x \leqslant z$ and $y \leqslant z$, and $x \not\uparrow y$ otherwise. A subset A of D is *directed* if for any $a, b \in A$ there is $c \in A$ with $a \leqslant c$ and $b \leqslant c$. $(D, \leqslant)$ is *complete*, if D has a smallest element, denoted by $\perp$, and any directed subset of D has a supremum in D. An element $x \in D$ is called *isolated* (or *compact*), if for any directed subset A of D for which $\sup A$ exists and $x \leqslant \sup A$ there is $y \in A$ with $x \leqslant y$. The set of all isolated points of D is denoted by D^0. Then D is *algebraic*, if for each $x \in D$ the set $\{d \in D^0 : d \leqslant x\}$ is directed and has x as supremum. Let $x, y \in D$. We write $x \multimap y$ if y covers x, i.e. if $x < y$ and there is no $z \in D$ with $x < z < y$. A *chain* from x to y is a sequence $x_0, \ldots, x_n$ in D such that $\;= x_0, y = x_n$ and $x_i \multimap x_{i+1}$ for each $i = 0, \ldots, n-1$. A *prime interval* of D is a pair (x, x') such that $x, x' \in D^0$ and $x \multimap x'$;

this pair is then denoted by $[x, x']$. For prime intervals we put $[x, x'] \prec [y, y']$ if $x \prec y$, $x' \prec y'$ and $y \neq x'$. If s, t are elements (prime intervals) of D, a *zigzag* from s to t is a sequence $s_0, \ldots, s_n$ of elements (prime intervals), respectively, of D such that $s = s_0$, $t = s_n$, and for all $0 \leq i < n$, either $s_i \prec s_{i+1}$ or $s_{i+1} \prec s_i$. We call two prime intervals $[x, x']$ and $[y, y']$ *equivalent*, denoted by $[x, x'] \asymp [y, y']$, if there exists a zigzag from $[x, x']$ to $[y, y']$. The equivalence class of $[x, x']$ is denoted by $[x, x']_{\asymp}$. For any $x \in D$, we put $s(x) = \{[z, z']_{\asymp} : z' \leq x\}$. Clearly $x \leq y$ implies $s(x) \subseteq s(y)$. Now we can state the formal definition of an event domain.

Definition 2.2. An *event domain* is an algebraic complete partial order $(D, \leq)$ satisfying the following conditions for any $x, x', y, y', z \in D^0$:

(F) $\{d \in D : d \leq x\}$ is finite;

(C) if $x \prec y$, $x \prec z$, $y \neq z$ and $y \uparrow z$, then $y \vee z$ exists and $y \prec y \vee z$, $z \prec y \vee z$;

(I) $[x, x'] \asymp [y, y']$ and $x \leq y$ imply $x' \leq y'$.

We say that an event structure E *generates* an event domain $(D, \leq)$, if $(D, \leq)$ and $(D(E), \subseteq)$ are order-isomorphic.

We note that condition (I) contains the following condition as an instance:

(R) $[x, x'] \asymp [y, y']$ and $x = y$ imply $x' = y'$.

Next we wish to check that the canonical event domain $(D(E), \subseteq)$ associated with an event structure E is indeed an event domain. As this is similar to the argument in [2, 10] for those event structures where Cons is induced by a symmetric binary relation of conflict on E, we leave most details to the reader. Note that in $(D(E), \subseteq)$ suprema are unions and any upper bounded subset of $D(E)$ has a supremum. If $[x, x'] \asymp [y, y']$ in $D(E)$, there is $e \in E$ with $x' = \dot\cup \{e\}$ and $y' = y \,\dot\cup\, \{e\}$ (here $A \,\dot\cup\, B$ always denotes a *disjoint* union); thus now $x \subseteq y$ implies $x' \subseteq y'$, and $(D(E), \subseteq)$ satisfies condition (I). Hence we obtain the following.

Proposition 2.3. *Let E be an event structure. Then* $(D(E), \subseteq)$ *is an event domain whose isolated elements are precisely the finite states of E.*

Now we wish to prove the converse of Proposition 2.3 that any event domain occurs as the canonical event domain of a suitably chosen event structure. The following result can be shown in the same way (using only axioms (F) and (C)) as the corresponding results in [2].

Lemma 2.4 ([2, Lemmas 2.2.3, 2.2.4, 2.2.6(1)]). *Let* $(D, \leq)$ *be an event domain, and let* $x, x', y \in D^0$.

(a) *If* $x \leq y$, *there exists a chain from x to y.*

(b) *If* $(y_n)_{n \leq p}$ *and* $(z_m)_{m \leq q}$ *are chains from x to y and e is any equivalence class of prime intervals, then*

$$|\{i : [y_i, y_{i+1}] \in e\}| = |\{j : [z_j, z_{j+1}] \in e\}|.$$

In particular, all chains from x to y have the same length.

(c) *If* $x \uparrow y$, *then* $x \vee y$ *exists and* $s(x \vee y) = s(x) \cup s(y)$, *and if moreover* $y < x \vee y$, $x' \prec x$ *and* $x' \leq y$, *then* $y \prec x \vee y$ *and* $[y, x \vee y] \succ\!\prec [x', x]$.

As a consequence of Lemma 2.4(b) note that if $x \in D^0$ and $(x_i)_{i \leq n}$ is any chain from $\perp$ to x, then $s(x) = \{[x_i, x_{i+1}]_{\succ\prec}: 0 \leq i < n\}$. Now let e be any equivalence class of prime intervals of D; we put $n(x, e) = |\{i: [x_i, x_{i+1}] \in e\}|$. By Lemma 2.4(b), this number is independent from the particular choice of the chain $(x_i)_{i \leq n}$. Next we wish to show that $n(x, e) \leq 1$. Observe that if $x, y \in D^0$, $n(y, e) > n(x, e)$ and $(x_i)_{i \leq m}$ is a zigzag from x to y, then induction on m shows that $[x_i, x_{i+1}] \in e$ for some $0 \leq i < m$ [2, Lemma 2.2.7].

Lemma 2.5. *Let* $(D, \leq)$ *be an event domain, and let* $x, x', y, y' \in D^0$ *such that* $x \prec x' \leq y \prec y'$. *Then* $\neg([x, x'] \succ\!\prec [y, y'])$.

Proof. Suppose we had $[x, x'] \succ\!\prec [y, y']$. Let $e = [x, x']_{\succ\prec}$. Let $([x_i, x_i'])_{i \leq n}$ be a zigzag from $[x, x']$ to $[y, y']$. Then $(x_i)_{i \leq n}$ is a zigzag from x to y, and as noted above we have $[x_i, x_{i+1}] \in e$ for some $i < n$. Since $[x_i, x_i'] \in e$, we obtain $x_i' = x_{i+1}$ by (I), contradicting $[x_i, x_i'] \prec [x_{i+1}, x_{i+1}']$. □

Next we associate with each event domain a canonical event structure.

Definition 2.6. Let $(D, \leq)$ be an event domain. We define an event structure $\mathscr{E}_D = (E_D, \mathrm{Cons}, \vdash)$ as follows:

(1) Let E_D be the set of all equivalence classes of prime intervals of D.

(2) Let Cons be the system of all finite subsets A of E_D for which there are representatives $x_a, x_a' \in D^0$ such that $a = [x_a, x_a']_{\succ\prec}$ for each $a \in A$ and the set $X = \{x_a': a \in A\}$ is bounded above in D.

(3) If $A \in \mathrm{Cons}$ and $e \in E_D$, we put $A \vdash e$ iff $e = [x, x']_{\succ\prec}$ and $s(x) \subseteq A$ for some $x, x' \in D^0$.

Then $\mathscr{E}_D = (E_D, \mathrm{Cons}, \vdash)$ is called the *canonical event structure associated with* $(D, \leq)$.

After these preparations, we can prove our first main result. Note that if $(D_1, \leq), (D_2, \leq)$ are two algebraic complete partial orders and $f: D_1 \to D_2$ maps $(D_1^0, \leq)$ isomorphically onto $(D_2^0, \leq)$ and is continuous (i.e. $f(\sup A) = \sup f(A)$ for any non-empty directed subset A of D_1), then f is an isomorphism from $(D_1, \leq)$ onto $(D_2, \leq)$.

Theorem 2.7. *For any event structure* $(E, \mathrm{Cons}, \vdash)$, $(D(E), \subseteq)$ *is an event domain. Conversely, let* $(D, \leq)$ *be an event domain and* $\mathscr{E}_D = (E_D, \mathrm{Cons}, \vdash)$ *the canonical event structure associated with* $(D, \leq)$. *Then the mapping*

$$s: (D, \leq) \to (D(E_D), \subseteq), \quad \textit{defined by } x \mapsto s(x), x \in D,$$

is an isomorphism.

Proof. The first assertion is immediate by Proposition 2.3. Now let $(D, \leqslant)$ be an event domain. Let $x \in D$; we claim that $s(x)$ is a state of $\mathscr{E}_D$. If $A \subseteq s(x)$ is finite, for each $a \in A$ there are $x_a, x'_a \in D^0$ such that $a = [x_a, x'_a]_{\sim}$ and $x'_a \leqslant x$. Hence $A \in \text{Cons}$. If $e \in s(x)$, there is $[y, y'] \in e$ with $y' \leqslant x$. Thus $s(y) \subseteq s(x)$ and $s(y) \vdash e$. By induction on the length of a chain from $\perp$ to y', this proves our claim.

Next we show that s maps $(D^0, \leqslant)$ isomorphically onto $(D^0(E_D), \subseteq)$. Let $x, y \in D^0$. Clearly $x \leqslant y$ implies $s(x) \subseteq s(y)$. Now suppose $s(x) \subseteq s(y)$. We claim that $x \leqslant y$ and prove this by induction on the length of a chain from $\perp$ to x. So let us assume that there is $x' \in D^0$ with $x' \prec x$ and $x' \leqslant y$. Let

$$y_0 \prec \cdots \prec y_p = x' \prec \cdots \prec y_n$$

be a chain from $\perp$ to y passing through x'. Since $[x', x] \in s(x) \subseteq s(y)$, by Lemma 2.4(b) we have $[x', x] \rightarrowtail [y_i, y_{i+1}]$ for some $i < n$. Then $p \leqslant i$ by Lemma 2.5. Now (I) implies $x \leqslant y_{i+1} \leqslant y$.

To show that s maps D^0 onto $D^0(E_D)$, let $S \subseteq E_D$ be a finite state. We can enumerate $S = \{e_1, \ldots, e_n\}$ such that

$$\forall i \leqslant n, \{e_j : j < i\} \vdash e_i.$$

Let $S' = \{e_j : j < n\}$ and assume by induction that $S' = s(x')$ for some $x' \in D^0$. Choose $X \subseteq S'$ minimal with respect to $X \vdash e_n$. Thus there are $z, z' \in D^0$ such that $X = s(z)$ and $[z, z'] \in e_n$. We claim that $z' \uparrow x'$. As S is consistent, there are $a_i, a'_i \in D^0$ such that $e_i = [a_i, a'_i]_{\sim}$ $(1 \leqslant i \leqslant n)$ and the set $A = \{a'_i : 1 \leqslant i \leqslant n\}$ has an upper bound $a \in D^0$. Hence

$$s(a) \supseteq \bigcup_{i=1}^{n} s(a'_i) \supseteq S = s(x') \cup s(z'),$$

so $a \geqslant x'$ and $a \geqslant z'$ by what we have shown in the previous paragraph. Thus, by Lemma 2.4(c), $z' \vee x' \in D^0$ and $S = s(z') \cup s(x') = s(z' \vee x')$.

Finally, observe that $s(x) = \bigcup \{s(x^0) : x^0 \in D^0, x^0 \leqslant x\}$ for each $x \in D$, hence $s : D \to D(E_D)$ is continuous. Consequently, s is an isomorphism from $(D, \leqslant)$ onto $(D(E_D), \subseteq)$. □

In the remainder of this section we wish to study the relationship between the event structures and domains of Winskel [9] (cf. [2]) and the event structures and event domains considered here. We will also show how Winskel's representation theorem can be derived from Theorem 2.7. The subsequent definition of conflict event structures, in which a symmetric binary relation of conflict is replaced by a consistency predicate, can be easily seen to be equivalent to Winskel's original one.

Definition 2.8. (a) An event structure $(E, \text{Cons}, \vdash)$ is called a *conflict event structure*, if for any finite subset A of E, A is consistent iff each subset B of A with precisely two elements is consistent.

(b) A *conflict event domain* is an algebraic complete partial order $(D, \leqslant)$ satisfying (F), (C) (cf. Definition 2.2) and for any $x, x', x'', y, y', y'' \in D^0$ the following axioms:

(R) $[x, x'] \rightarrowtail [x, x'']$ implies $x' = x''$;

(V) $[x, x'] \rightarrowtail [y, y'], [x, x''] \rightarrowtail [y, y'']$ and $x' \uparrow x''$ imply $y' \uparrow y''$.

It is easy to see that for any conflict event structure $(E, \text{Cons}, \vdash)$, $(D(E), \subseteq)$ is a conflict domain. Now we show:

Proposition 2.9. *Any conflict event domain is an event domain.*

Proof. Let $(D, \leqslant)$ be a conflict event domain. To check condition (I), let $x, x', y, y' \in D^0$ with $[x, x'] \rightarrowtail [y, y']$ and $x \leqslant y$. Then $s(x') = s(x) \cup \{[x, x']_{\rightarrowtail}\} \subseteq s(y')$, and now an argument as in [2, proof of Theorem 2.2.9] shows $x' \uparrow y$. If $y = x' \vee y$, trivially $x' \leqslant y'$. If $y < x' \vee y$, we have $y \prec x' \vee y$ and $[y, y'] \rightarrowtail [x, x'] \rightarrowtail [y, x' \vee y]$ by Lemma 2.4(c), hence $x' \leqslant x' \vee y = y'$ by (R). □

Now we show the following.

Corollary 2.10 [9]. *Let $(D, \leqslant)$ be a conflict event domain. Let E_D denote the set of equivalence classes of prime intervals of D, and define a binary relation $\#$ ("conflict") and* Cons*, $\vdash^*$ *as follows*:

(a) *Whenever $z, z', z'' \in D^0$ with $z \prec z', z''$ and $z' \not\uparrow z''$, then $[z, z']_{\rightarrowtail} \# [z, z'']_{\rightarrowtail}$.*

(b) *Let* Cons* *be the system of all finite subsets A of E_D such that whenever $e_1, e_2 \in A$, then $\neg(e_1 \# e_2)$.*

(c) *For any $x, x' \in D^0$ and $A \in \text{Cons}^*$ with $x \prec x'$ and $s(x) \subseteq A$, let $A \vdash^* [x, x']_{\rightarrowtail}$. Then $\mathscr{E}^* = (E_D, \text{Cons}^*, \vdash^*)$ is a conflict event structure, and s is an isomorphism from $(D, \leqslant)$ onto $(D(\mathscr{E}^*), \subseteq)$.*

Proof. Clearly $\mathscr{E}^*$ is an event structure. By Proposition 2.9, $(D, \leqslant)$ is an event domain. We first note that by [2, Lemmas 2.2.6, 2.2.8] $(D, \leqslant)$ has the following properties where all elements considered belong to D^0:

(1) If $x \not\uparrow y$, there are z, z', z'' with $z \prec z'$, $z \prec z''$ and $z' \not\uparrow z''$ such that $[z, z']_{\rightarrowtail} \in s(x)$ and $[z, z'']_{\rightarrowtail} \in s(y)$.

(2) If $x \prec x' \leqslant y \prec y'$ and z, z', z'' satisfy $[x, x'] \rightarrowtail [z, z']$ and $[y, y'] \rightarrowtail [z, z'']$, then $z' \neq z''$ and $z' \uparrow z''$.

Now we prove the converse of (1):

(3) If $[z, z']_{\rightarrowtail} \in s(x)$ and $[z, z'']_{\rightarrowtail} \in s(y)$, then $z' \not\uparrow z''$ implies $x \not\uparrow y$.

To check this, assume $x \uparrow y$. Then $x \vee y$ exists by Lemma 2.4(c). Let $(x_i)_{i \leqslant n}$ be a chain from $\perp$ to $x \vee y$. By Lemma 2.4(b), $[z, z'] \rightarrowtail [x_i, x_{i+1}]$ and $[z, z''] \rightarrowtail [x_j, x_{j+1}]$ for some $i, j < n$. But now $i = j$ implies $z' = z''$ by (R), and $i \neq j$ implies $z' \uparrow z''$ by (2), in both cases a contradiction.

(4) Let $A = \{x_1, \ldots, x_n\} \subseteq D^0$ such that any two elements of A have an upper bound in D. Then A has a supremum in D.

For this, suppose $i<n$ is maximal such that $x = x_1 \vee \cdots \vee x_i$ exists in D, but $x \vee x_{i+1}$ does not exist. Then $x \not\uparrow x_{i+1}$, and by (1) there are $[z, z']_{\prec} \in s(x)$, $[z, z'']_{\prec} \in s(x_{i+1})$ such that $z' \not\uparrow z''$. Then $[z, z']_{\prec} \in s(x_j)$ for some $j \leq i$ by Lemma 2.4(c) and $x_j \uparrow x_{i+1}$ by assumption, contradicting (3).

Now define Cons and $\vdash$ for E_D as in Definition 2.6, and put $\mathscr{E} = (E_D, \text{Cons}, \vdash)$. We show that $D(\mathscr{E}^*) = D(\mathscr{E})$; then the result follows from Theorem 2.7.

First, let A be a finite state of $\mathscr{E}^*$. For each $a \in A$ there are $x_a, x'_a \in D^0$ such that $a = [x_a, x'_a]_{\prec}$ and $s(x_a) \subseteq A$. Thus $s(x'_a) = s(x_a) \cup \{a\} \subseteq A$. Now for any $a_1, a_2 \in A$ we have $x'_{a_1} \uparrow x'_{a_2}$, since otherwise by property (1) there are $e_i \in s(x'_{a_i})$ $(i = 1, 2)$ with $e_1 \# e_2$, contradicting $e_1, e_2 \in A$. Now property (4) shows that $X = \{x'_a : a \in A\}$ is bounded above in D. Hence $A \in \text{Cons}$ and thus $A \in D(\mathscr{E})$.

Conversely, let A be a state of $\mathscr{E}$. Suppose $e_1 \# e_2$ for some $e_1, e_2 \in A$. Then $e_1 = [z, z']_{\prec}$, $e_2 = [z, z'']_{\prec}$ with $z' \not\uparrow z''$, and also $e_1 = [x, x']_{\prec}$, $e_2 = [y, y']_{\prec}$ with $x' \uparrow y'$. But then $[z, z']_{\prec} \in s(x')$ and $[z, z'']_{\prec} \in s(y')$, contradicting (3). Hence A is a state of $\mathscr{E}^*$. □

In view of Definitions 2.2 and 2.8 and Proposition 2.9 the question arises whether any algebraic complete partial order $(D, \leq)$ satisfying axioms (F), (C) and (R) is an event domain, i.e. whether in Definition 2.2 axiom (I) can be replaced by (R). That this is not true can be seen by examining the partial order $(D, \leq)$ with 13 elements shown in Fig. 2.

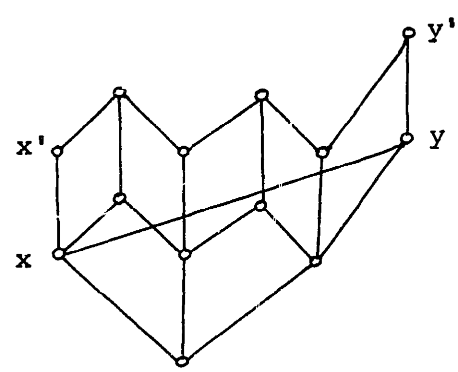

Fig. 2. $[x, x'] \prec [y, y']$, $x \leq y$ but not $x' \leq y'$.

3. Uniqueness properties of event structures

In this section we show that the canonical event structure E_D associated with an event domain $(D, \leq)$ has several properties which make it nice to behave. This allows us to derive a uniqueness criterion for event structures with given domains, and we also study the relationship between the structures E and $E_{D(E)}$, for any event structure E.

Definition 3.1. An event structure $\mathscr{E}=(E, \mathrm{Cons}, \vdash)$ will be called *canonical*, if the following conditions are satisfied:

(1) For each $x \in E$ there is a state X of E such that $x \in X$.

(2) Whenever $A \in \mathrm{Cons}$, there exists a state X of E such that $A \subseteq X$.

(3) If $e \in E$ and $A \subseteq E$ is finite and minimal with respect to $A \vdash e$, then $A \cup \{e\} \in D(E)$.

(4) Whenever $x, x', y, y' \in D^0(E)$ and $e \in E$ such that $x' = x \dot{\cup} \{e\}$ and $y' = y \dot{\cup} \{e\}$, then $[x, x'] \succ\!\prec [y, y']$ in $(D(E), \subseteq)$.

Intuitively, these conditions of Definition 3.1, with possibly the exception of condition (4), seem to be quite natural for event structures. The event structures E_1-E_4 given for Fig. 1 all satisfy conditions (1)-(3) of Definition 3.1, and E_1, E_3, E_4 also satisfy condition (4). Hence E_1, E_3, E_4 are canonical event structures. Now we show the following.

Proposition 3.2. *Let $(D, \leqslant)$ be an event domain and E_D the canonical event structure associated with $(D, \leqslant)$. Then E_D is a canonical event structure.*

Proof. Let $e \in E_D$. Then $e = [x, x']_{\succ\!\prec}$ for some $x, x' \in D^0$, and $s = s(x')$ is a state of E_D with $e \in s$. This proves condition (1) of Definition 3.1. Next, we check condition (3.1)(2). Let $A = \{e_1, \ldots, e_n\} \in \mathrm{Cons}(E_D)$. There are $x_i, x_i', z \in D^0$ such that $e_i = [x_i, x_i']_{\succ\!\prec}$ and $x_i' \leqslant z$ for all $i = 1, \ldots, n$. Hence $A \subseteq s(z)$ and $s(z)$ is a state of E_D. For condition (3.1)(3), let $e \in E_D$ and let $A \subseteq E_D$ be minimal with respect to $A \vdash e$. Then $A = s(x)$, $e = [x, x']_{\succ\!\prec}$ for some $x, x' \in D^0$. Hence $A \cup \{e\} = s(x') \in D(E_D)$.

To check condition (3.1)(4), let X, X', Y, Y' be finite states of E_D and $e \in E_D$ such that $X' = X \dot{\cup} \{e\}$ and $Y' = Y \dot{\cup} \{e\}$. By Theorem 2.7, the mapping $s : (D, \leqslant) \to (D(E_D), \subseteq)$ is an isomorphism. Choose $x, x', y, y' \in D^0$ with $s(x) = X$, $s(x') = X'$, $s(y) = Y$ and $s(y') = Y'$. Then $x \prec x'$ and $e \in X' \setminus X = s(x') \setminus s(x) = \{[x, x']_{\succ\!\prec}\}$, similarly $y \prec y'$ and $e = [y, y']_{\succ\!\prec}$. Hence $[x, x'] \succ\!\prec [y, y']$ in $(D, \leqslant)$, and thus $[X, X'] \succ\!\prec [Y, Y']$ in $(D(E_D), \subseteq)$. $\square$

Now let $\mathscr{E} = (E, \mathrm{Cons}, \vdash)$ and $\mathscr{E}^* = (E^*, \mathrm{Cons}^*, \vdash^*)$ be two event structures and $\varphi : E \to E^*$ a mapping. We say that φ *preserves consistency*, if $A \in \mathrm{Cons}$ implies $\varphi(A) \in \mathrm{Cons}^*$. Likewise, φ *preserves enabling*, if $A \subseteq E$, $e \in E$ and $A \vdash e$ imply $\varphi(A) \vdash \varphi(e)$. Finally, φ is an *isomorphism* from $\mathscr{E}$ onto $\mathscr{E}^*$, if φ is bijective and both φ and φ^{-1} preserve consistency and enabling. We write $\mathscr{E} \cong \mathscr{E}^*$ (or, also, simply $E \cong E^*$) if there exists an isomorphism from $\mathscr{E}$ onto $\mathscr{E}^*$. Next we study the relationship between the event structures E and $E_{D(E)}$.

Definition 3.3. Let E be an event structure. Define $\varphi : E_{D(E)} \to E$ by putting $\varphi([x, x']_{\succ\!\prec}) = e$ whenever $x, x' \in D^0(E)$ such that $x' = x \dot{\cup} \{e\}$. Then φ is called the *canonical mapping* from $E_{D(E)}$ into E.

Now we show that φ preserves the consistency and enabling relations and that as a set-valued mapping operating in the natural way on the states of $E_{D(E)}$, φ is inverse to s.

Proposition 3.4. *Let E be an event structure and $\varphi : E_{D(E)} \to E$ the canonical mapping. Then $\varphi(s(x)) = x$ for each state $x \in D(E)$. Also, φ preserves the consistency and enabling relations.*

Proof. Let $x = \{e_1, \ldots, e_n\} \in D^0(E)$. We may assume that the enumeration of the e_i is such that $\overline{e_i} := \{e_1, \ldots, e_i\} \in D(E)$ for each $i = 1, \ldots, n$. Put $\overline{e_0} := \emptyset$. Then $s(x) = \{[\overline{e_i}, \overline{e_{i+1}}]_{\sim} : i = 0, \ldots, n-1\}$. As $\varphi([\overline{e_i}, \overline{e_{i+1}}]_{\sim}) = e_{i+1}$, we obtain $\varphi(s(x)) = x$. For arbitrary elements $x \in D(E)$ note that $s(x) = \bigcup \{s(x^0) : x^0 \in D^0(E), x^0 \leqslant x\}$.

Now let $A \in \mathrm{Cons}(E_D)$. There are states x_i, x_i', z of E and elements $e_i \in E$ such that $x_i' = x_i \dot{\cup} \{e_i\} \subseteq z$ for all $i = 1, \ldots, n$ and $A = \{[x_i, x_i']_{\sim} : i = 1, \ldots, n\}$. Then $\varphi(A) = \{e_1, \ldots, e_n\} \subseteq z$, proving $\varphi(A) \in \mathrm{Cons}(E)$.

Next, let $A \subseteq E_D$ and $e \in E_D$ with $A \vdash e$. Choose $x, x' \in D^0$ such that $e = [x, x']_{\sim}$ and $s(x) \subseteq A$. Then $x' = x \dot{\cup} \{e'\}$ for some $e' \in E$. As shown above, we have $\varphi(s(x)) = x$ and $\varphi(e) = e'$. There is $B \subseteq x \subseteq \varphi(A)$ with $B \vdash e'$. Since $A \in \mathrm{Cons}(E_D)$, we obtain $\varphi(A) \in \mathrm{Cons}(E)$ as shown above. Hence $\varphi(A) \vdash \varphi(e)$. □

Next we state the main result of this section.

Theorem 3.5. *Let $\mathscr{E} = (E, \mathrm{Cons}, \vdash)$ be an event structure and $\varphi : E_{D(E)} \to E$ the canonical mapping.*

(a) *φ is surjective if and only if $\mathscr{E}$ satisfies condition* (3.1)(1).

(b) *φ is injective if and only if $\mathscr{E}$ satisfies condition* (3.1)(4).

(c) *φ is an isomorphism if and only if $\mathscr{E}$ is a canonical event structure.*

Proof. We write $D = D(E)$ for abbreviation.

(a) Assume $\mathscr{E}$ satisfies condition (3.1)(1), and let $e \in E$. Choose a state z of E with $e \in z$ and then finite states $x, x' \subseteq z$ such that $x' = x \dot{\cup} \{e\}$. Then $\varphi([x, x']_{\sim}) = e$. Hence φ is surjective. The converse is clear.

(b) Trivial.

(c) One implication is clear by Proposition 3.2. Now assume that $\mathscr{E}$ is a canonical event structure. By (a) and (b), φ is bijective, and by Proposition 3.4, φ preserves consistency and enabling. We show that φ^{-1} preserves consistency. Let $A = \{e_1, \ldots, e_n\} \in \mathrm{Cons}(E)$. There exists a state z of E with $A \subseteq z$. Choose states $x_i, x_i' \in D^0$ such that $x_i' = x_i \dot{\cup} \{e_i\} \subseteq z$ for each $i = 1, \ldots, n$. Then $\varphi^{-1}(A) = \{[x_i, x_i']_{\sim} : i = 1, \ldots, n\} \in \mathrm{Cons}(E_D)$.

Finally, we show that φ^{-1} preserves enabling. Let $A \subseteq E$ and $e \in E$ with $A \vdash e$. Choose $x \subseteq A$ minimal with respect to $x \vdash e$. Then $x' = x \cup \{e\}$ is a state of E. Hence $e \notin x$ and $x \in D$. Also, $\varphi(s(x)) = x$ by Proposition 3.4, and $\varphi([x, x']_{\sim}) = e$. Thus $s(x) \subseteq \varphi^{-1}(A)$. Since φ^{-1} preserves consistency, we obtain $\varphi^{-1}(A) \in \mathrm{Cons}(E_D)$ and hence $\varphi^{-1}(A) \vdash \varphi^{-1}(e)$. Thus φ is an isomorphism. □

We have two immediate consequences of Theorem 3.5(c).

Corollary 3.6. *Let E be an event structure. The following are equivalent:*
(1) *E is a canonical event structure.*
(2) $E \cong E_D$ *for some event domain* $(D, \leqslant)$.

Proof. (1)→(2): Apply Theorem 3.5(c) with $D = D(E)$.
(2)→(1): Immediate by Proposition 3.2. □

Finally, we state our uniqueness result for canonical event structures with given event domains.

Corollary 3.7. *Let* E_1, E_2 *be two canonical event structures which generate the same event domain* $(D, \leqslant)$. *Then* E_1 *and* E_2 *are isomorphic.*

Proof. Since $D(E_1) \cong D \cong D(E_2)$, by Theorem 3.5(c) we have $E_1 \cong E_{D(E_1)} \cong E_{D(E_2)} \cong E_2$. □

As a consequence we see that the mappings $E \mapsto D(E)$ and $D \mapsto E_D$ provide, up to isomorphism, inverse bijections between the classes of canonical event structures and event domains.

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