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On a Certain Class of Semilinear Volterra Diffusion Equations

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1. INTRODUCTION

In his study of the growth of biological populations Volterra [11] has proposed equations of the form

$$u_t(t)/u(t) = a - bu(t) - \int_0^t f(t-s)u(s)\,ds, \qquad t \geqslant 0, \tag{1.1}$$

which describe the growth of a single species whose population density at time t is $u(t)$. Here a and b are non-negative constants and f is a non-negative smooth function which is integrable on $[0, \infty)$. In (1.1), a logistic term (which expresses the crowding effect, etc.) is separated into two parts: a non-delay term bu, and a hereditary term represented by the Volterra integral $f * u(t) \equiv \int_0^t f(t-s)u(s)\,ds$. For the systematic investigation of (1.1), see Cushing [3], which contains a useful survey of results.

It seems very interesting to treat (1.1) in a spatially inhomogeneous situation. We assume that the species lives in a bounded domain Ω in R^n and diffuses spatially. Such a situation may be modeled, adding a diffusion term to (1.1), by the following equation for $u = u(x, t)$, $x = (x_1, x_2, ..., x_n) \in R^n$,

$$u_t = \Delta u + u(a - bu - f * u), \qquad x \in \Omega, \quad t \geqslant 0, \tag{1.2}$$

where $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$. At the boundary $\partial\Omega$ of Ω we impose the no-flux condition

$$\partial u/\partial n = 0, \qquad x \in \partial\Omega, \quad t \geqslant 0, \tag{1.3}$$

where $\partial/\partial n$ denotes the exterior normal derivative to $\partial\Omega$. Moreover, we consider the initial condition

$$u(x, 0) = u_0(x) \geqslant 0, \qquad x \in \Omega. \tag{1.4}$$

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Recently, Schiaffino [10] has treated the initial boundary value problem (1.2)–(1.4) with a, $b > 0$ and obtained an interesting result. In [10], it is assumed that $\min\{u_0(x),\ x \in \bar{\Omega}\} > 0$, $f(t) \geqslant 0$ is decreasing and that the hereditary term $f * u$ is dominated by the non-delay logistic term bu in the sense that

$$\int_0^\infty f(t)\,dt = \alpha < b.$$

Under these conditions, he has shown that the solution u of (1.2)–(1.4) satisfies

$$\lim_{t \to \infty} u(x, t) = \frac{a}{b + \alpha} \qquad \text{uniformly for} \quad x \in \bar{\Omega}.$$

The main purpose of the present paper is to study what influence the hereditary term has on the asymptotic behavior of solutions of (1.2) and (1.3). As is shown by Schiaffino, the asymptotic stability of the "equilibrium" $u = a/(b + \alpha)$ remains true as long as the effect of the hereditary term is, in a sense, smaller than that of the non-delay logistic term. However, in order to proceed to further investigation it is necessary to take account of qualitative properties of the kernel function f as well as its quantitative properties. We shall give some sufficient conditions for the global asymptotic stability of the equilibrium $u = a/(b + \alpha)$ in terms of the Laplace transform of the kernel function.

The plan of this paper is as follows. In Section 2 we shall state assumptions and some preliminary results. The main results are contained in Sections 3 and 4 (Theorems 3.2 and 4.2). Their proofs are based on the energy method with the use of an appropriate Lyapunov function. As a particular result, if $f \geqslant 0$ ($\not\equiv 0$) is non-increasing and convex, it is found that the hereditary term $f * u$ has such a stabilizing effect on the solution of (1.2)–(1.4) as a non-delay logistic term has. However, for general f, we cannot always expect the global asymptotic stability of the corresponding equilibrium. Under some circumstances, bifurcation of non-constant periodic solutions may take place. In Section 5 we shall give an example which exhibits a Hopf bifurcation.

2. Assumptions and Preliminaries

2.1. *Notation*

Throughout this paper, let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$.

For $1 \leqslant p \leqslant \infty$, $L^p(\Omega)$ denotes the Banach space of measurable functions u on Ω satisfying

$$\|u\|_p = \left\{\int_\Omega |u(x)|^p \, dx\right\}^{1/p} < \infty, \qquad \text{if} \quad 1 \leqslant p < \infty,$$

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)| < \infty, \qquad \text{if} \quad p = \infty.$$

In particular, if $p = 2$, $L^2(\Omega)$ becomes a Hilbert space with the usual inner product $(\cdot, \cdot)$. We sometimes write $\|\cdot\|$ instead of $\|\cdot\|_2$ if there is no confusion. For each $1 \leqslant p \leqslant \infty$ and integer $k \geqslant 1$, $W^{k,p}(\Omega)$ denotes the usual Sobolev space of measurable functions u on Ω such that u and its distributional derivatives up to order k belong to $L^p(\Omega)$.

Let I be any subinterval of $[0, \infty)$ and let X be any Banach space. Denote by $C(I; X)$ the space of X-valued strongly continuous functions on I. For any positive integer j, $C^j(I; X)$ denotes the space of functions $u \in C(I; X)$ such that u is j-times strongly continuously differentiable on I.

2.2. *Assumptions and Positive Kernel*

We shall state our assumptions on u_0 and f.

(A.1) $u_0(x) \geqslant 0$ $(\not\equiv 0)$ is a smooth function, say, of class $C^2(\bar{\Omega})$ satisfying $\partial u/\partial n = 0$ on $\partial\Omega$.

(A.2) $f(t) \geqslant 0$ $(\not\equiv 0)$ is a $C^1[0, \infty)$-function satisfying $f \in L^1(0, \infty)$ and $tf \in L^1(0, \infty)$.

In what follows, we set

$$\alpha = \int_0^\infty f(t) \, dt.$$

As typical examples of kernel functions, the following two types will be kept in mind:

$$f(t) = (\alpha/T) \exp(-t/T) \qquad \text{and} \qquad f(t) = (\alpha t/T^2) \exp(-t/T)$$

with $T > 0$. The first function takes its maximum value at $t = 0$, while the maximum of the second is attained at $t = T$.

Let us prepare the following terminology for the kernel function (see e.g., Barbu [2, Chapter 4] or Nohel and Shea [7]).

Definition. A kernel function f is called a *positive kernel* if

$$\int_0^T (f * u(t), u(t)) \, dt \geqslant 0,$$

for all $u \in L^2(0, T; L^2(\Omega))$ and $T > 0$. Furthermore, f is called a *strongly positive kernel* if there exist positive constants ε and γ such that $f(t) - \varepsilon \exp(-\gamma t)$ is a positive kernel.

The positivity of the kernel f is interpreted in terms of its Laplace transform. Define the Laplace transform $\hat{f}$ of f by

$$\hat{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt.$$

(If $f \in L^1(0, \infty)$, then $\hat{f}(p)$ is analytic for $\operatorname{Re} p > 0$ and continuous for $\operatorname{Re} p \geqslant 0$.) We have the following useful result due to Nohel and Shea [7, Theorem 2].

PROPOSITION 2.1. *Assume that* $f \in L^1(0, \infty)$. *Then*

(i) f *is a positive kernel if and only if* $\operatorname{Re} \hat{f}(i\eta) \geqslant 0$ *for every* $\eta \in R^1$.

(ii) f *is a strongly positive kernel if and only if there exists a positive constant* γ *such that* $\operatorname{Re} \hat{f}(i\eta) \geqslant \gamma/(1 + \eta^2)$ *for every* $\eta \in R^1$.

Finally we prepare the following lemma which will be used later.

LEMMA 2.2. *Let* $f \in L^1(0, \infty)$ *and* $u \in L^2_{\text{loc}}(0, \infty; L^2(\Omega))$. *Then, for each* $T > 0$,

$$\int_0^T (f * u(t), u(t))\, dt = \int_{-\infty}^{\infty} \operatorname{Re} \hat{f}(i\eta)\, \|(\mathscr{F} u_T)(\eta)\|_2^2\, d\eta,$$

where

$$\begin{aligned} u_T(x, t) &= u(x, t), \qquad && \text{for} \quad t \in [0, T], \\ &= 0, && \text{for} \quad t \in (-\infty, \infty) \backslash [0, T] \end{aligned}$$

and $\mathscr{F} u_T$ *denotes the Fourier transform of* u_T *with respect to t-variable*

$$(\mathscr{F} u_T)(x, \eta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\eta t} u_T(x, t)\, dt.$$

Proof. First observe

$$\int_0^T (f * u(t), u(t))\, dt = \int_{-\infty}^{\infty} (g_T(t), u_T(t))\, dt,$$

where $g_T(t) = \int_0^\infty f(s) u_T(t - s)\, ds$. Since $(\mathscr{F} g_T)(\eta) = \hat{f}(i\eta)(\mathscr{F} u_T)(\eta)$, Parseval's equality yields the conclusion. Q.E.D.

2.3. *Local Existence Results*

Now we consider the initial boundary value problem (1.2)–(1.4). Before stating local existence results for solutions, we observe the following fact: if a local solution $u(x, t)$ $(x \in \Omega,\ t \in [0, t_0])$ of (1.2)–(1.4) has a prolongation $\tilde{u}(x, t)$ $(x \in \Omega,\ t \in [0, t_1]$ with $t_1 > t_0)$, then $v(x, t) \equiv \tilde{u}(x, t + t_0)$ $(x \in \Omega,\ t \in [0, t_1 - t_0])$ satisfies (1.2) with a replaced by $a - \int_0^{t_0} f(t + t_0 - s)\, u(x, s)\, ds$. Hence it will be more convenient to get local existence results for

$$u_t = \Delta u + u\{a(x, t) - bu - f * u\}, \qquad x \in \Omega, \quad t \geqslant 0, \tag{2.1}$$

(in place of (1.2)).

Let $p > n$ be fixed. In $L^p(\Omega)$, define a closed linear operator A with dense domain $D(A)$ by

$$Au = -\Delta u, \quad D(A) = \{u \in W^{2,p}(\Omega);\ \partial u/\partial n = 0 \text{ on } \partial\Omega\}.$$

For each $0 \leqslant \mu \leqslant 1$, we introduce the fractional power spaces $X_\mu \equiv D(A^\mu)$ equipped with the graph norm of A^μ. Since it is well known that

$$X_\mu \subsetneq C^\nu(\bar{\Omega}) \qquad \text{if} \quad 0 \leqslant \nu < 2\mu - (n/p)$$

("$\subsetneq$" means that the inclusion is continuous), the following relation

$$X_\mu \subsetneq \{u \in C^1(\bar{\Omega});\ \partial u/\partial n = 0 \text{ on } \partial\Omega\} \tag{2.2}$$

holds if μ is chosen so close to 1 that $2\mu > 1 + (n/p)$ is satisfied. Moreover, $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t \geqslant 0}$ which has the estimate

$$\|A^\mu e^{-tA} u\|_p \leqslant (M_\mu / t^\mu)\, \|u\|_p, \quad 0 < t \leqslant 1,\ 0 \leqslant \mu \leqslant 1, \tag{2.3}$$

with some $M_\mu > 0$ $(M_0 = 1)$ (see e.g., Krein [4]).

Clearly, the initial boundary value problem (2.1), (1.3), and (1.4) can be reduced to the integral equation

$$\begin{aligned} u(t) &= e^{-tA} u_0 + \int_0^t e^{-(t-s)A} u(s)\, \{a(s) - bu(s) - f * u(s)\}\, ds \\ &\equiv (Su)(t). \end{aligned} \tag{2.4}$$

For the time being, suppose that $u_0 \in D(A)$, $f \in C^1[0, T]$, and $a \in C^\theta([0, T]; C(\bar{\Omega}))$ with $0 < \theta < 1$ (which denotes the space of $C(\bar{\Omega})$-valued functions v on $[0, T]$ such that $t \to v(t)$ is Hölder-continuous with exponent θ). By virtue of (2.2) and (2.3), it can be shown that, with a sufficiently small T_0

$(0 < T_0 \leqslant T)$, S is a contraction mapping sending a suitable ball in $C([0, T_0]; C(\bar{\Omega}))$ into itself. Hence the standard method based on Banach's fixed-point theorem allows us to conclude the existence and uniqueness of a solution $u \in C([0, T_0]; C(\bar{\Omega}))$ of (2.4). Since $u_0 \in D(A)$ and $a \in C^\theta([0, T]; C(\bar{\Omega}))$, one can also show, with use of the technics developed by Pazy [8, pp. 30–32], that $t \to u(t)$ is Hölder-continuous in $C(\bar{\Omega})$. This fact yields the Hölder continuity of $t \to u(t)\ \{a(t) - bu(t) - f * u(t)\}$ in $L^p(\Omega)$, so that we find in the usual manner that u actually satisfies (2.1), (1.3), and (1.4) in $L^p(\Omega)$.

Summarizing these results we have:

PROPOSITION 2.3. *If $u_0 \in D(A)$, $f \in C^1[0, T]$ and $a \in C^\theta([0, T]; C(\bar{\Omega}))$ with $0 < \theta < 1$, then there exists a positive constant T_0 $(\leqslant T)$ such that the initial boundary value problem* (2.1), (1.3), *and* (1.4) *has a unique solution $u \in C^1([0, T_0]; L^p(\Omega)) \cap C([0, T_0]; D(A))$.*

Moreover, u has the following properties:

(i) *$u(x, t) \geqslant 0$ for $x \in \bar{\Omega}$ and $t \in [0, T_0]$ if $u_0(x) \geqslant 0$ for $x \in \bar{\Omega}$.*

(ii) *If u_0 $(\geqslant 0)$ is not identically zero, then $u(x, t)$ is positive for $x \in \bar{\Omega}$ and $t \in (0, T_0]$.*

Remark 2.1. The non-negativity (positivity) of u in Proposition 2.3 is derived from the maximum principle for parabolic differential equations.

Remark 2.2. Let u be a maximal solution of (1.2)–(1.4) on $[0, T)$; in other words, there is no solution of (1.2)–(1.4) on $[0, T')$ if $T' > T$. If $\|u(t)\|_\infty$ is bounded on $[0, T) \cap [0, \tau]$ for any $\tau > 0$, then we can show $T = \infty$ by using a translation argument (cf. [8]).

3. RESULTS FOR VOLTERRA EQUATIONS WITH NON-DELAY LOGISTIC TERMS

Throughout this section, (A.1) and (A.2) are always assumed and a is a non-negative constant. Moreover, b is assumed to be a positive constant, which means the presence of a non-delay logistic term in (1.2).

We are interested in not only the existence of a bounded global solution for (1.2)–(1.4), but also its asymptotic behavior as $t \to \infty$. First we have:

THEOREM 3.1. *The initial boundary value problem* (1.2), (1.3), *and* (1.4) *has a unique solution $u \in C^1([0, \infty); L^p(\Omega)) \cap C([0, \infty); D(A))$ which satisfies*

$$0 \leqslant u(x, t) \leqslant \max\{\|u_0\|_\infty, a/b\} \equiv m, \qquad x \in \bar{\Omega}, \quad t \geqslant 0, \tag{3.1}$$

and

$$|\text{grad}\, u(x,t)| \leqslant M, \qquad x \in \bar{\Omega}, \quad t \geqslant 0, \tag{3.2}$$

where $|\text{grad}\, u|^2 = \sum_{i=1}^{n} (\partial u/\partial x_i)^2$ *and* M *is a positive constant.*

Proof. Since $u_0 \geqslant 0$, the non-negativity of u follows from Proposition 2.3. Moreover, the comparison theorem for parabolic differential equations enables us to derive an a priori estimate

$$u(x,t) \leqslant \max\{\|u_0\|_\infty, a/b\}, \qquad \text{for} \quad x \in \bar{\Omega}, \quad t \geqslant 0.$$

Hence, in view of Proposition 2.3 and Remark 2.2, we may conclude the existence and uniqueness of a global solution of (1.2)–(1.4) which satisfies (3.1).

It remains to prove (3.2). Let us first observe that u satisfies

$$u(t) = e^{-A} u(t-1) + \int_{t-1}^{t} e^{-(t-s)A} F(s)\, ds, \qquad \text{for} \quad t \geqslant 1, \tag{3.3}$$

where $F(t) = u(t)\, \{a - bu(t) - f * u(t)\}$. By (3.1), it is easy to show

$$\|F(t)\|_\infty \leqslant m\{a + (b + \alpha)m\} \equiv m_1, \qquad t \geqslant 0. \tag{3.4}$$

Operating A^μ $(0 \leqslant \mu < 1)$ to both sides of (3.3) and recalling (2.3), we have

$$\|A^\mu u(t)\|_p \leqslant \|A^\mu e^{-A} u(t-1)\|_p + \int_{t-1}^{t} \|A^\mu e^{-(t-s)A} F(s)\|_p\, ds$$

$$\leqslant M_\mu \left\{ \|u(t-1)\|_p + \int_{t-1}^{t} \frac{\|F(s)\|_p}{(t-s)^\mu}\, ds \right\}, \qquad t \geqslant 1,$$

which, together with (3.1) and (3.4), gives

$$\|A^\mu u(t)\|_p \leqslant M_\mu |\Omega|^{1/p} (m + (m_1/(1-\mu))), \qquad t \geqslant 1, \quad 0 \leqslant \mu < 1, \tag{3.5}$$

where $|\Omega|$ denotes the volume of Ω.

For $0 \leqslant t < 1$, use the following integral equation in place of (3.3):

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} F(s)\, ds.$$

Then one can show

$$\|A^\mu u(t)\|_p \leqslant \|A^\mu u_0\|_p + (1/(1-\mu))\, M_\mu m_1 |\Omega|^{1/p},$$
$$0 \leqslant t < 1, \qquad 0 \leqslant \mu < 1, \tag{3.5$'$}$$

in the same way as (3.5).

Consequently, it follows from (3.5) and (3.5)′, combined with (2.2), that there exists a positive constant M satisfying (3.2). Q.E.D.

Theorem 3.1 assures the existence of a bounded global solution of (1.2)–(1.4). We are ready to study its asymptotic behavior as $t \to \infty$. Set

$$\beta \equiv \inf\{\operatorname{Re} \hat{f}(i\eta);\ \eta \in R^1\}.$$

THEOREM 3.2. *Let $b+\beta>0$. Then the solution u of* (1.2)–(1.4) *satisfies*

$$\lim_{t\to\infty} u(x,t) = a/(b+\alpha) \qquad \text{uniformly for} \quad x \in \bar{\Omega}.$$

Remark 3.1. Theorem 3.2 extends the result of Schiaffino [10]. He has shown the global asymptotic stability of the "equilibrium" $u = a/(b+\alpha)$ when the non-delay logistic term bu is predominant over the hereditary term $f * u$ in the sense

$$\alpha = \int_0^\infty f(t)\,dt < b.$$

Our condition $b+\beta>0$ always holds if $\alpha<b$, because

$$\beta = \inf_{\eta\in R^1} \operatorname{Re}\hat{f}(i\eta) = \inf_{\eta\in R^1} \int_0^\infty \cos \eta t f(t)\,dt \geqslant -\alpha > -b.$$

However, this equilibrium may be globally asymptotically stable even if $b>\alpha$ does not hold. For example, if $f \in C^2[0,\infty)$ ($\not\equiv 0$) is a non-negative, non-increasing, and convex function, a simple calculation shows that $\operatorname{Re}\hat{f}(i\eta) \geqslant 0$ for all $\eta \in R^1$ (i.e., $\beta \geqslant 0$). In this situation, Theorem 3.2 asserts the global asymptotic stability of $a/(b+\alpha)$ independently of the size of kernel function f.

Proof of Theorem 3.2. If $a=0$, it is clear from the comparison theorem that $u(x,t)$ decays to zero (uniformly for $x\in\bar{\Omega}$) as $t\to\infty$.

In what follows, we assume $a>0$ and put $u_\infty = a/(b+\alpha) > 0$. Equation (1.2) may be rewritten as

$$u_t = \Delta u + u\left\{-b(u-u_\infty) - f*(u-u_\infty) + u_\infty \int_t^\infty f(s)\,ds\right\},$$

$$x\in\Omega, \qquad t\geqslant 0. \tag{3.6}$$

We introduce two non-negative functionals

$$E_0(u) = \int_\Omega \left\{u(x) - u_\infty - u_\infty \log \frac{u(x)}{u_\infty}\right\} dx, \tag{3.7}$$

and

$$E_1(u) = \tfrac{1}{2} \int_\Omega |\operatorname{grad} u(x)|^2 \, dx \tag{3.8}$$

(cf. Mimura and Nishida [6] and Williams and Chow [12], where similar functionals are used).

For simplicity, we shall prove Theorem 3.2 in the case $u_0 > 0$. For general $u_0 \geqslant 0$, the proof will be carried out with a slight modification, because $u(x, t)$ is positive for $x \in \Omega$ and $t > 0$ (Proposition 2.3(ii)). Differentiation of $E_0(u(t))$ with respect to t leads to

$$\begin{aligned}\frac{d}{dt} E_0(u(t)) &= \int_\Omega u_t(x, t) \left(1 - \frac{u_\infty}{u(x, t)}\right) dx \\ &= - u_\infty \int_\Omega \frac{|\operatorname{grad} u(x, t)|^2}{u(x, t)^2} dx - b \, \|u(t) - u_\infty\|^2 \\ &\quad - (f * (u - u_\infty)(t), u(t) - u_\infty) + \int_t^\infty f(s) \, ds (u_\infty, u(t) - u_\infty),\end{aligned} \tag{3.9}$$

where we have used (3.6). Integrate (3.9) over $[0, T]$ for any $T > 0$ and rearrange the resulting expression with the use of (3.1); then

$$\begin{aligned}&E_0(u(T)) + \frac{u_\infty}{m^2} \int_0^T \|\operatorname{grad} u(t)\|^2 \, dt + b \int_0^T \|u(t) - u_\infty\|^2 \, dt \\ &\quad + \int_0^T (f * (u - u_\infty)(t), u(t) - u_\infty) \, dt \\ &\leqslant E_0(u_0) + u_\infty (m + u_\infty) \, |\Omega| \int_0^\infty t f(t) \, dt \equiv K_0, \qquad T > 0.\end{aligned} \tag{3.10}$$

Observe that Lemma 2.2 yields

$$\begin{aligned}\int_0^T (f * (u - u_\infty)(t), u(t) - u_\infty) \, dt &= \int_{-\infty}^\infty \operatorname{Re} \hat{f}(i\eta) \, \|\mathscr{F}(u - u_\infty)_T(\eta)\|^2 \, d\eta \\ &\geqslant \beta \int_{-\infty}^\infty \|\mathscr{F}(u - u_\infty)_T(\eta)\|^2 \, d\eta \\ &= \beta \int_0^T \|u(t) - u_\infty\|^2 \, dt.\end{aligned} \tag{3.11}$$

(The last equality of (3.11) is due to Plancherel's theorem.) Therefore, it follows from (3.10) and (3.11) that

$$E_0(u(T)) + \frac{2u_\infty}{m^2}\int_0^T E_1(u(t))\,dt + (b+\beta)\int_0^T \|u(t)-u_\infty\|^2\,dt \leqslant K_0$$

$$\text{for every} \quad T > 0, \tag{3.12}$$

which, in particular, implies $E_1(u(\cdot)) \in L^1(0, \infty)$ and $\|u(\cdot)-u_\infty\|^2 \in L^1(0, \infty)$ (note $b+\beta > 0$).

Next, differentiating $E_1(u(t))$ with respect to t we have

$$\begin{aligned}
\frac{d}{dt}E_1(u(t)) &= (u_t(t), -\Delta u(t)) \\
&= -\|\Delta u(t)\|^2 + (\operatorname{grad}\{u(t)(a-bu(t)-f*u(t)\}, \operatorname{grad} u(t)) \\
&\leqslant -\|\Delta u(t)\|^2 + a\|\operatorname{grad} u(t)\|^2 \\
&\quad - (u(t)(f*\operatorname{grad} u)(t), \operatorname{grad} u(t)) \\
&\leqslant -\|\Delta u(t)\|^2 + a\,\|\operatorname{grad} u(t)\|^2 \\
&\quad + m(f*\|\operatorname{grad} u\|)(t)\,\|\operatorname{grad} u(t)\|,
\end{aligned}$$

where we have used the non-negativity of u. For any $T > 0$, integration of the above expression over $[0, T]$ gives

$$\begin{aligned}
&E_1(u(T)) + \int_0^T \|\Delta u(t)\|^2\,dt \\
&\quad\leqslant E_1(u_0) + 2a\int_0^T E_1(u(t))\,dt \\
&\qquad + m\int_0^T \left(\int_0^t f(t-s)\,\|\operatorname{grad} u(s)\|\,ds\right)\|\operatorname{grad} u(t)\|\,dt.
\end{aligned} \tag{3.13}$$

Since the L^2-norm on $[0, T]$ of $\int_0^t f(t-s)\,\|\operatorname{grad} u(s)\|\,ds$ is majorized by $\int_0^T f(t)\,dt\ \{\int_0^T \|\operatorname{grad} u(s)\|^2\,ds\}^{1/2}$ (use the fact that the convolution of an L^1 function with an L^2 function is an L^2 function), the right-hand side of (3.13) is bounded above by

$$E_1(u_0) + 2(a+am)\int_0^T E_1(u(t))\,dt.$$

Therefore, in view of (3.12), we get

$$E_1(u(T)) + \int_0^T \|\Delta u(t)\|^2\,dt \leqslant E_1(u_0) + \frac{(a+am)m^2K_0}{u_\infty}, \tag{3.14}$$

for any $T > 0$.

We are also able to prove

$$\left|\frac{d}{dt}\|u(t)-u_\infty\|^2\right| \leqslant 2\|\operatorname{grad} u(t)\|^2 + K_1\Big\{\|u(t)-u_\infty\|^2 + \|u(t)-u_\infty\|(f*\|u-u_\infty\|)(t) + \int_t^\infty f(s)\,ds\Big\}$$

and

$$\left|\frac{d}{dt}E_1(u(t))\right| \leqslant \|\Delta u(t)\|^2 + K_2\{\|\operatorname{grad} u(t)\|^2 + \|\operatorname{grad} u(t)\|\,(f*\|\operatorname{grad} u\|)(t)\}$$

with suitable constants K_1 and K_2. Hence, by virtue of (3.12) and (3.14), both $(d/dt)\,\|u(t)-u_\infty\|^2$ and $(d/dt)\,E_1(u(t))$ belong to $L^1(0,\infty)$; so that, since $\|u(t)-u_\infty\|^2 \in L^1(0,\infty)$ and $E_1(u(t)) \in L^1(0,\infty)$, we find

$$\lim_{t\to\infty}\|u(t)-u_\infty\| = \lim_{t\to\infty}\|\operatorname{grad} u(t)\| = 0. \tag{3.15}$$

Note the following inequality

$$\|u\|_p \leqslant \|u\|_\infty^{(p-2)/p}\,\|u\|_2^{2/p} \tag{3.16}$$

for $u\in L^\infty(\Omega)$ and $p\geqslant 2$. Both $\|u(t)-u_\infty\|_\infty$ and $\|\operatorname{grad} u(t)\|_\infty$ being bounded by (3.1) and (3.2), convergence properties (3.15) combined with (3.16) imply

$$\lim_{t\to\infty}\|u(t)-u_\infty\|_p = \lim_{t\to\infty}\|\operatorname{grad} u(t)\|_p = 0, \tag{3.17}$$

for any $p\geqslant 2$.

In order to complete the proof we invoke Sobolev's imbedding theorem; for $p>n$, there exists a positive number $C=C(\Omega,n,p)$ such that

$$\|u\|_\infty \leqslant C\{\|u\|_p + \|\operatorname{grad} u\|_p\} \equiv C\,\|u\|_{W^{1,p}} \tag{3.18}$$

for $u\in W^{1,p}(\Omega)$. The conclusion of the theorem is easily derived from (3.17) and (3.18). Q.E.D.

Remark 3.2. It seems that our condition $b+\beta>0$ is the best possible one for the global asymptotic stability of the equilibrium $u=u_\infty$ $(=a/(b+\alpha))$. For example, take $f(t)=(\alpha t/T^2)\exp(-t/T)$. Since $\operatorname{Re}\hat{f}(i\eta)=\alpha(1-\eta^2T^2)/(1+\eta^2T^2)^2 \geqslant -\alpha/8$, it follows from Theorem 3.2 that for $\alpha<8b$ any non-negative solution $(\not\equiv 0)$ of (1.2) and (1.3) converges to u_∞ (uniformly for $x\in\bar{\Omega}$) as $t\to\infty$. However, for $\alpha\geqslant 8b$, bifurcation of non-constant periodic solutions can take place (see Section 5).

4. Results for Volterra Equations without Non-Delay Logistic Terms

In this section we shall deal with initial boundary value problems for Volterra diffusion equations without non-delay logistic terms, i.e.,

$$u_t = \Delta u + u(a - f * u), \qquad x \in \Omega, \quad t \geqslant 0, \tag{4.1}$$

with (1.3) and (1.4). Conditions (*A*.1) and (A.2) are imposed on f and u_0. In general, we do not know whether problem (4.1), (1.3), and (1.4) has a bounded global solution or not. However, it will be shown that, if f is a (strongly) positive kernel, the hereditary term $f * u$ has a stabilizing effect on solutions to (4.1) and (1.3).

Put $u_\infty = a/\alpha$. We have

THEOREM 4.1. *Assume that f is a positive kernel. If $u_\infty \int_0^\infty tf(t)\,dt < 1$, then the initial boundary value problem* (4.1), (1.3), *and* (1.4) *has a unique solution $u \in C^1([0, \infty); L^p(\Omega)) \cap C([0, \infty); D(A))$ satisfying*

$$0 \leqslant u(x, t) \leqslant m, \qquad x \in \bar{\Omega}, \quad t \geqslant 0,$$

$$|\operatorname{grad} u(x, t)| \leqslant M, \qquad x \in \bar{\Omega}, \quad t \geqslant 0,$$

with some positive constants m and M.

Proof. Let u be a solution of (4.1), (1.3), and (1.4) on $[0, T]$ with any specific $T > 0$. By recalling the proof of Theorem 3.1, it suffices to derive an estimate (independent of T) for $\|u(t)\|_\infty$ to arrive at the conclusion.

If $a = 0$, then it is easy to see $\|u(t)\|_\infty \leqslant \|u_0\|_\infty$ by virtue of the comparison theorem.

Hereafter we shall assume $a > 0$ and prove the theorem in the case $u_0 > 0$; the proof for general $u_0 \geqslant 0$ is essentially the same. Note that (3.9) holds with $b = 0$. Since f is a positive kernel, integration of (3.9) yields

$$\begin{aligned} E_0(u(t)) &\leqslant E_0(u_0) + \int_0^t \int_\tau^\infty f(s)\,ds(u_\infty, u(\tau) - u_\infty)\,d\tau \\ &\leqslant E_0(u_0) + \int_0^t \int_\tau^\infty f(s)\,ds(u_\infty, u(\tau))\,d\tau \end{aligned} \tag{4.2}$$

for any $0 \leqslant t \leqslant T$, where $E_0(u)$ is defined by (3.7). Moreover, since

$$V(u) \equiv u - u_\infty - u_\infty \log(u/u_\infty)$$

is a non-negative convex function, we have, by Jensen's inequality,

$$E_0(u(t)) = \int_\Omega V(u(x,t))\,dx \geqslant |\Omega|\, V\left(\int_\Omega u(x,t)\,dx/|\Omega|\right). \tag{4.3}$$

Hence it follows from (4.2) and (4.3) that

$$\|u(t)\|_1 - u_\infty |\Omega| \log \|u(t)\|_1$$
$$\leqslant \int_\Omega \{u_0(x) - u_\infty \log(|\Omega|\, u_0(x))\}\,dx + \int_0^t \int_\tau^\infty f(s)\,ds(u_\infty, u(\tau))\,d\tau \tag{4.4}$$

for any $0 \leqslant t \leqslant T$.

Now set $m(t) = \max\{\|u(s)\|_1;\ 0 \leqslant s \leqslant t\}$. It follows from (4.4) that $m(t)$ satisfies

$$m(t) \leqslant K_1 \log m(t) + K_2 m(t) + K_3,$$

where $K_1 = u_\infty |\Omega|$, $K_2 = u_\infty \int_0^\infty tf(t)\,dt$ and $K_3 = \int_\Omega \{u_0(x) - u_\infty |\Omega| \log(|\Omega|\, u_0(x))\}\,dx$. Since $K_2 < 1$ by the assumption, we can deduce

$$m_1 \leqslant m(t) \leqslant m_2, \qquad 0 \leqslant t \leqslant T, \tag{4.5}$$

where m_1 and m_2 ($m_1 < m_2$) are positive numbers satisfying

$$(1 - K_2)m_i = K_1 \log m_i + K_3, \qquad i = 1, 2.$$

Hence, estimate (4.5) gives

$$\|u(t)\|_1 \leqslant m_2, \qquad 0 \leqslant t \leqslant T.$$

(Note that m_2 is independent of T.) We make use of the result of Alikakos [1, Theorem 3.1]: if $\sup_{t \geqslant 0} \|u(t)\|_1 \leqslant K$ with a finite positive constant K, then $\sup_{t \geqslant 0} \|u(t)\|_\infty$ is bounded above by a suitable positive constant depending on K and $\|u_0\|_\infty$. Thus we complete the proof. Q.E.D.

We are now in a position to study the asymptotic behavior of the solution u in Theorem 4.1.

THEOREM 4.2. *In addition to the assumptions of Theorem* 4.1, *suppose that f is a strongly positive kernel. Then the solution u in Theorem* 4.1 *satisfies*

$$\lim_{t \to \infty} u(x,t) = u_\infty \qquad \textit{uniformly for} \quad x \in \bar{\Omega}.$$

Proof. Since both $\|u(t)\|_\infty$ and $\|\operatorname{grad} u(t)\|_\infty$ are bounded for all $t \geqslant 0$, the idea of proof is almost the same as that of Theorem 3.2.

First we shall prove the case $a > 0$. Observe the following fact: for every $T > 0$, all the L^1-norms on $[0, T]$ of $\|\operatorname{grad} u(t)\|^2$, $(f * (u - u_\infty)(t), u(t) - u_\infty)$ and $\|\Delta u(t)\|^2$ are bounded above by a positive number K independent of T (see (3.10) and (3.13)). This fact enables us to prove

$$\lim_{t \to \infty} \|\operatorname{grad} u(t)\| = 0$$

in the same way as the proof of (3.15).

In order to complete the proof, it suffices to show

$$\lim_{t \to \infty} \|u(t) - u_\infty\| = 0. \tag{4.6}$$

From the definition of the strong positivity of f, there exist positive constants ε and γ such that $f(t) - \varepsilon e^{-\gamma t}$ is a positive kernel. Hence, for any $T > 0$,

$$\begin{aligned} K &\geqslant \int_0^T (f * (u - u_\infty)(t), u(t) - u_\infty)\, dt \\ &\geqslant \varepsilon \int_0^T (\Gamma(t), u(t) - u_\infty)\, dt, \end{aligned} \tag{4.7}$$

where

$$\Gamma(t) = \int_0^t e^{-\gamma(t-s)}(u(s) - u_\infty)\, ds.$$

Note the following;

$$\frac{1}{2}\frac{d}{dt}\|\Gamma(t)\|^2 = (\Gamma(t), u(t) - u_\infty) - \gamma \|\Gamma(t)\|^2.$$

For any $T > 0$, integration of this identity over $[0, T]$ leads to

$$\|\Gamma(t)\|^2 + 2\gamma \int_0^T \|\Gamma(t)\|^2\, dt = 2 \int_0^T (\Gamma(t), u(t) - u_\infty)\, dt \leqslant 2K/\varepsilon, \tag{4.8}$$

where (4.7) has been used. Since $\partial/\partial t\, \Gamma(t) = u(t) - u_\infty - \gamma\Gamma(t)$ is in $L^\infty(0, \infty; L^2(\Omega))$, $t \to \Gamma(t)$ is uniformly continuous on $[0, \infty)$ in $L^2(\Omega)$-norm. Hence it follows from (4.8) that

$$\lim_{t \to \infty} \Gamma(t) = 0 \qquad \text{in} \quad L^2(\Omega). \tag{4.9}$$

Moreover, since $t \to \partial\Gamma(t)/\partial t$ is also uniformly continuous on $[0, \infty)$ in

$L^2(\Omega)$-norm, we can show with the use of (4.9) and the mean value theorem that

$$\lim_{t\to\infty} \frac{\partial}{\partial t} \Gamma(t) = 0 \qquad \text{weakly in} \quad L^2(\Omega). \tag{4.10}$$

(See e.g., Barbu [2, Chapter 4] and Nohel and Shea [7], where similar technics are employed.)

It follows from (4.9) and (4.10) that

$$\lim_{t\to\infty} u(t) = u_\infty \qquad \text{weakly in} \quad L^2(\Omega)$$

and that, since $\{u(t)\}_{t>0}$ is bounded in $W^{1,2}(\Omega)$,

$$\lim_{t\to\infty} u(t) = u_\infty \qquad \text{weakly in} \quad W^{1,2}(\Omega). \tag{4.11}$$

Since $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$, (4.11) gives (4.6).

It remains to prove the case $a = 0$:

$$u_t = \Delta u - u(f * u), \qquad x \in \Omega, \quad t \geqslant 0. \tag{4.1)'$$

Integration of (4.1)' over $\Omega \times [0, T]$ with any $T > 0$ yields

$$\int_\Omega u(x, t)\, dx + \int_0^T \int_\Omega u(x, t)(f * u(x, \cdot))(t)\, dx\, dt = \int_\Omega u_0(x)\, dx. \tag{4.12}$$

Furthermore, by taking the $L^2(\Omega)$-inner product of (4.1)' with $u(t)$, it is easily seen that, for every $T > 0$,

$$\|u(t)\|^2 + \int_0^T \|\operatorname{grad} u(t)\|^2\, dt \leqslant \|u_0\|^2, \tag{4.13}$$

which insures $\|\Delta u(\cdot)\|^2 \in L^1(0, \infty)$ (see (3.13)). Making use of (4.12) and (4.13), we have only to repeat the preceding arguments to complete the proof. Q.E.D.

Remark 4.1. Theorems 4.1 and 4.2 assert that, if f is a strongly positive kernel, the equilibrium is (globally) asymptotically stable. In this sense, the hereditary term $f * u$ with a strongly positive kernel has a stabilizing effect as a nondelay logistic term has.

Especially, take a $C^2[0, \infty)$-function f ($\not\equiv 0$) such that

$$(-1)^k f^{(k)}(t) \geqslant 0, \qquad t \in [0, \infty), \quad k = 0, 1, 2.$$

It is well known that f is a strongly positive kernel (see Barbu [2] or Nohel and Shea [7]). As is stated above, the effect of time delay of such a function is not so significant.

5. Remarks on Bifurcation of Periodic Solutions

In the previous sections we have discussed the global asymptotic stability of a suitable equilibrium. In particular, if a kernel function $f \in C^2[0, \infty) \cap L^1(0, \infty)$ ($\not\equiv 0$) is non-negative, non-increasing and convex, then the hereditary term $f * u$ represents a 'weak" delay in the sense that the maximum response to the growth rate is due to current population density and past densities have a decreasing influence. In such a situation, the effect of time delay is not so significant (see Remarks 3.1 and 4.1).

In this section we shall give some remarks about effects which generic delay kernels have on the asymptotic behavior of solutions to (1.2) and (1.3). As a special kernel function, we take $f(t) = (\alpha t/T^2) \exp(-t/T)$ and regard α as a positive parameter. This is the case when the maximum influence on growth rate response at any time t is due to population density at the previous time $t - T$. It is already shown that any non-trivial solution $u \geqslant 0$ of (1.2) and (1.3) satisfies

$$\lim_{t \to \infty} u(x, t) = u_\infty \ (\equiv a/(b + \alpha)) \qquad \text{uniformly for } x \in \bar{\Omega},$$

if $\alpha < 8b$ (Remark 3.2). What will happen if $\alpha \geqslant 8b$?

In what follows, a and b are assumed to be positive. Observe that, if u is a solution of (1.2), then $\tilde{u}(t) \equiv u(t + T)$ satisfies (1.2) with $f * u(t)$ replaced by $\int_{-T}^{t} f(t-s)\, \tilde{u}(s)\, ds$. Therefore, letting $T \to \infty$ we may consider

$$u_t = \Delta u + u\left(a - bu - \int_{-\infty}^{t} f(t-s) u(s)\, ds\right), \qquad x \in \Omega, \quad t \in R^1, \quad (5.1)$$

in place of (1.2). Put $v = u - u_\infty$ and neglect second-order terms with respect to v; this linearization procedure leads us to

$$v_t = \Delta v - u_\infty \left(bv + \int_{-\infty}^{t} f(t-s) v(s)\, ds\right), \qquad x \in \Omega, \quad t \in R^1. \quad (5.2)$$

We shall seek solutions of (5.2) with zero Neumann condition in the form

$$\sum_{k=0}^{\infty} T_k(t)\, \phi_k(x),$$

where ϕ_k $(k = 0, 1, 2,...)$ are eigenfunctions corresponding to the eigenvalues λ_k (with $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$) associated with the following eigenvalue problem

$$-\Delta u = \lambda u, \qquad \text{in} \quad \Omega,$$
$$\partial u/\partial n = 0, \qquad \text{on} \quad \partial\Omega.$$

It is easy to verify that T_k $(k = 0, 1, 2,...)$ satisfy

$$\dot{T}_k(t) + (bu_\infty + \lambda_k)T_k(t) + u_\infty \int_{-\infty}^{t} f(t-s)T_k(s)\,ds = 0, \tag{5.3}$$

where "$\cdot$" means d/dt. For (5.3), Miller [5] has shown that $T_k = 0$ is asymptotically stable if and only if

$$p + (bu_\infty + \lambda_k) + u_\infty \hat{f}(p) \neq 0 \qquad \text{for} \quad \operatorname{Re} p \geqslant 0;$$

in other words,

$$\begin{aligned} D_k(p) \equiv T^2p^3 + \{2 + (bu_\infty + \lambda_k)T\}Tp^2 + \{1 + 2(bu_\infty + \lambda_k)T\}p + (a + \lambda_k) \\ \neq 0 \qquad \text{for} \quad \operatorname{Re} p \geqslant 0. \end{aligned} \tag{5.4}$$

Moreover, a straightforward application of the Hurwitz criterion assures that (5.4) is equivalent to

$$\{(bu_\infty + \lambda_k)T + 2\}\{2(bu_\infty + \lambda_k)T + 1) - (a + \lambda_k)T > 0. \tag{5.5}$$

(When $\alpha < 8b$, (5.5) always holds true for every $k = 0, 1, 2,....$) A simple calculation shows that for every $k \geqslant 1$ (5.5) is true if α is sufficiently close to $8b$, while for $k = 0$ (5.5) is violated at $\alpha = 8b$ and $aT = 9$.

Hereafter, we assume $aT = 9$ for convenience. With the use of a new parameter $\mu = 9b/(b + \alpha)$ (which means that $\alpha = 8b$ corresponds to $\mu = 1$), $D_0(p) = 0$ is rewritten as

$$p^3 + ((2 + \mu)/T)p^2 + ((1 + 2\mu)/T^2)p + (9/T^3) = 0. \tag{5.6}$$

For $u = 1$, (5.6) has three roots $\{-3/T, \pm i\sqrt{3}/T\}$, which, in particular, implies that (5.3) (with $k = 0$) has non-constant periodic solutions $\{\exp(\pm i(\sqrt{3}/T)t)\}$. Denote by $p(\mu)$ the root of (5.6) such that $p(1) = i\sqrt{3}/T$. Since $\operatorname{Re} p'(1) < 0$, this will be a case to which the Hopf bifurcation theory is applicable.

Taking account of the preceding arguments, we shall find spatially homogeneous solutions of (5.1); i.e.,

$$u_t = u(a - bu - \int_{-\infty}^{t} f(t-s)u(s)\,ds), \qquad t \in R^1,$$

which may be written with a new unknown function $v = u - u_\infty$,

$$v_t = -(v + u_\infty)\left(bv + \int_{-\infty}^{t} f(t-s)v(s)\,ds\right), \qquad t \in R^1. \tag{5.7}$$

We shall follow the argument used by Cushing [3, Chapter 5]. By putting

$$v_1 = v, \qquad v_2 = \int_{-\infty}^{t} f(t-s)v(s)\,ds, \qquad \text{and} \qquad \dot{v}_3 = v_2,$$

the single equation (5.7) is reduced to the system

$$\begin{aligned} \dot{v}_1 &= -bu_\infty v_1 - u_\infty v_2 - v_1(bv_1 + v_2), \\ \dot{v}_2 &= v_3, \\ \dot{v}_3 &= (a/T^2)v_1 - (1/T^2)v_2 - (2/T)v_3. \end{aligned}$$

Note that the characteristic equation of the matrix

$$\begin{bmatrix} -bu_\infty & -u_\infty & 0 \\ 0 & 0 & 1 \\ a/T^2 & -1/T^2 & -2/T \end{bmatrix} \qquad u_\infty = a/(b+a),$$

is given by (5.6) with $aT = 9$ and $\mu = 9b/(b+a)$.

Consequently, the Hopf bifurcation theory (see e.g., Poore [8]) asserts that, for sufficiently small $\varepsilon_1 > 0$, Eq. (5.1) has non-constant periodic solutions of the form

$$u = \frac{\mu(\varepsilon)}{bT} + \varepsilon v\left(\frac{t}{1+\varepsilon\eta(\varepsilon)}, \varepsilon\right), \qquad \mu \equiv \frac{9b}{b+a} = 1 - \varepsilon\delta(\varepsilon), \tag{5.8}$$

where η and δ are differentiable functions on $[-\varepsilon_1, \varepsilon_1]$ satisfying $\eta(0) = \delta(0) = 0$, and where $v(t, \varepsilon)$ ($\not\equiv 0$) is a $2\pi T/\sqrt{3}$-periodic function of t for each $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. Moreover, making use of the result of Poore [8, Theorem 4.1] we can show after some tedious calculations that

$$\eta'(0) > 0 \qquad \text{and} \qquad \delta'(0) > 0.$$

Since $\mu = 1 - \varepsilon\delta(\varepsilon) = 1 - \varepsilon^2\delta'(0) + o(\varepsilon^2)$ as $\varepsilon \to 0$, the bifurcated periodic orbit (5.8) exists in a neighborhood of $(u, \mu) = (1/bT, 1)$ only for $\mu < 1$, which corresponds to $a > 8b$. The period of the solution $u = u(t, \varepsilon)$ in (5.8) is

$$(2\pi T/\sqrt{3})(1 + \varepsilon\eta(\varepsilon)) = (2\pi T/\sqrt{3})(1 + \varepsilon^2\eta'(0) + o(\varepsilon^2)) \qquad \text{as} \quad \varepsilon \to 0$$

and increases from $2\pi T/\sqrt{3}$.

Remark 5.1. Let $f(t) = (at/T^2)\exp(-t/T)$ with $\alpha \geqslant 8b$. Suppose that (5.5) holds for every $k = 0, 1, 2, \ldots$; or, equivalently,

$$(2 - aT)\alpha^2 + b(4 + 3aT)\alpha + 2b^2(aT + 1)^2 > 0.$$

In this case, we can prove that $u = u_\infty$ ($\equiv a/(b + \alpha)$) is locally asymptotically stable for (1.2)–(1.4).

On the other hand, if $\alpha = \alpha_0$ satisfies

$$(2 - aT)\alpha_0^2 + b(4 + 3aT)\alpha_0 + 2b^2(aT + 1)^2 = 0,$$

then it is possible to show that non-constant periodic solutions of (5.1) and (5.2) bifurcate from $(u, \alpha) = (u_\infty, \alpha_0)$.

These results will be discussed elsewhere

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