A New Kind of Inequality for Bessel Functions

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We study a new type of inequality for Bessel functions. This is an analog of an inequality for Legendre polynomials, which plays an important role in studying the nonlinear Boltzmann equation. As an application of the Bessel case we treat the spherical functions associated with Minkowski space.

Let $J_0(x)$ denote the usual Bessel function of order zero. We want to prove the following relation:

**Theorem I.** Take $a, b, c > 0$ such that

$$a^2 = b^2 + c^2. \quad (1)$$

Then

$$J_0(a) + 1 \geq J_0(b) + J_0(c). \quad (2)$$

A similar property holds for Bessel functions of higher order, and such an extension appears as Theorem I' below.

Before taking up the proof of the above result we comment briefly upon its meaning.

The function $(x, y) \rightarrow J_0(\lambda(x^2 + y^2)^{1/2})$ is the regular eigenfunction of the Laplacian in $\mathbb{R}^2$, which depends only on the radial distance, has eigenvalue $-\lambda^2$ and is normalized by $J_0(0) = 1$.

Thus a better way of expressing (2) is by saying that the following relation, among radially symmetric eigenfunctions of the Laplacian, holds

$$F_{\lambda_1}(r) + 1 \geq F_{\lambda_2}(r) + F_{\lambda_3}(r), \quad (2')$$

provided the eigenvalues corresponding to the right hand side add up to the eigenvalue corresponding to the left hand side.

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A property identical to (2) but involving Legendre polynomials turns out to play an important role in relation to Boltzmann's equation.

For the case of a Maxwell gas the eigenvalues of the linearized operator are completely known, see [5], and are expressed as integrals involving Legendre polynomials.

The speed of approach to equilibrium for a gas described by Boltzmann's equation is clearly governed by the top negative eigenvalue, and usually one does not pay attention to the rest of the spectrum.

Now it develops that a careful study of the actual nonlinear equation requires a detailed study of certain order relation among the eigenvalues (see [1]). This in turn can be done using the relations among Legendre polynomials referred to above (also see Theorem II). We want to present the corresponding property for Bessel functions in the hope that somebody will find it useful. Another interesting problem would be to find a better proof that would make a more clever use of the property of these functions as spherical functions for the corresponding symmetric spaces. See Helgason [3] as a reference for symmetric spaces.

In this line we present another instance of a family of inequalities which ought to hold for interesting symmetric spaces. We deal with the space $SO_0(3, 1)/SO(3)$, the so-called Minkowski space. This case, as we will see, follows easily from the Bessel case. The case of $SO_0(2, 1)/SO(2)$ which leads to Legendre functions cannot be treated so simply and it is not presented here.

A word of caution is probably appropriate here. Using the representation

$$J_0(r) = \frac{1}{\pi} \int_0^\pi e^{ir \cos \phi} \, d\phi,$$

one might be tempted to deduce our inequality (2) from a stronger relation involving the integrands in (3).

But it is not hard to see that the relation

$$\cos a + 1 \geq \cos b + \cos c$$

does not hold under the sole assumption (1).

The same temptation arises when one proves the analog of (2) involving Legendre polynomials (see Theorem II), but in that case part of (4) can be put to use and the final result comes out, relying heavily on the discretness of the spectrum.

For the Bessel case the spectrum of the Laplace Beltrami operator is continuous and a different method is needed.

The fact that the spectrum is continuous presents in general serious troubles but for the case at hand we can resort to the famous result of Mehler–Heine relating Legendre polynomials with Bessel functions [4].
This formula is a manifestation of the fact that a sphere of larger and larger radius appears, close to the north pole, as being a flat plane.

The ingredients for the proof of Theorem I are (a) Theorem II below, (b) the formula of Mehler-Heine referred to above (12) and (c) a lemma that allows us to use these two facts successfully.

**Theorem II.** Let \( n, r, s \) be nonnegative integers such that

\[
n(n + 1) = r(r + 1) + s(s + 1). \tag{5}
\]

Then for \( 0 < \theta < \pi/2 \), we have

\[
P_n(\cos \theta) + 1 > P_r(\cos \theta) + P_s(\cos \theta). \tag{6}
\]

This result is proved in a previous note [2] by the author. The proof consists mainly of exploiting the Mehler representation formula

\[
P_n(\cos \theta) = \frac{2}{\pi} \int_0^\theta \frac{\cos(n + 1/2) t}{(2 \cos t - 2 \cos \theta)^{1/2}} \, dt
\]

for values of \( \theta \) which are not too large in \([0, \pi/2]\). Beyond certain value \((\theta - \pi(r + 1/2)^{-1})\) the proof is accomplished using some detailed properties of Legendre polynomials. If \( r < 5 \) the proof cannot be handled in this way and here the discretness of the spectrum is essential. One simply verifies that for \( r < 5 \) there are only two cases to be considered (i.e., \( r = s = 2, n = 3 \) and \( r = 3, s = 5, n = 6 \)) and those ones you verify directly. For higher dimensions, the proof is carried out by simple inspection.

Clearly no such easy trick will work when one has a continuum of instances to consider.

Now we turn to an important tool in the proof of Theorem I.

**Lemma.** Given real numbers \( a, b, c \) such that

\[
a^2 = b^2 + c^2,
\]

there exist three sequences of natural numbers \( a_n, b_n, c_n \) such that

\[
a_n(a_n + 1) = b_n(b_n + 1) + c_n(c_n + 1). \tag{7}
\]

The ratios \( b_n/a_n \) and \( c_n/a_n \) converge to \( b/a \) and \( c/a \) respectively. \( \tag{8} \)

**Proof.** Rewrite the diophantine equation

\[
t(t + 1) = r(r + 1) + s(s + 1) \tag{9}
\]
Then the solutions of (9) are obtained from all possible choices of the integers

\[ m, n, p, q \text{ such that} \]

\[
\begin{align*}
2t + 1 &= mq + np, \\
1 &= nq - mp, \\
2r + 1 &= nq + mp, \\
2s + 1 &= mq - np.
\end{align*}
\]

Thus we get for the ratio \( s/t \) the value

\[
\frac{s}{t} = \frac{m(q + p) - n(q + p)}{m(q + p) + n(p - q)} = \frac{m}{n} \left( \frac{q}{p} + 1 \right) - \left( \frac{q}{p} + 1 \right) = \frac{m}{n} \left( \frac{q}{p} + 1 \right) + \left( 1 - \frac{q}{p} \right) .
\]

Now take two unbounded sequences \( p_i, q_i \) so that \( p_i, q_i \) are relatively
prime and \( q_i/p_i \) converges to \( 1 < x < \infty \). Clearly one can determine two
sequences \( n_i, m_i \) such that \( 1 = n_i q_i - m_i p_i \). Define \( t_i, r_i, s_i \) in agreement
with (10) and notice that \( m_i/n_i \) converges to \( x \). From (11) we see that \( s_i/t_i \)
approaches the value

\[
\frac{x(x + 1) - (1 + x)}{x(x + 1) + (1 - x)} = \frac{(1 + x)(x - 1)}{x^2 + 1} .
\]

As \( x \) ranges in \( [1, \infty) \) this function takes any value in \( [0, 1] \). Thus for any
given value of the ratio \( 0 < b/a < 1 \) we find three sequences of nonnegative
integers \( a_n, b_n, c_n \) such that (7) is satisfied, and besides

\[
b_n/a_n \rightarrow b/a .
\]

Then (8) follows because

\[
\left( 1 + \frac{1}{a_n} \right) - \frac{b_n}{a_n} \left( \frac{b_n}{a_n} + \frac{1}{a_n} \right) + \frac{c_n}{a_n} \left( \frac{c_n}{a_n} + \frac{1}{a_n} \right)
\]

and one concludes easily from here that \( c_n/a_n \) also converges, to the value \( c/a \)
a fortiori.
Proof of Theorem I. Bring in now the celebrated theorem
\[
\lim_{n \to \infty} \frac{1}{n^3} P_{a_n} \left( \cos \frac{x}{n} \right) = \left( \frac{2}{x} \right)^a J_a(x),
\]
which holds uniformly for \(|x| < R\), \(R\) fixed.

If \(a^2 = b^2 + c^2\), define sequences \(a_n, b_n, c_n\) as in the previous lemma. For \(i\) large enough, Theorem II gives the relation
\[
P_{a_i} \left( \cos \frac{a}{a_i} \right) + 1 \geq P_{b_i} \left( \cos \frac{a}{b_i} \frac{b}{a_i} \right) + P_{c_i} \left( \cos \frac{a}{c_i} \frac{c}{a_i} \right)
\]
as \(i\) grows to infinity. Using the uniform convergence mentioned above we get
\[
J_0(a) + 1 \geq J_0(b) + J_0(c)
\]
as required.

For the sake of completeness we state now

**Theorem I'.** Take \(a, b, c \geq 0\) such that \(a^2 = b^2 + c^2\) then
\[
\Gamma \left( \frac{n}{2} \right) J_{\frac{n-2}{2}}(a) + 1 \geq \Gamma \left( \frac{n}{2} \right) J_{\frac{n-2}{2}}(b) + \Gamma \left( \frac{n}{2} \right) J_{\frac{n-2}{2}}(c),
\]
For the proof of this theorem one simply copies each of the steps carried out for the case \(n = 2\), with the obvious modifications. In the last step one needs to replace Legendre functions by the spherical functions of \(SO(n+1)/SO(n)\). For these ultraspherical polynomials, the required inequality becomes clear by inspection and this case is then essentially simpler than \(n = 2\).

**Minkowski Space**

The case \(n = 3\) in the previous theorem leads to an amusing application. We have
\[
J_{1/2}(x) = \frac{\sin x}{(x/2)^{1/2}},
\]
and thus we get from (13) the relation
\[
\frac{\sin a}{a} + 1 \geq \frac{\sin b}{b} + \frac{\sin c}{c}.
\]
Using this inequality we obtain as a bonus the corresponding inequality for the so-called Minkowski space obtained from the proper homogeneous Lorentz group $SO_0(3, 1)$ dividing by $SO(3)$.

The spherical functions in this case are particularly simple

$$F_t(r) = \frac{\sinh r(t + \frac{1}{2})}{(t + \frac{1}{2}) \sinh r}, \quad r \geq 0,$$

the corresponding eigenvalue is

$$(t - \frac{1}{2})(t + \frac{3}{2})$$

and the parameter $t$ moves in the subset

$$\{t : -\frac{3}{2} \leq t \leq \frac{1}{2}\} \cup \{t : \Re t = -\frac{1}{2}\}$$

of the complex plane. Put $t = -\frac{1}{2} + i\tau$, so that the eigenvalue corresponding to $F_t$ is $-1 - \tau^2$, and the eigenfunction inequality we want to prove reduces to

$$\frac{\sinh i\tau r}{i\tau \sinh r} + 1 \geq \frac{\sinh i\tau_1 r}{i\tau_1 \sinh r} + \frac{\sinh i\tau_2 r}{i\tau_2 \sinh r}$$

under the assumption

$$\tau^2 = 1 + \tau_1^2 + \tau_2^2. \quad (15)$$

Assume first that $\tau_1$, $\tau_2$ and $\tau$ are real, so that we are in the vertical part of the spectrum. We want to infer from (15) the relation

$$\frac{\sin \tau r}{r^\tau} + \frac{\sin r}{r} \geq \frac{\sin \tau_1 r}{r^\tau_1} + \frac{\sin r}{r_1}, \quad r \geq 0.$$

Define $\tau_3$ such that $\tau_3^2 = \tau_1^2 + \tau_2^2$, then (14) gives

$$\frac{\sin \tau_3 r}{r^\tau_3} + 1 \geq \frac{\sin \tau_1 r}{r^\tau_1} + \frac{\sin \tau_2 r}{r^\tau_2}.$$

From (15) we have $\tau^2 = 1 + \tau_3^2$, and thus

$$\frac{\sin \tau r}{r^\tau} + 1 \geq \frac{\sin \tau_3 r}{r^\tau_3} + \frac{\sin r}{r}.$$

Finally if you combine these last two relations you get

$$\frac{\sin \tau r}{r^\tau} + \frac{\sin r}{r} \geq \left(\frac{\sin \tau}{r} + \frac{\sin r}{r} - 2\right) + \frac{\sin \tau_1 r}{r^\tau_1} + \frac{\sin \tau_2 r}{r^\tau_2}$$

as required.
The case when one at least of the $\tau$'s is not real is easily taken care of if one notices that

$$a^2 + 1 = \beta^2 + \gamma^2 \geq 0$$

implies

$$a^{2n} + 1 \geq \beta^{2n} + \gamma^{2n}.$$

**Addendum.** Richard Askey has obtained a proof of the inequalities reported here, as well as some extensions, which does not make use of the relation with Legendre polynomials.

I want to thank him for a very lively correspondence and the remark that condition (1) in this paper cannot be relaxed to $a^2 \leq b^2 + c^2$.

**REFERENCES**