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Practical Stability of Impulsive Functional Differential Equations in Terms of Two Measurements

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Abstract—In this paper, we consider practical stability of impulsive functional differential equations in terms of two measurements. Some sufficient conditions of uniform practical stability for functional differential equation with impulses are obtained by using piecewise continuous Lyapunov functions and Razumikhin techniques. An example illustrates the effectiveness of the proposed result.

Keywords—Practical stability, Impulsive functional differential equation, Lyapunov function, Razumikhin technique.

1. INTRODUCTION

In many cases, some well-designed, asymptotically stable control schemes cannot work as expected. One reason is that the domain of attraction is too small. A way to overcome this problem is to use practical stability. The practical stability only needs to stabilize a system into a region of phase space. So it has a significant practice. In recent years, the qualitative properties in the mathematical theory on impulsive differential system have been very important, are interested and developed by a large number of mathematicians, see [1–13]. In [2,3,9,14,15], the authors have obtained some results for practical stability of differential equations or impulsive systems, but there are rare results for impulsive functional differential equations.

In [10], the authors have gotten some results for the uniform stability of impulsive delay differential equations. However, in the present paper, we consider more general stability for impulsive functional differential equations—practical stability. By means of piecewise continuous Lyapunov functions and Razumikhin techniques, we establish some criteria for uniform practical stability of impulsive functional differential equations in terms of two measurements. Since in [10], there is only one measurement, and it considers only stability not practical stability, our result is more general.

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The rest of this paper is organized as follows. In Section 2, we present some notations and definitions. In Section 3, some sufficient conditions of uniform practical stability for functional differential equation with impulses are obtained; an example is also discussed in this section to illustrate the theorem. Finally, conclusion remarks are given in Section 4.

2. PRELIMINARIES

Consider the impulsive functional differential problem

$$\dot{x} = f(t, x_t), \qquad \text{for } t \geq t_0, \quad t \neq t_k, \tag{2.1}$$

$$x(t_k) = x\left(t_k^-\right) + I_k\left(x\left(t_k^-\right)\right), \qquad \text{for } k \in N, \tag{2.2}$$

$$x(t + t_0) = \varphi(t), \qquad \text{for } t \in [-\tau, 0], \tag{2.3}$$

in which $x \in R^n$, $f : [0, \infty) \times D \to R^n$, D is an open set in $PC([-\tau, 0], R^n)$, where $\tau > 0$ and $PC([-\tau, 0], R^n) = \{\phi : [-\tau, 0] \to R^n, \phi(t)$ is continuous everywhere except a finite number of points $\hat{t}$ at which $\phi(\hat{t}^+)$ and $\phi(\hat{t}^-)$ exist and $\phi(\hat{t}^+) = \phi(\hat{t})\}$, $I_k \in C(R^n, R^n)$ for $k \in Z^+$, $t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$, where $t_k \to \infty$, for $t \to \infty$, $t_0 \geq 0$, $\tau = \text{const.} > 0$, $x_t \in D$ is defined by $x_t(s) = x(t + s)$, $-\tau \leq s \leq 0$. For ϕ in $PC([-\tau, 0], R^n)$, the norm of ϕ is defined by $|\phi| = \sup\{\|\phi(\theta)\| : -\tau \leq \theta \leq 0\}$, where $\|\cdot\|$ is a norm in R^n. Let $R_\tau^+ = [-\tau, \infty)$.

Throughout this paper, we introduce the following conditions.

(a) For $t \in [t_0 - \tau, t_0]$, the solution $x(t; t_0, \varphi)$ coincides with the function $\varphi(t - t_0)$.
(b) $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in $PC([-\tau, 0], R^n)$.
(c) Functions $I_k : R^n \to R^n$, $k = 1, 2, \ldots$, are such that the inequality $\|x + I_k(x)\| < H$ holds if $\|x\| \leq H$ and $I_k(x) \neq 0$, where $H = \text{const.} > 0$.
(d) $f(t, 0) \equiv 0$, $I_k(0) = 0$.

Under the upper conditions, we can see that there is a unique solution of problem (2.1)–(2.3) through (t_0, φ).

We denote the solution of impulsive functional differential problem (2.1)–(2.3) by $x(t; t_0, \varphi)$ and $J(t_0, \varphi)$—the maximal interval of the type $[t_0 - \tau, \beta)$ in which $x(t; t_0, \varphi)$ is defined.

We using the following notations:

$$S(\rho) = \{x \in R^n : \|x\| < \rho\},$$

$$\Gamma^n = \left\{h \in C\left[R^+ \times R^n, R^+\right] : \forall t \in R^+, \inf_x h(t, x) = 0\right\},$$

$$\Gamma_\tau^n = \left\{h \in C\left[R_\tau^+ \times R^n, R^+\right] : \forall t \in R_\tau^+, \inf_x h(t, x) = 0\right\}.$$

We introduce the following definitions.

DEFINITION 1. *(See [2].) Function $V : [0, \infty) \times S(\rho) \to R^+$ belongs to class v_0 if*

(A1) *V is continuous on each of the sets $[t_{k-1}, t_k) \times S(\rho)$ and for all $x \in S(\rho)$ and*

$$k \in N, \qquad \lim_{(t,y) \to (t_k^-, x)} V(t, y) = V(t_k^-, x) \text{ exists;}$$

(A2) *V is locally Lipschitzian in $x \in S(\rho)$ and for all $t \geq t_0, V(t, 0) \equiv 0$.*

DEFINITION 2. *(See [2].) Let $V \in v_0$ for any $(t, x) \in [t_{k-1}, t_k) \times S(\rho)$, the right-hand derivative $V'(t, x(t))$ along the solution of problem (2.1)–(2.3) is defined by*

$$V'(t, x(t)) = \lim_{h \to 0^+} \sup \frac{\{V(t + h, x(t + h)) - V(t, x(t))\}}{h}.$$

DEFINITION 3. *(See [3].) Suppose that $h_0 \in \Gamma_\tau^n$, $x_t \in PC\{[-\tau, 0], R^n\}$, for any $t \in R^+$, we define*

$$\tilde{h}_0(t, x_t) = \sup_{-\tau \le \theta \le 0} h_0(t+\theta, x_t(\theta)).$$

DEFINITION 4. *(See [2].) Let $h_0 \in \Gamma_\tau^n$, $h \in \Gamma^n$. Then, the impulsive functional differential problem (2.1)–(2.3) is said to be*

(S1) *$(\tilde{h}_0, h)$-practically stable, if given (u, v) with $0 < u < v$, we have $\tilde{h}_0(t_0, x_{t_0}) < u$ implies $h(t, x(t)) < v$, $t \ge t_0$ for some $t_0 \in R^+$;*

(S2) *$(\tilde{h}_0, h)$-uniformly practically stable if (S1) holds for every $t_0 \in R^+$.*

3. MAIN RESULTS

We shall establish, in this section, theorems that provide sufficient conditions for uniform practical stability of the impulsive functional differential problem (2.1)–(2.3).

Let the sets K, K_1 be defined as

$$K = \left\{w \in C\left(R^+, R^+\right) : \text{ strictly increasing and } w(0) = 0\right\},$$
$$K_1 = \left\{\varphi \in C\left(R^+, R^+\right) : \text{ increasing and } \varphi(s) < s \text{ for } s > 0\right\}.$$

THEOREM 1. *Let the following conditions hold*

(i) *$0 < u < v$ are given;*

(ii) *$h_0 \in \Gamma_\tau^n$, $h \in \Gamma^n$, $h(t, x) \le \phi(\tilde{h}_0(t, x_t))$ with $\phi \in K$, whenever $\tilde{h}_0(t, x_t) < u$;*

(iii) *there exists a function $V \in v_0$ such that $\beta(h(t, x)) \le V(t, x) \le \alpha(h_0(t, x))$ for $(t, x) \in [t_0 - \tau, \infty) \times S(\rho)$, where $\alpha, \beta \in K$, $h_0 \in \Gamma_\tau^n$;*

(iv) *$V(t, x(t)) \ge \sup\{V(t+s, x(t+s)) : s \in [-\tau, 0]\}$, implies that $V'(t, x(t)) < 0$;*

(v) *$V(t_k, x(t_k^-) + I_k(x(t_k^-))) \le (1 + c_k) V(t_k^-, x(t_k^-))$, where $c_k \ge 0$ and $\sum_{k=1}^\infty c_k < \infty$;*

(vi) *$\phi(u) < v$ and $M\alpha(u) < \beta(v)$, where $\prod_{k=1}^\infty (1 + c_k) = M$.*

Then, the impulsive functional differential problem (2.1)–(2.3) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly practically stable.

PROOF. From Section 2, we know that for any $t_0 \in R^+$, there is a unique solution of problem (2.1)–(2.3) through (t_0, φ). We denote the solution of impulsive functional differential problem (2.1)–(2.3) by $x(t; t_0, \varphi)$.

Without loss of generality, we can assume that $t_0 < t_1$. Since $\sum_{k=1}^\infty c_k < \infty$, it follows that $1 \le M < \infty$.

If $(t_0, x_{t_0}) \in R^+ \times PC([-\tau, 0], R^n)$ such that $\tilde{h}_0(t_0, x_{t_0}) < u$. Then, by Conditions (ii) and (vi)

$$h(t_0, x(t_0)) \le \phi\left(\tilde{h}_0(t_0, x_{t_0})\right) < \phi(u) < v.$$

We then prove that

$$V(t, x(t)) \le M\alpha(u), \qquad \forall\, t \ge t_0. \tag{3.1}$$

For any $t \in (t_0 - \tau, t_0]$, there exists a $\theta \in (-\tau, 0]$, such that $t = t_0 + \theta$, then from Definition 3 and Condition (iii), we know that for $t \in (t_0 - \tau, t_0]$

$$\begin{aligned} &h_0(t, x(t)) = h_0(t_0 + \theta, x(t_0 + \theta)) = h_0(t_0 + \theta, x_{t_0}(\theta)) \le \tilde{h}_0(t_0, x(t_0)) < u, \\ &V(t, x(t)) \le \alpha(h_0(t, x)) \le \alpha(u). \end{aligned} \tag{3.2}$$

Next, we prove that

$$V(t, x(t)) \le \alpha(u), \qquad t_0 \le t < t_1. \tag{3.3}$$

If (3.3) does not hold, then there exists a $\hat{t} \in [t_0, t_1)$ such that $V(\hat{t}, x(\hat{t})) > \alpha(u)$. Let $\bar{t} = \inf\{t \mid V(t, x(t)) > \alpha(u),\ t \in [t_0, t_1)\}$. It is obvious that $V(\bar{t}, x(\bar{t})) = \alpha(u)$, $V'(\bar{t}, x(\bar{t})) \ge 0$, and from (3.2) $V(\bar{t}+s, x(\bar{t}+s)) \leq \alpha(u) = V(\bar{t}, x(\bar{t}))$ for $s \in [-\tau, 0]$. By Condition (iv), we have $V'(\bar{t}, x(\bar{t})) < 0$, a contradiction so (3.3) holds.

By Condition (v), we have

$$V(t_1, x(t_1)) = V\left(t_1, x\left(t_1^-\right) + I_k\left(x\left(t_1^-\right)\right)\right) \leq (1+c_1)V\left(t_1^-, x\left(t_1^-\right)\right) \leq (1+c_1)\alpha(u).$$

Next, we prove that

$$V(t, x(t)) \leq (1+c_1)\alpha(u), \qquad t_1 \leq t < t_2. \tag{3.4}$$

If (3.4) does not hold, then there exists a $\hat{u} \in [t_1, t_2)$ such that $V(\hat{u}, x(\hat{u})) > (1+c_1)\alpha(u)$. Let $\bar{u} = \inf\{t \mid V(t, x(t)) > (1+c_1)\alpha(u),\ t \in [t_1, t_2)\}$. It is obvious that $V(\bar{u}, x(\bar{u})) = (1+c_1)\alpha(u)$, $V'(\bar{u}, x(\bar{u})) \geq 0$, and from (3.2),(3.3) $V(\bar{u}+s, x(\bar{u}+s)) \leq (1+c_1)\alpha(u) = V(\bar{u}, x(\bar{u}))$ for $s \in [-\tau, 0]$. By Condition (iv), we have $V'(\bar{u}, x(\bar{u})) < 0$, a contradiction so (3.4) holds.

By Condition (v), we have

$$V(t_2, x(t_2)) = V\left(t_2, x\left(t_2^-\right) + I_k\left(x\left(t_2^-\right)\right)\right) \leq (1+c_2)\, V\left(t_2^-, x\left(t_2^-\right)\right) \leq (1+c_1)\,(1+c_2)\,\alpha(u).$$

By similar arguments as before, we can prove that for $k = 1, 2, \ldots$

$$V(t, x(t)) \leq (1+c_1)(1+c_2)\ldots(1+c_k)\alpha(u), \qquad t_k \leq t < t_{k+1}, \tag{3.5}$$

which together with (3.3), we have that

$$V(t, x(t)) \leq M\alpha(u), \qquad t \geq t_0. \tag{3.6}$$

By Condition (vi), we have

$$V(t, x(t)) \leq M\alpha(u) < \beta(v), \qquad t \geq t_0.$$

So, from Condition (iii), we get

$$h(t, x(t)) \leq \beta^{-1}(V(t, x(t))) < \beta^{-1}(\beta(v)) = v, \qquad t \geq t_0.$$

Thus, the impulsive functional differential problem (2.1)–(2.3) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly practically stable. The proof of Theorem 1 is complete.

THEOREM 2. *Assume the following conditions hold*

(i) $0 < u < v$ *are given;*
(ii) $h_0 \in \Gamma_\tau^n$, $h \in \Gamma^n$, $h(t, x) \leq \phi(\tilde{h}_0(t, x_t))$ *with* $\phi \in K$, *whenever* $\tilde{h}_0(t, x_t) < u$.
(iii) *There exists a function* $V \in v_0$ *such that* $\beta(h(t, x)) \leq V(t, x) \leq \alpha(h_0(t, x))$ *for*

$$(t, x) \in [t_0 - \tau, \infty) \times S(\rho), \qquad \text{where } \alpha, \beta \in K, \quad h_0 \in \Gamma_\tau^n.$$

(iv) *There exists a function* $\psi \in K_1$ *such that for any solution* $x(t)$ *of problem (2.1)–(2.3),* $\psi^{-1}(V(t, x(t))) > \sup\{V(t+s, x(t+s)) : s \in [-\tau, 0]\}$, *implies that* $V'(t, x(t)) \leq g(t)$ $w(V(t, x(t)))$, *where* $g, w : [t_0 - \tau, \infty) \to R^+$, *locally integrable. Also, for all* $k \in Z^+$ *and* $x \in S(\rho)$,

$$V\left(t_k, x\left(t_k^-\right) + I_k\left(x\left(t_k^-\right)\right)\right) \leq \psi\left(V\left(t_k^-, x\left(t_k^-\right)\right)\right).$$

(v) *There exists a constant* $A > 0$ *such that* $\int_{t_{k-1}}^{t_k} g(s)\, ds < A$, $k \in Z^+$. *Also, for any*

$$\mu > 0, \qquad \int_{\mu}^{\psi^{-1}(\mu)} \frac{ds}{w(s)} \geq A \text{ is valid;}$$

(vi) $\phi(u) < v$ *and* $\alpha(u) < \psi(\beta(v))$.

Then, the impulsive functional differential problem (2.1)–(2.3) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly practically stable.

PROOF. From Section 2, we know that for any $t_0 \in R^+$, there is a unique solution of problem (2.1)–(2.3) through (t_0, φ). We denote the solution of impulsive functional differential problem (2.1)–(2.3) by $x(t; t_0, \varphi)$.

If $(t_0, x_{t_0}) \in R^+ \times PC([-\tau, 0], R^n)$ such that $\tilde{h}_0(t_0, x_{t_0}) < u$. Then, by Conditions (ii) and (vi)

$$h(t_0, x(t_0)) \leq \phi\left(\tilde{h}_0(t_0, x_{t_0})\right) < \phi(u) < v.$$

We then prove that

$$V(t, x(t)) \leq \psi^{-1}(\alpha(u)), \qquad \forall t \geq t_0. \tag{3.7}$$

For any $t \in (t_0 - \tau, t_0]$, there exists a $\theta \in (-\tau, 0]$, such that $t = t_0 + \theta$, then from Definition 3, we know that for $t \in (t_0 - \tau, t_0]$

$$h_0(t, x(t)) = h_0(t_0 + \theta, x(t_0 + \theta)) = h_0(t_0 + \theta, x_{t_0}(\theta)) \leq \tilde{h}_0(t_0, x(t_0)) < u$$

since $\psi \in K_1$, from Condition (iii), we have for $t \in (t_0 - \tau, t_0]$

$$V(t, x(t)) \leq \alpha(h_0(t, x)) \leq \alpha\left(\tilde{h}_0(t_0, x(t_0))\right) < \alpha(u) < \psi^{-1}(\alpha(u)). \tag{3.8}$$

Next, we prove that

$$V(t, x(t)) \leq \psi^{-1}(\alpha(u)), \qquad t_0 \leq t < t_1. \tag{3.9}$$

If (3.9) does not hold, then there exists a $\hat{s} \in [t_0, t_1)$ such that

$$V(\hat{s}, x(\hat{s})) > \psi^{-1}(\alpha(u)) > \alpha(u) > V(t_0, x(t_0)).$$

Let $\bar{s} = \inf\{t \mid V(t, x(t)) > \psi^{-1}(\alpha(u)),\ t \in [t_0, t_1)\}$, then $V(\bar{s}, x(\bar{s})) = \psi^{-1}(\alpha(u))$, since $V(t_0, x(t_0)) < \alpha(u)$, we have $\bar{s} > t_0$, and for $\bar{s} < t \leq \hat{s}$, $V(t, x(t)) > \psi^{-1}(\alpha(u))$. From (3.8) and the definition of $\bar{s}$, we also have for $t_0 - \tau \leq t \leq \bar{s}$, $V(t, x(t)) \leq \psi^{-1}(\alpha(u))$. Since $\alpha(u) < \psi^{-1}(\alpha(u))$, $V(t_0, x(t_0)) < \alpha(u)$, $V(\bar{s}, x(\bar{s})) = \psi^{-1}(\alpha(u))$, and $V(t, x(t))$ is continuous in $[t_0, t_1)$, it follows that there exists a $s_1 \in [t_0, \bar{s})$, such that $V(s_1, x(s_1)) = \alpha(u)$ and for $s_1 \leq t < \bar{s}$, $V(t, x(t)) \geq \alpha(u)$.

Since for $t_0 - \tau \leq t \leq \bar{s}$, $V(t, x(t)) \leq \psi^{-1}(\alpha(u))$, for $s_1 \leq t < \bar{s}$, $V(t, x(t)) \geq \alpha(u)$ and $s_1 \in [t_0, \bar{s})$, then for $t \in [s_1, \bar{s}]$ and $s \in [-\tau, 0]$, we have

$$V(t + s, x(t + s)) \leq \psi^{-1}(\alpha(u)) \leq \psi^{-1}(V(t, x(t))).$$

In view of Condition (iv), we have for $t \in [s_1, \bar{s}]$,

$$V'(t, x(t)) \leq g(t) w(V(t, x(t))) \tag{3.10}$$

an integration of (3.10) over $(s_1, \bar{s})$, by Condition (v), we have

$$\int_{V(s_1, x(s_1))}^{V(\bar{s}, x(\bar{s}))} \frac{dx}{w(x)} \leq \int_{s_1}^{\bar{s}} g(t)\, dt \leq \int_{t_0}^{t_1} g(t)\, dt < A.$$

On the other hand,

$$\int_{V(s_1, x(s_1))}^{V(\bar{s}, x(\bar{s}))} \frac{dx}{w(x)} = \int_{\alpha(u)}^{\psi^{-1}(\alpha(u))} \frac{dx}{w(x)} \geq A$$

a contradiction so (3.9) holds.

By Condition (iv) and (3.9), we have

$$V(t_1, x(t_1)) = V\left(t_1, x\left(t_1^-\right) + I_k\left(x\left(t_1^-\right)\right)\right) \le \psi\left(V\left(t_1^-, x\left(t_1^-\right)\right)\right) \le \alpha(u).$$

Next, we prove that

$$V(t, x(t)) \le \psi^{-1}(\alpha(u)), \qquad t_1 \le t < t_2. \tag{3.11}$$

If (3.11) does not hold, then there exists a $\hat{r} \in [t_1, t_2)$ such that

$$V(\hat{r}, x(\hat{r})) > \psi^{-1}(\alpha(u)) > \alpha(u) \ge V(t_1, x(t_1)).$$

Let $r_1 = \inf\{t \mid V(t, x(t)) > \psi^{-1}(\alpha(u)),\ t \in [t_1, t_2)\}$ then $V(r_1, x(r_1)) = \psi^{-1}(\alpha(u))$, since $V(t_1, x(t_1)) \le \alpha(u) < \psi^{-1}(\alpha(u))$, we have $r_1 > t_1$, and for $r_1 < t \le \hat{r}$, $V(t, x(t)) > \psi^{-1}(\alpha(u))$. From inequalities (3.8),(3.9) and the definition of r_1, we have for $t_0 - \tau \le t \le r_1$, $V(t, x(t)) \le \psi^{-1}(\alpha(u))$. Since $\alpha(u) < \psi^{-1}(\alpha(u))$, $V(t_1, x(t_1)) \le \alpha(u)$, $V(r_1, x(r_1)) = \psi^{-1}(\alpha(u))$, and $V(t, x(t))$ is continuous in $[t_1, t_2)$, it follows that there exists a $r_2 \in [t_1, r_1)$, such that $V(r_2, x(r_2)) = \alpha(u)$ and for $r_2 \le t < r_1$, $V(t, x(t)) \ge \alpha(u)$.

Since for $t_0 - \tau \le t \le r_1$, $V(t, x(t)) \le \psi^{-1}(\alpha(u))$, for $r_2 \le t < r_1$, $V(t, x(t)) \ge \alpha(u)$ and $r_2 \in [t_1, r_1)$, then for $t \in [r_2, r_1]$ and $s \in [-\tau, 0]$, we have

$$V(t+s, x(t+s)) \le \psi^{-1}(\alpha(u)) \le \psi^{-1}(V(t, x(t))).$$

In view of Condition (iv), we have for $t \in [r_2, r_1]$,

$$V'(t, x(t)) \le g(t) w(V(t, x(t))) \tag{3.12}$$

an integration of (3.12) over (r_2, r_1), by Condition (v), we have

$$\int_{V(r_2, x(r_2))}^{V(r_1, x(r_1))} \frac{dx}{w(x)} \le \int_{r_2}^{r_1} g(t)\, dt \le \int_{t_1}^{t_2} g(t)\, dt < A.$$

On the other hand,

$$\int_{V(r_2, x(r_2))}^{V(r_1, x(r_1))} \frac{dx}{w(x)} = \int_{\alpha(u)}^{\psi^{-1}(\alpha(u))} \frac{dx}{w(x)} \ge A.$$

A contradiction so (3.11) holds.

By Condition (iv), we have

$$V\left(t_2, x\left(t_2\right)\right) = V\left(t_2, x\left(t_2^-\right) + I_k\left(x\left(t_2^-\right)\right)\right) \le \psi\left(V\left(t_2^-, x\left(t_2^-\right)\right)\right) \le \alpha(u).$$

By similar arguments as before, we can prove that for $k = 1, 2, \ldots$

$$V(t, x(t)) \le \psi^{-1}(\alpha(u)), \qquad t_{k-1} \le t < t_k$$

and

$$V(t_k, x(t_k)) \le \alpha(u).$$

Since $\alpha(u) < \psi^{-1}(\alpha(u))$, it follows by Conditions (vi) and (iii) that

$$\begin{aligned} &V(t, x(t)) \le \psi^{-1}(\alpha(u)) < \beta(v), \\ &h(t, x(t)) \le \beta^{-1}(V(t, x(t))) < \beta^{-1}(\beta(v)) < v, \qquad t \ge t_0. \end{aligned}$$

The proof of Theorem 2 is complete.

EXAMPLE. Consider practical stability of following equation similar to that given in [10] in terms of two measurements

$$\begin{aligned} \dot{x} &= -a(t)x(t) + b(t)x(t-\tau), && \text{for } t \geq 0, \quad t \neq t_k, \\ x(t_k) &= cx\left(t_k^-\right), && \text{for } k \in N \end{aligned} \tag{3.13}$$

in which $x \in R^n$, $0 < c < 1$, $\tau > 0$, $a(t), b(t) \in C[R^+, R^+]$, $a(t) \geq a$, $b(t) \leq b$, $(1 + 1/c^2)b - 2a > 0$. Denote $x \in R^n$ by $x = (x_1, x_2, \dots, x_n)$.

Let $h(t,x) = \|x\|_1 = \sum_{i=1}^n |x_i|$, $h_0(t,x) = \|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$. For the definition of $\tilde{h}_0$, we know that $\tilde{h}_0(t, x_t) = \sup_{-\tau \leq \theta \leq 0} h_0(t+\theta, x(t+\theta)) = \sup_{-\tau \leq \theta \leq 0} \|x(t+\theta)\|_\infty = |x_t|_\infty$.

For given (u, v), with $0 < u < (1/n\sqrt{n})cv$, if the following assumptions hold:

(H1) $t_k - t_{k-1} < -2\ln c/(-2a + (1 + (1/c^2))b)$.

(H2) $|x_t|_\infty < u$ implies that for any $s \in [-\tau, 0]$, $\|x(t)\|_1 < (1/c)\|x(t+s)\|_1$ holds.

Then, equation (3.13) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly practically stable.

PROOF. We choose the functions in Theorem 2 as follows: $V(t, x(t)) = x^T(t)x(t)$, $\psi(t) = c^2 t$, $\beta(x) = (1/n^2)x^2$, $\alpha(x) = nx^2$, $w(t) = t$, $g(t) = -2a + (1 + 1/c^2)b$, $\phi(t) = (n/c)t$.

(1) If $\tilde{h}_0(t, x_t) < u$, we have for any $s \in [-\tau, 0]$,

$$h(t,x) = \|x\|_1 < \frac{1}{c}\|x(t+s)\|_1 \leq \frac{n}{c}\|x(t+s)\|_\infty \leq \frac{n}{c}|x_t|_\infty = \phi\left(\tilde{h}_0(t, x_t)\right),$$

Condition (ii) in Theorem 2 is satisfied.

(2) Since $(1/n^2)\|x\|_1^2 \leq \|x\|_2^2 \leq n\|x\|_\infty^2$, then $\beta\ (h(t,x)) \leq V(t,x) \leq \alpha(h_0(t,x))$ holds.

(3) For any solution $x(t)$ of (3.13) such that

$$\sup\{V(t+s, x(t+s)) : s \in [-\tau, 0]\} < \psi^{-1}(V(t, x(t)))$$

we have clearly that

$$\sup\left\{x^T(t+s)x(t+s) : s \in [-\tau, 0]\right\} < \frac{1}{c^2}x^T(t)x(t).$$

Thus,

$$\begin{aligned} V'(t, x(t)) &= \left(-a(t)x^T(t) + b(t)x^T(t-\tau)\right)x(t) + x^T(t)(-a(t)x(t) + b(t)x(t-\tau)) \\ &= -2a(t)x^T(t)x(t) + 2b(t)x^T(t-\tau)x(t) \\ &\leq -2a(t)x^T(t)x(t) + b(t)\left(x^T(t-\tau)x(t-\tau) + x^T(t)x(t)\right) \\ &\leq \left[-2a + \left(1 + \frac{1}{c^2}\right)b\right]x^T(t)x(t) = g(t)w(V(t, x(t))). \end{aligned}$$

It is also holds that

$$V\left(t_k, x\left(t_k^-\right) + I_k\left(x\left(t_k^-\right)\right)\right) = V\left(t_k, cx\left(t_k^-\right)\right) = c^2 x^T\left(t_k^-\right)x\left(t_k^-\right) = \psi\left(V\left(t_k^-, x\left(t_k^-\right)\right)\right).$$

So Condition (iv) in Theorem 2 is satisfied.

(4) From the choice of function $g(t)$ and $w(t)$, we obtain that

$$\int_{t_{k-1}}^{t_k} g(s)\,ds = \left[-2a + \left(1 + \frac{1}{c^2}\right)b\right](t_k - t_{k-1}) < -2\ln c$$

and

$$\int_q^{\psi^{-1}(q)} \frac{ds}{w(s)} = \int_q^{q/c^2} \frac{ds}{s} = -2\ln c,$$

let $A = -2\ln c > 0$, then Condition (v) in Theorem 2 is satisfied.

(5) Since $0 < u < v$, $u < (1/n\sqrt{n})cv$, it is obvious that $\phi(u) = (n/c)u < (1/\sqrt{n})v \leq v$, $\alpha(u) = nu^2 < n(1/n^3)c^2v^2 = (1/n^2)c^2v^2 = \psi(\beta(v))$ Condition (vi) of Theorem 2 is satisfied.

From (1)–(5), we know that all conditions in Theorem 2 are satisfied. So for given (u, v), with $0 < u < (1/n\sqrt{n})cv$, equation (3.13) with respect to (u, v) is $(\tilde{h}_0, h)$-uniformly practically stable.

4. CONCLUSION

In this paper, by using piecewise continuous Lyapunov functions and Razumikhin techniques, we have got some criterions of uniform practical stability for impulsive functional differential equations in terms of two measurements. We also use an example to illustrate the theorem. We can see that impulses do contribute to the system's practical stability property.

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