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Hamiltonicity and colorings of arrangement graphs

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Abstract

We study connectivity, Hamilton path and Hamilton cycle decomposition, 4-edge and 3-vertex coloring for geometric graphs arising from pseudoline (affine or projective) and pseudocircle (spherical) arrangements. While arrangements as geometric objects are well studied in discrete and computational geometry, their graph theoretical properties seem to have received little attention so far. In this paper we show that they provide well-structured examples of families of planar and projective-planar graphs with very interesting properties. Most prominently, spherical arrangements admit decompositions into two Hamilton cycles; this is a new addition to the relatively few families of 4-regular graphs that are known to have Hamiltonian decompositions. Other classes of arrangements have interesting properties as well: 4-connectivity, 3-vertex coloring or Hamilton paths and cycles. We show a number of negative results as well: there are projective arrangements which cannot be 3-vertex colored. A number of conjectures and open questions accompany our results.

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1. Introduction

We study connectivity, vertex and edge coloring and Hamiltonicity properties for classes of geometric graphs arising from finite collections of pseudolines (resp., pseudocircles) in the Euclidean and Projective planes or on the sphere **S**. Our objects of study, known as arrangement graphs in the computational or discrete geometry literature, are 4-regular and planar (or projective-planar). They arise in connection with many combinatorial or algorithmic questions involving finite sets of planar lines or (via polar-duality) points (see [7]).

We proceed to a systematic study of these properties and report a number of positive and negative results, as well as a few still open questions which resisted our methods. Our most striking result, described in Section 3, is the existence of two Hamilton path (2HP) and cycle (2HC) decompositions for spherical arrangements, obtained via a short and easy to describe construction based on wiring diagrams.

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Finding Hamilton paths and cycles in graphs is an NP-hard problem, even for planar graphs, and even for arrangement graphs of Jordan curves (see [15]). It is known that 4-connected planar graphs always have a Hamilton cycle ([26], see also [25,22]). The same property holds for 4-connected projective-planar graphs [24]. It is therefore interesting to see if the Hamilton cycles could be explicitly constructed for particular classes of graphs. We have such a simple construction for spherical arrangements and odd projective arrangements.

2HP and 2HC decompositions for 4-regular graphs have been widely studied in the graph theory community [4], but there are relatively few positive results. For instance, Hamiltonian decompositions are known to exist in 4-regular Cayley graphs [1], and also in line graphs of Hamiltonian cubic graphs [19]. Our pseudocircle and separating-circle arrangement graphs provide new significant examples.

Coloring vertices of planar graphs with few (3 or 4) colors is known either via the Four Color Theorem or for particular classes of planar graphs (such as 3-colorability of outerplanar graphs and triangle-free planar graphs). 4-edge colorability of 4-regular planar graphs arising from arrangements of planar curves is known only for special cases. There are some graph theoretical conjectures (see [16]) about 4-edge colorings of certain circle arrangements: a simple proof of them would imply a simple proof for the Four Color Theorem (see also [17, p. 45]). Although our 4-edge coloring result for spherical arrangement graphs does not seem to lead in the direction of Jaeger and Shank's conjecture, some ideas might prove relevant.

The paper is organized as follows. In Section 2 we present the definitions, preliminaries and basic results on connectivity, coloring and Hamiltonicity pertaining to our three geometric models: projective, Euclidean and spherical. In Section 3, we present the wiring diagram technique for constructing Hamilton path and cycle decompositions for spherical arrangements and partial results in the projective setting. Open problems and conjectures follow the natural flow of the paper.

Arrangement graphs are defined precisely in the next section. We remark that there is no connection between our arrangement graphs and another family of graphs that bears the same name and which play a role in the field of network design [6,21].

2. Arrangement graphs: preliminaries

The general objects of our study are arrangement graphs arising from finite sets of curves obeying specific intersection rules and which live in the Euclidean or projective plane or on the two-dimensional sphere. In this section, we introduce three classes of arrangements and their corresponding arrangement graphs. We illustrate the definitions by examples and provide proofs of some elementary structural properties concerning connectivity and coloring.

2.1. Projective lines

Arrangements of straight lines are among the most basic objects one may study in the real projective plane **P**. Accordingly they have been and still are studied under a vast variety of aspects. See the overviews by Grünbaum [13] and Erdős and Purdy [8] for further pointers to the field. Many combinatorial properties of arrangements of lines do not depend on the fact that the lines are straight, but rather on the nature of their incidence properties. This leads to the natural generalization, first done by Levi [20], to arrangements of pseudolines. See [11] for a comprehensive survey.

An *arrangement of pseudolines* in the projective plane **P** is a family of simple closed curves (called *pseudolines*) such that every two curves have exactly one point in common, where they cross. If no point belongs to more than two of the (pseudo) lines the arrangement is called *simple*, otherwise it is *non-simple*.

Pseudoline arrangements provide generic models for the (purely combinatorial) oriented matroids of rank 3 (see [2]). In this paper we will work only with this model. A few simplifying assumptions: we will work only with simple arrangements. We also simplify the terminology by dropping the *pseudo* prefix from *pseudoline*: all the results of this paper hold in this more general context, and straightness of lines is no issue.

With an arrangement we associate the cell complex of vertices, edges and two-dimensional regions into which the lines of the arrangement decompose the underlying space **P**. Arrangements are *isomorphic* provided their cell complexes are isomorphic. A projective arrangement graph is the graph of vertices and edges of an arrangement of pseudolines. See Fig. 1 for an example.

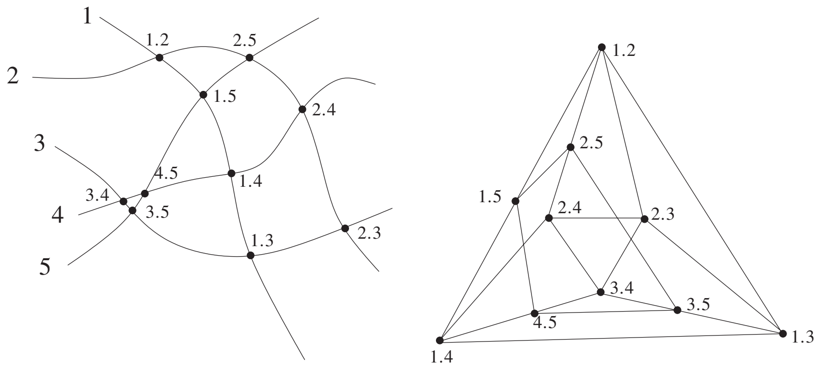

Fig. 1. A projective arrangement of pseudolines and its graph.

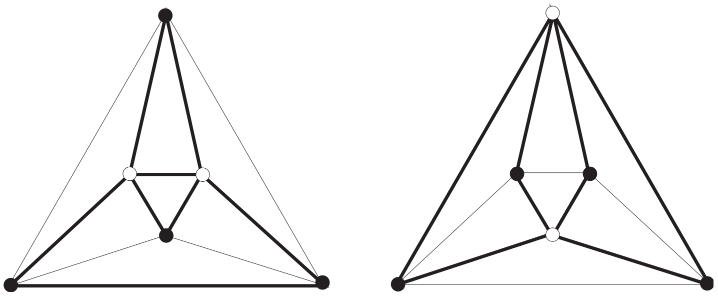

Fig. 2. Four paths between two vertices of H: adjacent vertices, non-adjacent vertices.

Let G be the graph of a simple projective arrangement of $n \geqslant 4$ lines. The following list collects some basic facts about G:

- G is 4-regular.
- G has $\binom{n}{2}$ vertices and $n(n-1)$ edges.
- G is planar only for $n = 4$ but always projective-planar.

A less trivial result is given in the next proposition.

Proposition 1. *The graph of a simple projective arrangement of $n \geqslant 4$ lines is* 4*-connected.*

Proof. Let G be such a graph and u, v any two vertices of G. To show 4-connectedness we will exhibit four internally disjoint paths connecting u and v in G. In the arrangement $\mathscr{A}$ defining G let p_1, p_2 be the lines through u and let q_1, q_2 be the lines through v. If $B = \{p_1, p_2, q_1, q_2\}$ contains only three lines augment B by an arbitrary fourth line. Now consider the graph H of the arrangement of the four lines in B. Note

- The vertices of H are also vertices of G and u and v are vertices of H.
- To an edge e of H connecting vertices w and w' there is a path connecting w and w' in G such that all edges of this path are supported by the line supporting e. Call this the canonical path of e.
- The canonical paths corresponding to the edges of H are pairwise internally disjoint, i.e., they can only meet at endpoints.

From these observations it follows that four disjoint paths between u and v in H can be lifted to disjoint paths in G by replacing edges by their canonical paths. Fortunately, there is only one projective arrangement of four lines and hence only one projective arrangement graph H with six vertices. This graph is the skeleton graph of the octahedron. By the high regularity of this graph there are only two cases to consider, see Fig. 2. $\square$

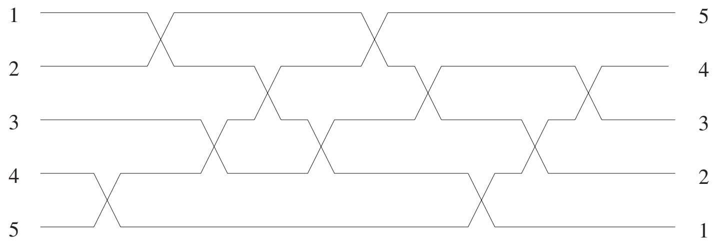

Fig. 3. Wiring diagram of an arrangement of five pseudolines.

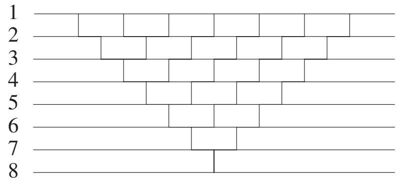

Fig. 4. Wiring diagram of the cyclic arrangement of eight lines.

Particularly nice pictures of arrangements of pseudolines and of their graphs are given by the *wiring diagrams* introduced in Goodman [10] (see also [12,9] and Fig. 3). In this representation the n curves are restricted to n wires with different y-coordinates, except for some local switches where adjacent lines cross. These switches are the vertices of the graph. The half-edges extending to the left and right of the picture have to be identified in reverse order, as the numbers indicate in Fig. 3. Sometimes a further simplification is made in drawings of wiring diagrams and the switches are only indicated by vertical segments, as in Fig. 4.

The *cyclic arrangement* of n lines is the arrangement where line i has the crossings with the other lines in the order $1, 2, \ldots, i-1, i+1, \ldots, n$. The *vee-shape wiring diagram* of the cyclic arrangement is the diagram where the crossings form a triangle of bricks (see Fig. 4).

We close this introductory section on projective arrangement graphs with some remarks on colorings.

By Vizing's theorem the edge chromatic number of a projective arrangement graph is either four or five. If it is 4 every color class has to consist of $n(n-1)/4$ edges. This is only possible if $n \equiv 0, 1 \pmod 4$.

Conjecture 1. *The necessary condition* $n \equiv 0, 1 \pmod 4$ *is sufficient for the* 4*-edge colorability of projective arrangement graphs*.

With respect to the chromatic number we observe the following:

- $\chi(G) \geqslant 3$ for every projective arrangement graph G. This is because G always contains a triangle (actually it contains at least n triangles, [20]).
- The graph of the cyclic arrangement of five lines has $\chi = 4$. We also have found an arrangement of six lines with $\chi = 4$.
- The graph of the cyclic arrangement has $\chi = 3$ for every $n > 5$. To see this for $n \equiv 0 \pmod 3$ color all vertices (switches) in each column of the vee-shape wiring diagram with the same color, start with 1 and repeat using 1,2,3 in cyclic order.

  For $n \equiv 1 \pmod 3$ do as in the previous case but recolor the right leg of the vee-shape as 32 312 312... 312 1. If $n \equiv 2 \pmod 6$ color columns in order 123 123 ... 123 121**3**2 123 ... 123 12. Finally, if $n \equiv 5 \pmod 6$ color columns in order 123 ... 123 132**3** 123 ... 123 121321, and recolor the right leg of the vee as 32 123 ... 123 21. (In the last two cases the digit in boldface corresponds to the column containing the apex of the vee.)

An upper bound of 4 for the chromatic number of every arrangement graph is straightforward because of the degree: just use Brooks' theorem (see [3]). Results about Euclidean arrangement graphs will allow us to find a 4-coloring very efficiently.

Theorem 2. *The chromatic number of projective arrangement graphs is at most four. A 4-coloring can be efficiently found by a simple linear (in the number of vertices) time algorithm.*

2.2. Euclidean lines

Given an arrangement $\{p_0, p_1, \ldots, p_n\}$ of $n+1$ lines in the projective plane we may specify a line p_0 as the "line at infinity". This induces the Euclidean arrangement of the n lines $\{p_1, \ldots, p_n\}$ in $\mathbf{E} = \mathbf{P} \backslash p_0$.

The graph of an Euclidean arrangement is the graph of the bounded edges of the arrangement. A nice thing about Euclidean arrangement graphs is that they come with a natural planar embedding. The parameters of the graph G of a simple Euclidean arrangement of $n \geqslant 4$ lines are as follows:

- G has minimum degree 2 and maximum degree 4.
- G has $\binom{n}{2}$ vertices.
- G has $n(n-2)$ edges.
- G is 2-connected.

As in the case of projective arrangement graphs the wiring diagram is a useful form of representing Euclidean arrangement graphs. To illustrate the power of this tool we give two examples concerning colorings.

Proposition 3. *The edge-chromatic number of an Euclidean arrangement graph is four.*

Proof. Consider a wiring diagram W of the arrangement defining G. Note that an edge e of G is assigned to a single wire, let $w(e)$ be the number of this wire counted from top to bottom. Color the edges on each odd numbered wire alternating with colors 1 and 2 and the edges on even numbered wires alternating with colors 3 and 4. The coloring thus obtained is a legal edge coloring of G. $\square$

Proposition 4. *The chromatic number of an Euclidean arrangement graph G is three.*

Proof. Consider a wiring diagram W of the arrangement defining G and let the left-to-right orientation of W induce an orientation on the edges of G. Note the following facts about this oriented graph $\overrightarrow{G}$:

- $\overrightarrow{G}$ is acyclic.
- The indegree and the outdegree of vertices of $\overrightarrow{G}$ are at most 2.

A 3-coloring of G is obtained by coloring the vertices in the order given by a topological sorting of $\overrightarrow{G}$. When it comes to color v at most two neighbors (the in-neighbors) of v have been colored. Hence, one of the three available colors can legally be assigned to v. $\square$

The two coloring results are exemplified in Fig. 5. The vertex coloring was obtained by coloring from left to right and assigning colors in order of preference 1–2–3.

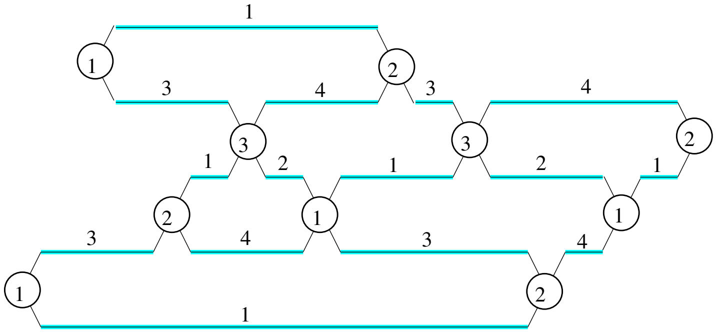

Fig. 5. An Euclidean arrangement graph with 3-vertex coloring and 4-edge coloring.

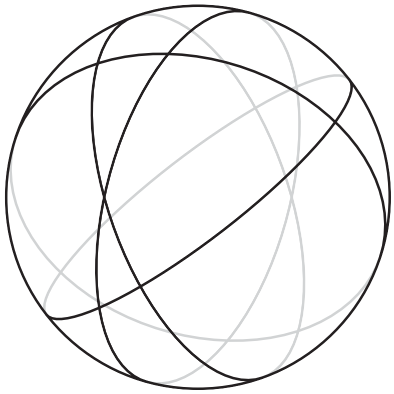

Fig. 6. An arrangement of four circles on the sphere.

Proof (*Theorem 2*). Let $\{p_0, p_1, \ldots, p_n\}$ be a projective arrangement and G its graph. Declare p_0 the line at infinity and consider the Euclidean arrangement $\{p_1, \ldots, p_n\}$ with graph G'. Note that G' is an induced subgraph of G. The vertices of G which are not in G' form an n-cycle $C = (v_1, v_2, \ldots, v_n)$ (the edges of G supported by p_0) and every vertex of C has exactly two neighbors in G'. Fix a coloring of G' with colors $\{1, 2, 3\}$ (see Proposition 4) and for every vertex $v_i \in C$ choose a color $c_i \in \{1, 2, 3\}$ which has not been used for a neighbour of v_i in G'.

If n is even we complete a 4-coloring of G by coloring the vertices of C of even index i with color c_i and those of odd index with a new color 4.

If n is odd and it is possible to choose the c_i such that there is an i with $c_i \neq c_{i+1}$, w.l.o.g. $i = 1$, then we complete the 4-coloring of G by coloring v_1 with c_1 and the other vertices of odd index with color 4 and those of even index i with c_i.

In the remaining case the two neighbors of all vertices of C in G' use the same two colors, say 1 and 2, so that $c_i = 3$ for all i. In this situation we choose a vertex x in G' which has two neighbors on C (this is possible since there exist triangles with a side on p_0, see [20]). W.l.o.g. we may assume that these are the vertices v_1 and v_2. Recolor x with color 4 and change c_1 to the old color of x. This brings us back to the previous case and completes the proof. □

2.3. Circles on the sphere

Arrangements of pseudocircles on the sphere **S** consist of a family $\{c_1, \ldots, c_n\}$ of simple closed curves (called *circles*) such that

- every two circles have exactly two points in common at which they cross,
- for three different indices $i, j, k \in \{1, \ldots, n\}$ circle c_k separates the two intersections of c_i and c_j.

The motivating examples for arrangements of circles are arrangements of great circles on the sphere. In this case **S** is a sphere centered at the origin and the circles are the intersections of planes containing the origin with **S**. In Fig. 6 such an arrangement of four circles on the sphere is shown.[3]

If we identify points on the frontside of the sphere with their antipodal counterparts on the backside we obtain a projective arrangement of n lines. If we remove the horizon-circle we obtain two isomorphic Euclidean arrangements.

Let G be the graph of a simple circle arrangement of $n \geqslant 3$ circles. We summarize some elementary facts about G:

- G is 4-regular.
- G has $n(n-1)$ vertices and $2n(n-1)$ edges.
- G is planar.

[3] Thanks to Cinderella [23] for this picture.

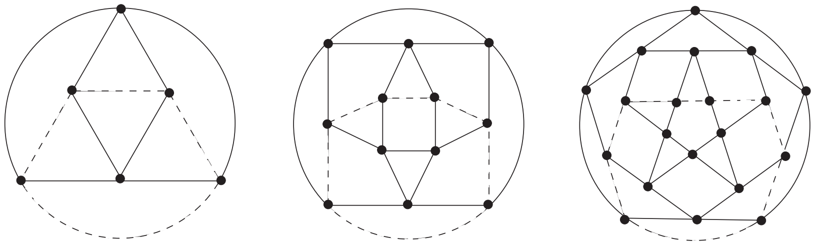

Fig. 7. Circle arrangement graphs of tree, four and five circles.

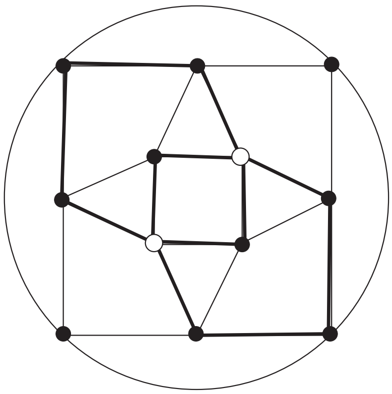

Fig. 8. Four connecting paths for the white vertices.

In Fig. 7, we show planar embeddings of the unique simple circle arrangement graphs of tree, four and five circles. In each case one of the circles is bold-dashed, the other circles can be obtained by rotations. The connectivity of circle arrangement graphs is as high as the degree allows:

Proposition 5. *The graph of a simple circle arrangement of $n \geqslant 3$ circles is* 4*-connected.*

Proof. Given the graph G of a simple circle arrangement and two vertices u, v of G we exhibit four internally disjoint paths connecting u and v. Let $B = \{c_1, c_2, c_3, c_4\}$ be the circles defining the two vertices. We distinguish three cases depending on the size of B. If $|B| = 2$, i.e., if the two vertices are antipodal the four paths are given by the four arcs connecting u and v along the two cycles. If $|B| = 3$ the three cycles induce the first graph of Fig. 7 and the two vertices are adjacent in this graph. Since the graph is isomorphic to the graph of Fig. 2 we can refer to that figure which shows the four paths. In the last case $|B| = 4$ the two vertices are the non-adjacent vertices of a quadrilateral face of the induced graph of the four circles (this is the second graph of Fig. 7). Its symmetry allows us to assume that the quadrilateral is the central one of the drawing, in which case the four paths can be chosen as shown in Fig. 8. □

Wiring diagrams are again a useful representation for this class of arrangements. This is how the wiring diagram of an arrangement of n great circles can be obtained: imagine the sphere to be a globe with the great circles drawn onto it. Now observe the shadow of the frame while the sphere moves on a full rotation around its axis. Label the circles such that in the initial position they occur in the order $1, 2, \ldots, n$ and start drawing them on n wires. When the frame passes a crossing the two circles involved in it change their order and in the wiring diagram a switch has to be drawn. After a half rotation every two circles have interchanged their order. Hence all circles are in reversed order $n, \ldots, 2, 1$. The second half of the rotation is an upside down copy of the first half. After the full rotation the frame reaches its initial position. Fig. 9 shows the wiring diagram of a circle arrangement with the two halfs emphasized. To read the graph of a circle arrangement from the wiring diagram the half-edges extending to the left and right have to be identified in the same order as the numbers indicate in Fig. 9.

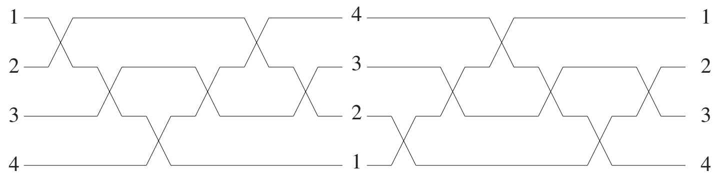

Fig. 9. Two copies of the wiring diagram of a projective arrangement glued together, the second copy taken upside down, give a wiring diagram of a circle arrangement.

The process described above for the construction of the wiring diagram is known as *sweeping* an arrangement. With some care in technical details it can be shown that arrangements of pseudocircles on the sphere are sweepable and also admit wiring diagrams which decompose into two halfs, one being the mirror image of the other (see [9] for related results). The diagram shown in Fig. 9 has the additional property that from left to right the first crossing of every circle $c_i, i \neq 1$, is the crossing with circle c_1. Every circle arrangement has a wiring diagram with this property, which we call the *one-down property*. To transform an arbitrary diagram into one with the one-down property, move all the switches which block the visibility of circle c_1 from the left to the right side.

Using a diagram with the one-down property we will show in Section 3 that the edge set of a circle arrangement graph can be decomposed into 2HCs. Since each Hamiltonian cycle has $n(n-1)$ edges, an even number, we may alternatingly use colors 1 and 2 for the edges of one of the Hamiltonian cycles and colors 3 and 4 for the edges of the other Hamiltonian cycle. This proves the following proposition as a corollary.

Proposition 6. *Circle arrangement graphs are* 4-*edge colorable*.

Concerning vertex colorings, we have a conjecture and an efficient procedure for 4-coloring. The existence of such a coloring is implied by Brooks' theorem, but our procedure is much simpler.

Conjecture 2. *Circle arrangement graphs are* 3-*vertex colorable*.

We have verified this conjecture for all cyclic arrangements of circles. These are the arrangements obtained from Fig. 4 by gluing a mirror image of the corresponding wiring diagram.

Proposition 7. *Circle arrangement graphs are* 4-*vertex colorable*.

Proof. Let $\{c_0, c_1, \ldots, c_n\}$ be a circle arrangement and G its graph. Declare c_0 to be the equator and consider the Euclidean arrangements on the two hemispheres of $\mathbf{S}\backslash c_0$. Let G' and G'' be the graphs of these arrangements. The vertices of G which are not in G' or G'' form an $2n$-cycle $C = (v_1, v_2, \ldots, v_{2n})$ (the edges of G supported by c_0) and every vertex of C has exactly one neighbor in G' and one in G''. Fix colorings of G' and G'' with colors $\{1, 2, 3\}$ (see Proposition 4) and for every vertex $v_i \in C$ choose a color $\gamma_i \in \{1, 2, 3\}$ which has not been used for a neighbor of v_i in $G' \cup G''$.

Since n is even we complete a 4-coloring of G by coloring the vertices of even index i on the cycle C with color γ_i and those of odd index with a new color 4. $\square$

There are several generalizations of circle arrangements. We mention two of them.

- *Separating circle arrangements* consist of a family $\{c_1, \ldots, c_n\}$ of simple closed curves in the plane or on the sphere (called *circles*) such that: (1) every two circles cross exactly twice and (2) for any two different indices $i, j \in \{2, \ldots, n\}$ circle c_1 separates the two intersections of c_i and c_j.
- *Digon-free circle arrangements* consist of a family $\{c_1, \ldots, c_n\}$ of simple closed curves in the plane or on the sphere (called *circles*) such that: (1) every two circles cross exactly twice and (2) the arrangement contains no cell with only two edges and two vertices (digon).

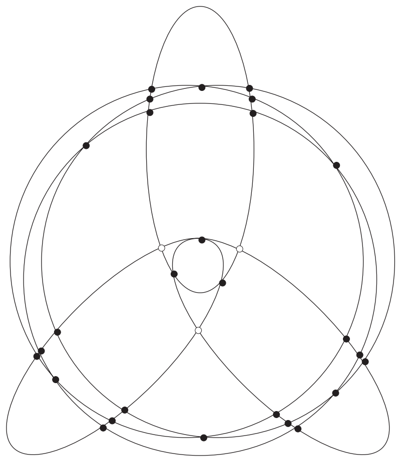

Fig. 10. A digon-free arrangement with a cutset of size three.

All the results we have for circle arrangement graphs still hold for the class of separating circle arrangement graphs. Digon-free circle arrangements have been studied by Grünbaum [13]. They are much more general and have less favorable properties. E.g., in Fig. 10 a digon-free arrangement is shown whose graph is only 3-connected. The unrestricted class of all 2-intersecting systems of closed curves has the disadvantage that the resulting graphs may have double edges.

3. Hamilton paths and cycles in arrangement graphs

In this section we study Hamiltonicity properties of spherical and projective arrangements. The Euclidean case has been settled in [5] with a negative answer (there are non-Hamiltonian Euclidean line arrangement graphs).

As shown in the previous section, both the pseudocircle and the projective arrangements are 4-connected. A well-known theorem of Tutte [26] on 4-connected planar graphs guarantees a Hamilton cycle. An even stronger result follows from Thomassen's [25] strengthening of Tutte's theorem: every 4-connected planar graph is Hamilton connected (there exists a Hamilton path connecting any two prescribed vertices).

Theorem 8. *Every spherical arrangement graphs has a Hamilton cycle and is Hamilton connected.*

Thomas and Yu's theorem on 4-connected projective-planar graphs [24] implies a similar result for projective arrangements.

Theorem 9. *Every projective arrangement graph is Hamiltonian.*

We now proceed to strengthen these results with explicit constructions. For spherical arrangements, we find not just one, but two such Hamilton paths and cycles, which, moreover, yield a decomposition of the edges of the graph.

3.1. Pseudocircle and separating circle arrangements

Theorem 10. *Every pseudocircle arrangement and separating circle arrangement can be decomposed into two edge-disjoint Hamilton paths* (*plus two extra edges*), *and the decomposition can be found efficiently*.

Proof. The construction is based on the representation of these arrangements as wiring diagrams. As shown in the previous section, we can assume that the wiring diagram representation has the *one-down property*, as in Fig. 9. The

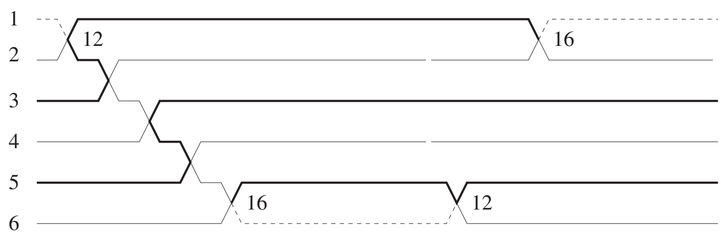

Fig. 11. Two Hamilton paths in a pseudocircle arrangement.

construction of the 2HPs, bold and thin, is described in Fig. 11 for six wires, but it can be easily generalized to any number of wires by repeating the pattern of colors going up along the switches on line 1. The figure needs some explanations, as it looks incomplete: we did not draw all the switches corresponding to the vertices of the arrangement. We did this to draw the attention to the structure of the construction and avoid cluttering the picture. A continuously colored line along a wire of the wiring diagram denotes a path in the arrangement graph, whose edges are colored in that color and which goes along the edges incident with that wire and touches all vertices connected to them. Remember that the one-down property, and the choice of the wiring diagram drawing, insured that there are no switches left of the one-down switches.

The pictured illustrates a key element of the construction: the *one-down* property. the bold, resp., thin Hamilton paths walk along the edges of a level (wire) (visiting all vertices adjacent to it) then go down by two levels at the switches (vertices) corresponding to pseudocircle 1 (the one going *one-down*).

The crucial observation is that the bold (resp., thin) path never touches the same vertex twice, and visits them all, therefore guaranteeing Hamiltonicity. The correctness of the construction follows from the following easy to establish properties.

- Each switch, except the one involving pseudocircle 1, is touched by the bold path on an odd-numbered wire and by a thin path on an even-numbered wire.
- Each edge (with the two exceptions left uncolored (dashed)) is colored either bold or thin.
- All bold edges are connected in a path, and so are the thin edges.
- A path in one color never visits the same vertex twice, and covers all the switches (vertices). □

Since the spherical and projective graphs are 4-regular graphs, removing a Hamilton cycle (guaranteed by Theorem 8) leaves a 2-regular graph. It is a remarkable feature of the pseudocircle arrangements that we can in fact *partition* the edges of the graph into 2HCs.

Theorem 11. *Every pseudocircle arrangement can be decomposed into two edge-disjoint Hamilton cycles, and the decomposition can be found efficiently.*

Proof. For the proof of this theorem we concentrate on the crossings involving circles 1 and 2 in the first half of the wiring diagram (cf. Fig. 9). We again assume the one-down property. Now let $x_1, x_2, \ldots, x_{n-1}, x_1, \ldots$ be the sequence of switches involving line 2, clearly $x_1 = 1$. Line 2 is going down at each of the crossings with $x_2, \ldots, x_{n-1}$ and is going down again at the second crossing with line 1 (this is the first switch of the second half of the diagram). Since our figures display the first n switches of line 1 and the first n switches of line 2 the sections of wire between two switches on line 1, resp., line 2 are not incident to any other switch. This observation is critical to the verification of the construction since whenever the bold or thin cycle runs on two adjacent wires one of the two sides corresponds to line 1 or 2. Figs. 12 and 13 show special instances of the construction with seven and eight lines. The figures are generic for the even and the odd case, other instances are obtained by insertion or deletion of pairs of strand going up four levels, see Fig. 14.

Correctness of the constructive pattern follows from the following properties:

- Each switch is touched by the bold path on an even-numbered wire and by a thin path on an odd-numbered wire.
- Each edge is colored either bold or thin.

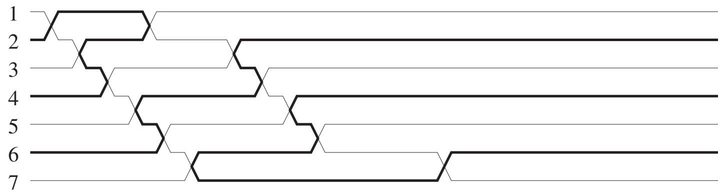

Fig. 12. Two Hamilton cycles in a pseudocircle arrangement of seven lines.

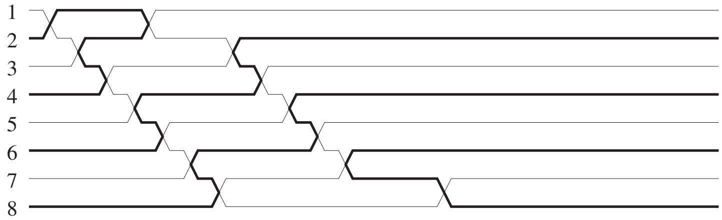

Fig. 13. Two Hamilton paths in a pseudocircle arrangement of eight lines.

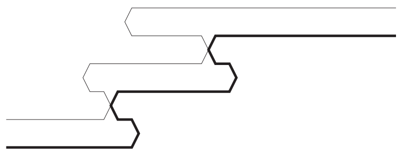

Fig. 14. Two strands moving up four levels.

- All bold edges are connected in a cycle, and so are the thin edges.
- The path of bold (resp., thin) edges never visits the same vertex twice, and covers all the switches.

As an additional feature of the construction, notice that the 2HCs only cross at the two intersection points of circles 1 and 2. □

Since these arguments do not depend on how the switches are arranged on the wires, our argument generalizes to a wider class of 4-regular planar graphs. Each 4-regular planar graph can be decomposed into closed curves crossing properly (not necessarily simple). Some of these graphs can be drawn as wiring diagrams (*leveled*): this is a necessary condition. To make the previous construction of 2HC decomposition work, they also have to have two *one-down* strands of these curves, as in Figs. 12 and 13.

3.2. Projective arrangements

Theorem 12. *Every projective arrangement with an odd number of pseudolines can be decomposed into two edge-disjoint Hamilton paths* (*plus two unused edges*), *and the decomposition can be found efficiently*.

Proof. The proof is based on a construction for which examples with $n = 7$ and 9 are depicted in Figs. 15 and 16. The construction uses the switches of line 1 to allow each path to go up. The two dashed edges are unused, the others partition the graph into 2HPs. The correctness follows from similar properties as described for pseudocircles. □

Unfortunately, we have not been able to extend this general type of argument in the case of an even number of lines. Since projective arrangement graphs are also 4-regular, one could expect again the existence of Hamilton cyclic

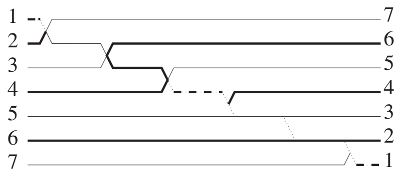

Fig. 15. Two Hamilton path in a projective arrangement of seven lines.

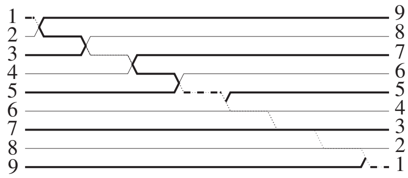

Fig. 16. Two Hamilton path in a projective arrangement of nine lines.

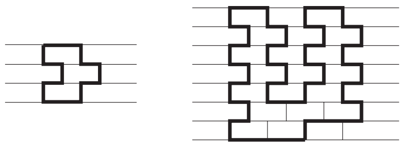

Fig. 17. Two Hamilton cycles in cyclic projective arrangements with a number n of lines such that $n \equiv 0 \pmod 4$.

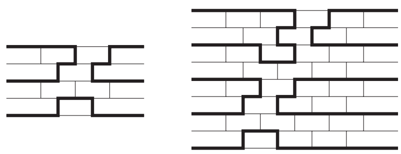

Fig. 18. Two HC for $n \equiv 1 \pmod 4$.

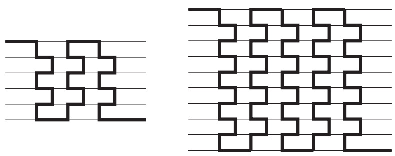

Fig. 19. Two HC for $n \equiv 2 \pmod 4$.

decompositions: this is not the case. Harborth and Möller [14] have found a counterexample. However, Hamilton decompositions do exist for cyclic arrangements.

Proposition 13. *For every $n \geqslant 4$, the graph of the cyclic projective arrangement admits a decomposition into two edge-disjoint Hamilton cycles.*

Proof. The proof is based on a construction which depends on the value of n modulo 4 and is illustrated in Figs. 17–20. Note that, after a suitable shift of some of the intersection points, the arrangements depicted in the figures are actually the same as those defined in Fig. 4. □

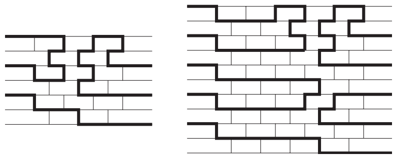

Fig. 20. Two HC for $n \equiv 3 \pmod 4$.

4. Conclusion

2HP and 2HC decompositions show a high degree of structure in the geometric arrangement graphs. We have exhibited a general technique for constructing such decompositions based on wiring diagrams. It would be interesting to extend this study to 1-skeletons of arrangements in higher dimensions, where some of the tools we used (wiring diagrams, sweeps) are not available.

Several other directions for further research are open, besides the various conjectures already described in the paper. It would be interesting to count the number of 2HP and 2HC decompositions of spherical arrangements, or to characterize those graphs for which our technique of 2HP and 2HC construction works. It might be possible to generalize these techniques to classes of $2k$-regular graphs, including 1-skeletons of rank $k+1$ pseudosphere arrangements. We leave these problems open for further investigations.

Finally, we would like to add a few comments on algorithmic issues. Arrangement graphs of circles on the sphere can be recognized efficiently. Since the graphs are 4-connected they have unique embeddings, from which we define circles by going straight through each vertex. The verification of the incidence properties is straightforward. It is interesting to note that for projective arrangement graphs this idea would fail: there are 5-connected projective-planar graphs with many embeddings, see [18]. Concerning the vertex-coloring of projective arrangements, an interesting problem is to find a polynomial time algorithm for deciding whether χ is equal to 3 or 4.

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