JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 40, 174–182 (1972)

Periodic Solutions of Nonlinear Differential Systems*

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Submitted by Kenneth L. Cooke

Received April 19, 1971

1. Let $f(t, x)$ be a continuous function on $[0, T] \times R^n$ with its range contained in R^n. In this paper, we give sufficient conditions in order that the differential system

$$x' = f(t, x) \tag{1}$$

has at least one periodic solution $x(t)$, i.e., a solution $x(t)$ such that $x(0) = x(T)$.

Let "$\leqslant$" denote the usual partial ordering of R^n and also the linear ordering of R. We say that $f(t, x)$ is of type K (after Kamke [5]) in a set $D \subset R^n$ if for all $x, y \in D$, $x = (x_1 ,..., x_n)$, $y = (y_1 ,..., y_n)$, with $x \leqslant y$ and $x_i = y_i$, it is true that $f_i(t, x) \leqslant f_i(t, y)$, where $f = (f_1 ,..., f_n)$, $i = 1,..., n$.

We shall establish the following result.

THEOREM 1. *Let there exist continuous functions $\alpha, \beta : [0, T] \to R^n$ such that $\alpha(t) \leqslant \beta(t)$, $0 \leqslant t \leqslant T$, $\alpha(0) \leqslant \alpha(T)$, $\beta(0) \geqslant \beta(T)$ and*

$$D_-\alpha(t) \leqslant f(t, \alpha(t)), \qquad D^-\beta(t) \geqslant f(t, \beta(t)), \qquad 0 < t \leqslant T. \tag{2}$$

Let f be of type K on the set $\{x : \alpha(t) \leqslant x \leqslant \beta(t), 0 \leqslant t \leqslant T\}$. Then there exists a periodic solution $x(t)$ of (1) with $\alpha(t) \leqslant x(t) \leqslant \beta(t)$, $0 \leqslant t \leqslant T$. By D_- and D^-, respectively, we mean the lower left and upper left Dini differential operators.

A similar result has been established by Knobloch [6] who assumes f to be Lipschitz continuous in x, but he does not assume f to be of type K. On the other hand, the solutions of differential inequalities used in [6] satisfy a set of conditions different from ours.

In proving Theorem 1, we proceed as follows: We first establish a special case where f is assumed to be Lipschitz continuous with respect to x and the

* Research supported by NSF Contract GP-11555.

inequalities in (2) are assumed to be strict. Then we use approximation arguments to obtain the more general result from the special one.

In Section 3, we apply Theorem 1 to get periodic solutions of certain linear systems and then use these existence results together with multivalued fixed-point theory to obtain the existence of periodic solutions of certain quasilinear systems which are not necessarily of type K. Finally, in Section 4, we indicate how Theorem 1 may be extended to certain classes of functional differential equations—in particular, to differential-difference and integro-differential equations.

2. Before proving Theorem 1, we consider the following special case.

THEOREM 2. *Let $f : [0, T] \times R^n \to R^n$ be continuous and satisfy a Lipschitz condition with respect to x on compact subsets of $[0, T] \times R^n$. Further, let there exist continuous functions $\alpha, \beta : [0, T] \to R^n$ such that $\alpha(t) < \beta(t)$, $\alpha(0) \leqslant \alpha(T)$, $\beta(0) \geqslant \beta(T)$ and*

$$D_{-}\alpha(t) < f(t, \alpha(t)), \qquad D^{-}\beta(t) > f(t, \beta(t)), \qquad 0 < t \leqslant T.$$

If, in addition, f is of type K on $\{x : \alpha(t) \leqslant x \leqslant \beta(t), 0 \leqslant t \leqslant T\}$, then there exists a periodic solution $x(t)$ of (1) such that

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad 0 \leqslant t \leqslant T.$$

Proof. Let $[\alpha, \beta] = \{(t, x) : \alpha(t) \leqslant x \leqslant \beta(t), 0 \leqslant t \leqslant T\}$. Then $[\alpha, \beta]$ is a compact subset of $[0, T] \times R^n$. Since f satisfies a Lipschitz condition with respect to x on this set, the initial value problem

$$x' = f(t, x), \qquad x(0) = x_0$$

has a unique solution $x(t; x_0)$ for every x_0 with $\alpha(0) \leqslant x_0 \leqslant \beta(0)$. It follows from Kamke [5] (see also Walter [7]) that $x(t; x_0)$ exists on $[0, T]$ and that

$$\alpha(t) \leqslant x(t, x_0) \leqslant \beta(t), \qquad 0 \leqslant t \leqslant T.$$

Define the mapping

$$S : \{x_0 : \alpha(0) \leqslant x_0 \leqslant \beta(0)\} \to \{z : \alpha(T) \leqslant z \leqslant \beta(T)\} \subseteq \{x_0 : \alpha(0) \leqslant x_0 \leqslant \beta(0)\}$$

by

$$S(x_0) = x(T; x_0).$$

Then S is continuous. It follows from Brouwer's Fixed Point Theorem that S has a fixed point, proving Theorem 2.

Proof of Theorem 1. Let α and β be as in Theorem 1. Define the function $F(t, x)$ in the following way. For each $t \in [0, T]$ and $i = 1,..., n$, let

$$F_i(t, x) = \begin{cases} f_i(t, \bar{x}) - \dfrac{x_i - \beta_i(t)}{1 + |x_i|}, & \text{if} \quad x_i > \beta_i(t) \\ f_i(t, \bar{x}), & \text{if} \quad \alpha_i(t) \leqslant x_i \leqslant \beta_i(t) \\ f_i(t, \bar{x}) - \dfrac{x_i - \alpha_i(t)}{1 + |x_i|}, & \text{if} \quad x_i < \alpha_i(t), \end{cases}$$

where $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$ and

$$\bar{x}_i = \begin{cases} \beta_i(t), & \text{if} \quad x_i > \beta_i(t) \\ x_i, & \text{if} \quad \alpha_i(t) \leqslant x_i \leqslant \beta_i(t) \\ \alpha_i(t), & \text{if} \quad x_i < \alpha_i(t). \end{cases}$$

The function $F(t, x)$, so defined, is continuous and bounded on $[0, T] \times R^n$ and furthermore is of type K everywhere. Also a function $x(t)$,

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad 0 \leqslant t \leqslant T,$$

is a solution of (1) if and only if it is a solution of

$$x' = F(t, x). \tag{3}$$

Let $\epsilon = (\epsilon_1, ..., \epsilon_n)$ be a constant vector with $\epsilon_i > 0$, $i = 1,..., n$. Let $A(t) = \alpha(t) - \epsilon$, $B(t) = \beta(t) + \epsilon$. Then

$$F_i(t, A(t)) = f_i(t, \alpha(t)) + \frac{\epsilon_i}{1 + |A_i(t)|} > f_i(t, \alpha(t)),$$

and hence

$$D_{-}A(t) = D_{-}\alpha(t) \leqslant f(t, \alpha(t)) < F(t, A(t)).$$

Similarly,

$$D^{-}B(t) > F(t, B(t)).$$

Consider now the function $F(t, x)$ on the set

$$\mathscr{A} = \{(t, x) : A(t) \leqslant x \leqslant B(t), 0 \leqslant t \leqslant T\}$$

and let $\delta > 0$ be such that

$$D_{-}A_i(t) - F_i(t, A(t)) < -\delta < 0 < \delta < D^{-}B_i(t) - F(t, B(t)), \quad i = 1,..., n.$$

Using an approximation argument (see, e.g., Hartman [4, pp. 6, 7]), we may find a function $G(t, x)$ which satisfies a Lipschitz condition with respect to x, is of type K and is such that $| G_i(t, x) - F_i(t, x)| < \delta/2$, $i = 1,..., n$ for all $(t, x) \in \mathscr{A}$. This in turn implies that

$$D_-A(t) < G(t, A(t)) \qquad \text{and} \qquad D^-B(t) > G(t, B(t)).$$

Hence by Theorem 2, the equation $x' = G(t, x)$ has a periodic solution $x(t)$ such that $A(t) \leqslant x(t) \leqslant B(t)$. Using the Ascoli–Arzela Theorem (i.e., pick a monotone decreasing sequence of δ's converging to 0, etc.) we conclude that Eq. (3) has a periodic solution $x(t)$ such that $A(t) \leqslant x(t) \leqslant B(t)$. This in turn is true for every positive vector ϵ. Using Ascoli–Arzela once more, we conclude that there exists a periodic solution $x(t)$ of (3) such that $\alpha(t) \leqslant x(t) \leqslant \beta(t)$. By the remark at the beginning of the proof we conclude that $x(t)$ is a periodic solution of (1).

When trying to apply Theorem 1, one would, of course, investigate first whether α and β may be chosen to be constants; if this is the case, Theorem 1 takes the following form:

COROLLARY 3. *Let there exist constant vectors α, β with $\alpha \leqslant \beta$ such that*

$$f(t, \beta) \leqslant 0 \leqslant f(t, \alpha) \qquad (0 = (0, 0,..., 0))$$

and let f be of type K on the set $\{x : \alpha \leqslant x \leqslant \beta\}$. Then there exists a periodic solution $x(t)$ of (1) with $\alpha \leqslant x(t) \leqslant \beta$.

To illustrate Corollary 3, we consider the following example:

Let h and g be continuous functions on R such that $h(0), g(0) \geqslant 1$ and $h(1), g(1) \leqslant -2$. Let p and q be positive real numbers and consider the two-dimensional system

$$\begin{aligned} x' &= h(x) + y^p + \sin t \\ y' &= x^q + g(y) + \cos t. \end{aligned} \tag{4}$$

The hypotheses on h and g imply that $\alpha = (0, 0)$ and $\beta = (1, 1)$ satisfy the desired differential inequalities; further, since p and q are positive, the right side of (4) is of type K for $0 \leqslant x, y \leqslant 1$. Hence (4) has a 2π periodic solution $(x(t), y(t))$ such that $0 \leqslant x(t), y(t) \leqslant 1$.

3. In this section, we give an application of Theorem 1 to quasilinear systems of the form

$$x' = A(t)\, x + g(t, x), \tag{5}$$

where g is not necessarily of type K, but where $A(t)$ is a continuous $n \times n$ matrix defined on $[0, T]$ and $A(t)$ is of type K, i.e., if $A(t) = (a_{ij}(t))$, then $a_{ij}(t) \geqslant 0$, $i \neq j$, $i = 1,..., n$, $j = 1,..., n$. Thus when $A(t)$ is a matrix of type K, then for any function $\varphi : [0, T] \to R^n$, the function $A(t)\,x + \varphi(t)$ is of type K, as defined in Section 1.

THEOREM 4. *Let $A(t)$ be a continuous $n \times n$ matrix of type K which is such that*

$$\sum_{j=1}^{n} a_{ij}(t) < 0, \qquad i = 1,\ldots, n. \tag{6}$$

Then for any continuous function $\varphi : [0, T] \to R^n$, there exists a solution $x(t)$ of

$$x' = A(t)\,x + \varphi(t) \tag{7}$$

such that $x(0) = x(T)$.

Proof. By (6), we can find constants α, β, $\alpha < 0 < \beta$, such that

$$\alpha \sum_{j=1}^{n} a_{ij}(t) + \varphi_i(t) \geqslant 0 \geqslant \beta \sum_{j=1}^{n} a_{ij}(t) + \varphi_i(t), \qquad i = 1,\ldots, n.$$

The constant vectors $(\alpha,\ldots, \alpha)$ and $(\beta,\ldots, \beta)$ then satisfy the desired differential inequalities and we conclude that there exists a solution $x(t)$ of (7) with $x(0) = x(T)$ and $\alpha \leqslant x_i(t) \leqslant \beta$, $i = 1,\ldots, n$, $0 \leqslant t \leqslant T$.

Remark. In case A is a constant $n \times n$ matrix, Theorem 4, of course, follows immediately from the Floquet theory, because the conditions

$$a_{ii} < 0, \qquad a_{ij} \geqslant 0, \qquad i \neq j, \qquad \text{and} \qquad \sum_{j=1}^{n} a_{ij} < 0,$$

guarantee that all eigenvalues of A must have negative real parts (all eigenvalues of A must be in the union of the discs $| \lambda - a_{ii} | \leqslant \sum_{j \neq i} a_{ij}$). Hence the unperturbed equation $x' = Ax$ has as its only periodic solution the trivial one which in turn implies the existence of a unique periodic solution of (7) for every φ.

However, in order to be able to apply the Floquet theory when $A(t)$ is not a constant $n \times n$ matrix, we must compute the fundamental matrix solution $X(t)$ of $x' = A(t)\,x$ and check whether or not $X(T)$ has 1 as an eigenvalue. Or alternatively, check whether $A(t)$ is an asymptotically stable matrix (here we consider $A(t)$ to be defined on $(-\infty, \infty)$ having period T), for if $A(t)$ is an asymptotically stable matrix, then the unperturbed system has as its only periodic solution the trivial one. Again it is true that all eigenvalues of $A(t)$ have negative real parts and since $A(t)$ is periodic, the real parts of the eigenvalues are bounded away from zero. It is well-known (see, e.g., Hahn [3,

p. 307]) that this is not sufficient to guarantee that $A(t)$ is a stable matrix, and thus the uniqueness of a periodic solution of $x' = A(t)\,x$ can, in general, not be deduced. If on the other hand, the matrix $M(t) = \frac{1}{2}(A^T(t) + A(t))$ has all its eigenvalues with negative real parts, then the trivial solution of $x' = A(t)\,x$ is asymptotically stable (the function $v = |\,x\,|^2$ serves as a Lyapunov function) and hence in this case, we again can deduce the existence and uniqueness of a periodic solution of (7) for every φ, by other means.

If $A(t)$ is such that for every φ, the system (7) has a unique periodic solution $x(t, \varphi)$ with $|\,x(t, \varphi)| \leqslant k\,|\,\varphi(t)|$, where k is a constant depending on $A(t)$ only, then the existence of a periodic solution of the quasilinear system (5) can easily be established by means of the Schauder Fixed Point Theorem (provided it satisfies a certain growth condition with respect to x).

In the absence of uniqueness, this approach is, of course, no longer feasible. Using Theorem 4, we, nevertheless, can obtain the existence of a periodic solution of (5) without *a priori* knowledge of uniqueness.

THEOREM 5. *Let the hypotheses of Theorem 4 hold. Let $g(t, x)$ be continuous on $[0, T] \times R^n$ into R^n. Further assume that $|\,g(t, x)| \leqslant \delta\,|\,x\,|$ for all $|\,x\,|$ sufficiently large, where $\delta > 0$ is such that*

$$\sum_{j=1}^{n} a_{ij}(t) \leqslant -\delta < 0, \qquad i = 1,\dots, n.$$

Then there exists a periodic solution of (5).

Proof. Choose $r > 0$ large enough so that $|\,f(t, x)| \leqslant \delta\,|\,x\,|$, for $|\,x\,| \geqslant r$. Let $M = \max\{|\,f(t, x)| : |\,x\,| \leqslant r, t \in [0, T]\}$. Choose $\beta \geqslant r$ such that $\delta\beta \geqslant M$ and let $\alpha = -\beta$. Let $P = \{\varphi : [0, T] \to R^n : \varphi(0) = \varphi(T), \|\,\varphi\,\| = \max\{|\,\varphi(t)| : t \in [0, T]\} \leqslant \beta\}$. Then P is a closed convex subset of the Banach space $C([0, T], R^n)$.

For every $\varphi \in P$, consider the linear equation

$$x' = A(t)\,x + g(t, \varphi(t)). \tag{8}$$

One may easily show that

$$\alpha \sum_{j=1}^{n} a_{ij}(t) + g_i(t, \varphi(t)) \geqslant 0 \geqslant \beta \sum_{j=1}^{n} a_{ij}(t) + g_i(t, \varphi(t)), \qquad i = 1,\dots, n.$$

Hence by Theorem 4, there exists at least one periodic solution $x(t)$ of (8) with $\|\,x\,\| \leqslant \beta$. Denote by $S(\varphi)$ the set of all such solutions of (8). Then the mapping S, so defined, is a multivalued mapping on P satisfying the following properties:

(i) For each $\varphi \in P$, $S(\varphi)$ is a convex subset of P.

(ii) $\{\varphi_n\}, \{\Psi_n\} \subseteq P$, $\| \varphi_n - \varphi \| \to 0$, $\| \Psi_n - \Psi \| \to 0$, $\Psi_n \in S(\varphi_n)$, implies $\Psi \in S(\varphi)$.

(iii) $\overline{S(P)}$ is compact.

These properties follow easily from the linearity of (8) and the Ascoli–Arzela Theorem. We may now apply the Eilenberg–Montgomery Fixed Point Theorem [1] (a generalization of the Schauder Theorem to set-valued mappings) to conclude that S has a fixed point in P, i.e., there exists $\varphi \in P$ such that $\varphi \in S(\varphi)$. Fixed points of S, however, are periodic solutions of (S).

4. In this section, we indicate how some of the previous results may be generalized to functional differential equations.

Let $h(t)$ be a continuous nonnegative function defined on $[0, T]$, let $\tau = \max\{h(t) : 0 \leqslant t \leqslant T\}$. If $x(t)$ is a continuous R^n-valued function defined on $[-\tau, T]$, then for every $t \in [0, T]$, we define the function x_t by

$$x_t(\theta) = x(t + \theta), \qquad -h(t) \leqslant \theta \leqslant 0.$$

By C_t, we shall mean the set of continuous R^n-valued functions on $[-h(t), 0]$. Consider the functional differential equation

$$x'(t) = f(t, x(t), x_t), \qquad 0 \leqslant t \leqslant T, \tag{9}$$

where for each t, $f(t, \cdot, \cdot) : R^n \times C_t \to R^n$. We say that $f(t, \cdot, \cdot)$ is of type K if for every $x, y \in R^n$ with $x_j \leqslant y_j$, $i \neq j$, $x_i = y_i$, $j = 1,..., n$ and for each $u, v \in C_t$ with $u \leqslant v$ (i.e., $u(\theta) \leqslant v(\theta)$, $-h(t) \leqslant \theta \leqslant 0$)

$$f_i(t, x, u) \leqslant f_i(t, y, v).$$

Let C_t be endowed with the supnorm $\| \cdot \|_t$ and assume that f satisfies a Lipschitz condition of the type

$$| f(t, x, u) - f(t, y, v)| \leqslant k(| x - y | + \| u - v \|_t).$$

It is then well known (see, e.g., Driver [2]) that for every $\varphi \in C_0$, the initial value problem

$$\begin{aligned} x'(t) &= f(t, x(t), x_t) \\ x_0 &= \varphi \end{aligned} \tag{10}$$

has a unique solution whenever f is continuous in the sense that if $x(t)$ is a continuous function on $[-\tau, T]$, then $f(t, x(t), x_t)$ is a continuous function of t.

LEMMA 6. *Let there exist continuous functions* $\alpha, \beta : [-\tau, T] \to R^n$ *with* $\alpha(t) \leqslant \beta(t)$, $-\tau \leqslant t \leqslant T$, *such that*

$$D_{-}\alpha(t) < f(t, \alpha(t), \alpha_t), \qquad D^{-}\beta(t) > f(t, \beta(t), \beta_t), \qquad 0 < t \leqslant T.$$

Then, for every $\varphi \in C_0$, $\alpha_0 \leqslant \varphi \leqslant \beta_0$, *the initial value problem* (10) *has a unique solution* $x(\varphi)$ *such that* $\alpha_t \leqslant x_t(\varphi) \leqslant \beta_t$, $0 \leqslant t \leqslant T$.

Using this lemma, we may now easily establish the following theorem.

THEOREM 7. *Let* $h(0) = h(T)$ *and let* α *and* β *be as in Lemma* 6 *and satisfy in addition* $\alpha_0 \leqslant \alpha_T$, $\beta_0 \geqslant \beta_T$. *Then there exists a solution* $x(t)$ *of* (9) *such that* $x_0 = x_T$ *and* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$, $t \in [0, T]$.

Proof. Let $\varphi \in C_0$ be such that $\alpha_0 \leqslant \varphi \leqslant \beta_0$. Define the mapping S on C_0 into itself (note $C_0 = C_T$) by

$$S(\varphi) = x_T(\varphi).$$

It is easy to see that S, so defined, satisfies the hypotheses of the Schauder Theorem and we conclude that S has a fixed point.

Remark. Using approximation arguments as in Section 2, one may now verify that in the case of differential difference equations

$$x' = f(t, x(t), x(t-h)),$$

the Lipschitz condition on f may be removed and the strict differential inequalities may be replaced by weak ones.

Further, similar results may be established for equations without previous history, i.e., $0 \leqslant h(t) \leqslant t$, without requiring that $h(T) = h(0)$, where we seek solutions $x(t)$ satisfying the condition $x(0) = x(T)$.

For purposes of illustration, we consider the following example.

Let all functions considered be continuous. Let $g : [0, T] \times R \to R$, $k : [0, T]^2 \to R$, $k \geqslant 0$, $h : [0, T] \times R \to R$, where h is nondecreasing in its second argument.

Consider the integro-differential equation

$$x'(t) = g(t, x(t)) + \int_0^t k(t, s)\, h(s, x(s))\, ds. \tag{11}$$

Assume that

$$g(t, \beta) + \int_0^t k(t, s)\, h(s, \beta)\, ds \leqslant 0 \leqslant g(t, \alpha) + \int_0^t k(t, s)\, h(s, \alpha)\, ds$$

for some constants α and β, $\alpha \leqslant \beta$. Then there exists a solution $x(t)$ of (11) such that $x(0) = x(T)$ and $\alpha \leqslant x(t) \leqslant \beta$, $0 \leqslant t \leqslant T$.

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